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Generalized Common Neighbor Polynomial of Graphs

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Abstract: Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered vertex pair of distinct vertices of G. The i-common neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v$ and $|N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n-2$. The polynomial

$$N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$$

is defined as the common neighbor polynomial of G. In this paper we introduce the generalized i-common neighbor set and the generalized common neighbor polynomial of a graph.

Key Words: Generalized i-common neighbor set, generalized common neighbor polynomial.

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§1. Introduction

A group of individuals who belong to various social, economical and occupational status, may show consensus in some areas. The similarity strength of such groups can be measured by the number of areas in which they are mutually interested. Visualizing the situation graphically, the individuals and different areas of interest may represented by nodes of the bipartite sets A and B of a bipartite graph and a node in A is connected to a node in B if the corresponding individual have the particular area of interest. Then the similarity strength of a group of r individuals is given by the number of common neighbors shared by the corresponding nodes. These concepts motivated the authors to define generalized i-common neighbor sets and common neighbor polynomial of graphs.

Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered vertex pair of distinct vertices of G. The i-common neighbor set of G is defined by the present authors as $N(G, i) := \{(u, v) : u, v \in V, u \neq v$ and $|N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n-2$. The polynomial $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$ is defined as the common

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neighbor polynomial of G [2].

A family Δ of finite subsets of a set V is a simplicial complex [6] if it satisfy the condition that whenever $\sigma \in \Delta$ and $\tau \subset \sigma$ then, $\tau \in \Delta$. If $\sigma \in \Delta$ is of cardinality $k+1$, then σ is called a k-simplex and every $\tau \subset \sigma$ is a face of the simplex. The dimension of a simplex is one less than its cardinality. If a simplex is not a proper subset of any other simplexes in the complex, then it is a facet of the complex.

In this paper we introduce the generalized i-common neighbor set and the generalized common neighbor polynomial of graphs. Also we derive the generalized common neighbor polynomial of some well known graph classes. Moreover, we define the simplicial complex of a graph G and introduce the cluster of a vertex $v \in G$ as a simplicial complex of G. These concepts are used to deduce some interesting properties of generalized i-common neighbor sets.

§2. Main Results

In this section we first introduce the definition of *generalized i-common-neighbor set* and then define the *generalized common neighbor polynomial* of a graph. Moreover, we discuss some properties of generalized $i-$ common neighbor sets and also derive the generalized common neighbor polynomial of some well known graph classes. Also we express generalized common neighbor sets using the theory of simplicial complexes in order to deduce some interesting properties of the sets.

2.1 Generalized Common Neighbor Sets and Common Neighbor Polynomial of Graphs

Definition 2.1 *Let $G(V,E)$ be a graph of order n. Let $\mathscr{L}_r$ denotes the set of all unordered r-tuples of distinct elements of V. For $0 \leq i \leq n-r$, the generalized i-common neighbor set of G is defined as follows:*

$$N_r(G,i) = \{(u_1, u_2, \cdots, u_r) \in \mathscr{L}_r : |\cap_{k=1}^{r} N(u_k)| = i\}.$$

Definition 2.2 *Let G be a graph of order n. For $0 < r \leq n$ the generalized common neighbor polynomial, $N_r[G;x]$, of G is defined as*

$$N_r[G;x] = \sum_{i=0}^{(n-r)} |N_r(G,i)| x^i.$$

Throughout this paper, r denotes an integer such that $1 \leq r \leq n$. We observe the following simple properties of $N_r[G;x]$:

(i) $N_2[G;x] = N[G;x]$, the common neighbor polynomial of the graph G;

(ii) $N_r[G;x]$ is a polynomial of degree at most $(n-r)$;

(iii) Isomorphic graphs have same generalized common neighbor polynomials;

(*iv*) $N_r(G,i)=\phi$ for $n-r+1\le i\le n$;
(*v*) $N_r[G;1]=\sum_{i=1}^{n-r}|N_r(G,i)|=\binom{n}{r}$;
(*vi*) $N_r[G;0]$ gives the number of elemnts in $\mathscr{L}_r$ having no common neighbors.

Theorem 2.3 *For any graph G, we have $|N_r(G,0)|\le|N_s(G,0)|$ if $r\le s\le n$.*

Proof It is enough to show that corresponding to each r-tuple of vertices in $N_r(G,0)$, there are one or more s-tuples of vertices in $N_s(G,0)$. Let $(u_1,u_2,\cdots,u_r)\in N_r(G,0)$. Let $u_{r+1},u_{r+2},\cdots,u_s$ be any $s-r$ vertices in $V-\{u_1,u_2,\cdots,u_r\}$. Then the s-tuple of vertices $(u_1,u_2,\cdots,u_r,u_{r+1},\cdots,u_s)$ have no common neighbors in G since the first r vertices have no common neighbors in G. Then $(u_1,u_2,\cdots,u_r,u_{r+1},\cdots,u_s)\in N_s(G,0)$. This completes the proof. □

Theorem 2.4 *For any graph G, if $(u_1,u_2,\cdots,u_r)\in N_r(G,i)$, then $(u_1,u_2,\cdots,u_r,u_{r+1},\cdots,u_s)\notin N_s(G,j)$, where $r<s$ and $0<i<j$.*

Proof Let $(u_1,u_2,\cdots,u_r)\in N_r(G,i)$. Let $u_{r+1},u_{r+2},\cdots,u_s$ be any $s-r$ vertices in $V-\{u_1,u_2,\cdots,u_r\}$ such that $(u_1,u_2,\cdots,u_r,u_{r+1},\cdots,u_s)\in N_s(G,j)$ where $r<s$ and $0<i<j$. Then the vertices $u_1,u_2,\cdots,u_r,\cdots,u_s$ have j common neighbors in G where $j>i$. In particular, the vertices $u_1,u_2,\cdots,u_r$ have at least j common neighbors in G, a contradiction since $j>i$ and $(u_1,u_2,\cdots,u_r)\in N_r(G,i)$. □

Theorem 2.5 *For a complete graph K_n $(n\ge r)$,we have*

$$N_r[K_n;x]=\binom{n}{r}x^{n-r}.$$

Proof The proof follows from the fact that any r-tuple of vertices of K_n have $(n-r)$ common neighbors and there are $\binom{n}{r}$ such r-tuples of vertices. □

Theorem 2.6 *For a path graph P_n, we have $N_r[P_n;x]=\binom{n}{r}$, $r\ge 3$.*

Proof The result follows from the fact that no r-tuple of vertices in P_n where $r\ge 3$ have common neighbors in P_n. □

Theorem 2.7 *For a cycle graph C_n, we have $N_r[C_n;x]=\binom{n}{r}$, $r\ge 3$.*

Proof The result follows from the fact that no r-tuple of vertices in C_n where $r\ge 3$ have common neighbors in C_n. □

Theorem 2.8 *For a complete bipartite graph $K_{m,n}$, we have*

$$N_r[K_{m,n};x]=\binom{m}{r}x^n+\binom{n}{r}x^m+\sum_{j=1}^{r-1}\binom{m}{j}\binom{n}{r-j}.$$

Proof Let M, N be the bipartite sets of vertices of $K_{m,n}$ where $|M| = m$ and $|N| = n$. We consider the following 3 cases according to the selection of vertices in the $r-$tuple $(u_1, u_2, \cdots, u_r)$.

Case 1. $u_k \in M$ for $1 \leq k \leq r$.

In this case, each of the $r-$tuple of vertices $(u_1, u_2, \cdots, u_r)$ have n common neighbors contributing the term $\binom{m}{r}x^n$ in the generalized common neighbor polynomial.

Case 2. $u_k \in N$ for $1 \leq k \leq r$.

In this case, each of the $r-$tuple of vertices $(u_1, u_2, \cdots, u_r)$ have m common neighbors contributing the term $\binom{n}{r}x^m$ in the generalized common neighbor polynomial.

Case 3. After a sufficient rearrangement of terms, let $u_k \in M$ for $1 \leq k \leq j$ and $u_k \in N$ for $j + 1 \leq k \leq r$.

For each j where $1 \leq j \leq r - 1$, the $r-$tuple of vertices $(u_1, u_2, \cdots, u_r)$ has no common neighbor in $K_{m,n}$ and there are $\binom{m}{j}\binom{n}{r-j}$ such $r-$ tuples.

This completes the proof. □

Corollary 2.9 *For a star graph $K_{1,n}$, we have $N_r[K_{1,n}; x] = \binom{n}{r}x + \binom{n}{r-1}$ for $r \geq 2$.*

A bistar graph $B_{n,n}$is the union of two star graphs $K_{1,n}$ with centres u and v together with a new edge uv.

Theorem 2.10 *For a bistar graph $B_(n, n)$ we have*

$$N_r[B_{n,n}; x] = 2\binom{n+1}{r}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1}\binom{n}{m}\binom{n}{r-m} + \delta_{r2},$$

where $\delta_{rj} = \begin{cases} 1 & if\ r = j, \\ 0 & if\ r \neq j. \end{cases}$

Proof Let $S = \{s_1, s_2, \cdots, s_n\}$ and $T = \{t_1, t_2, \cdots, t_n\}$ be the pendent vertices of the star graphs with center vertices u and v respectively, which together with the edge uv constitute the bistar graph $B_{n,n}$. Let $(u_1, u_2, \cdots, u_r)$ be any r-tuple of vertices of $B_{n,n}$. We consider different cases according to the selection of vertices in the r-tuple where $r > 2$.

Case 1. $u_i \in S$ or $u_i \in T$ for all $i \in \{1, 2, \cdots, r\}$.

All the r-tuple of vertices under this case have exactly one common neighbor u or v according as $u_i \in S$ or $u_i \in T$. Hence this case contribute the term $2\binom{n}{r}x$ to the generalized common neighbor polynomial of $B_{n,n}$.

Case 2. For $i \in \{1, 2, \cdots, r\}$, $u_i = v$ for exactly one i and all other $u_i \in S$.

The r-tuple of vertices under this case have exactly one common neighbor u and there are $\binom{n}{r-1}$ such r-tuples thereby contributing the term $\binom{n}{r-1}x$ to $N_r[B_{n,n}; x]$.

Case 3. For $i \in \{1, 2, \cdots, r\}$, $u_i = u$ for exactly one i and all other $u_i \in T$.

By a similar argument as in Case 2, the r-tuples in this case also contributes the term $\binom{n}{r-1}x$ to $N_r[B_{n,n}; x]$.

Case 4. For $i \in \{1, 2, \cdots, r\}$, $u_i = u$ or $u_i = v$ for exactly one i where all other $u_i \in S$ or $\in T$ respectively.

All the r-tuple of vertices under this case have no common neighbors and there are $2\binom{n}{r-1}$ such r-tuples.

Case 5. After an appropriate rearrangement of terms of the r-tuple, let $u_1, u_2, \cdots, u_m \in S$ and $u_{m+1}, u_{m+2}, \cdots, u_r \in T$ where $1 \leq m \leq r-1$.

All the r-tuple of vertices under this case have no common neighbors and this case contribute the term $\sum_{m=1}^{r-1} \binom{n}{m}\binom{n}{r-m}$ to $N_r[B_{n,n}; x]$.

It follows that

$$\begin{aligned} N_r[B_{n,n}; x] &= 2\binom{n}{r}x + 2\binom{n}{r-1}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1}\binom{n}{m}\binom{n}{r-m} \\ &= 2\binom{n+1}{r}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1}\binom{n}{m}\binom{n}{r-m} \end{aligned}$$

This completes the proof with a sufficient remark that when $r = 2$, there is a pair of vertices (u, v) having no common neighbors. □

Theorem 2.11 *Every graph G contains $|N_r(G, i)|$ number of complete bipartite subgraphs $K_{i,r}$ where $1 \leq i \leq n - r$.*

Proof Note that corresponding to each $r-$tuples of vertices $(u_1, u_2, \cdots, u_r) \in N_r(G, i)$, the vertices $u_1, u_2, \cdots, u_r$ together with their i common neighbors constitute a complete bipartite subgraph $K_{i,r}$. Hence the result follows. □

Theorem 2.12 *The generalized common neighbor polynomial of a graph G is non constant if and only if there exists a star $K_{1,r}$ in G where $1 \leq r \leq n$.*

Proof Let $N_r[G; x]$ be a non constant polynomial of degree $m \geq 1$. Then there exists an r-tuple of vertices $(u_1, u_2, \cdots, u_r)$ in G which has at least one common neighbor, say w in G. Then w together with the vertices $u_1, u_2, \cdots, u_r$ produces a star $K_{1,r}$ in G.

Conversely let there exists a star $K_{1,r}$ in G where $1 \leq r \leq n$. Let $u_1, u_2, \cdots, u_r$ be the pendent vertices of $K_{1,r}$. Then the center of the star graph $K_{1,r}$ is a common neighbor of the r-tuple $(u_1, u_2, \ldots, u_r)$. The result follows from the fact that $N_r(G, i) \neq \phi$ for some $i \geq 1$. □

Corollary 2.13 *If a graph G doesn't contain any star graph $K_{1,r}$ as a subgraph where $1 \leq r \leq n$, then the generalized common neighbor polynomial $N_r[G; x] = \binom{n}{r}$.*

Theorem 2.14 *The generalized common neighbor polynomial $N_r[G; x]$ of a graph G is of degree*

$k \geq 1$ *if and only if* k *is the largest integer such that* G *has a complete bipartite subgraph* $K_{r,k}$.

Proof Assume that $N_r[G;x]$ of a graph G is of degree $k \geq 1$. Then, $|N_r(G,k)| \neq \phi$. Take $(u_1, u_2, \cdots, u_r) \in N_r(G,k)$. Then the vertices $u_1, u_2, \cdots, u_r$ together with their k common neighbors constitute a complete bipartite subgraph $K_{r,k}$ of G. Moreover, if G contains $K_{r,j}$ as a subgraph,where $j \geq k+1$, then G contains an r-tuple of vertices having j common neighbors where $j \geq k+1$ which is a contradiction since $N_r[G;x]$ is of degree k. This proves the necessary part of the theorem.

Conversely, we assume that k is the largest integer such that G has a complete bipartite subgraph $K_{r,k}$. If possible, let $N_r[G;x]$ is of degree $j \geq k+1$. Then G contains an r-tuple of vertices having at least $k+1$ common neighbors. These r vertices together with their $k+1$ common neighbors constitute a complete bipartite subgraph $K_{r,k+1}$ of G which is a contradiction to the assumption. □

Definition 2.15 *Two graphs* G *and* H *are said to be* CNP_r *equivalent if* $N_r[G;x] = N_r[H;x]$. *The set of all graphs which are* CNP_r *equivalent to* G *is denoted by* $[G]_{\mathcal{N}_r}$.

Theorem 2.16 *For any graph* G, $\overline{G} \in [G]_{\mathcal{N}_r}$ *if and only if there are* $|N_r(G,i)|$ *number of* $r-$*tuple of vertices in* G *which dominate* $n-i$ *vertices of* G.

Proof First, suppose that $\overline{G} \in [G]_{\mathcal{N}_r}$. Then $|N_r(G,i)| = \left|N_r(\overline{G},i)\right|$ for $0 \leq i \leq n-r$. Let $(u_1, u_2, \cdots, u_r) \in N_r(\overline{G},i)$. Since the vertices $u_1, u_2, \cdots, u_r$ have only i common neighbors in $\overline{G}$, all the remaining $n-i$ vertices in G are adjacent to at least one of the vertices in $\{u_1, u_2, \cdots, u_r\}$. Then $\{u_1, u_2, \cdots, u_r\}$ dominate exactly $n-i$ vertices of G. Since $|N_r(G,i)| = |N_r(\overline{G},i)|$, it follows that G has $|N_r(G,i)|$ number of $r-$tuple of vertices which dominate $n-i$ vertices of G.

Conversely assume that there are $|N_r(G,i)|$ number of $r-$tuple of vertices in G which dominate $n-i$ vertices of G. From the proof of first part of the theorem, the $r-$tuples of vertices in G which dominate exactly $n-i$ vertices of G are those which belongs to $N_r(\overline{G},i)$. It follows that $|N_r(G,i)| = |N_r(\overline{G},i)|$ and hence $N_r[G;x] = N_r[\overline{G};x]$. This completes the proof.□

Corollary 2.17 *Let* G *be a graph of order* n. *If* $\overline{G} \in [G]_{\mathcal{N}_r}$, *then* $|N_r(G,0)|$ *gives the number of dominating sets in* G *of order* r.

Lemma 2.18 *Let* G *be a connected graph with* $n > 3$ *vertices. If all the pairs of edges of* G *have a common end vertex, then* G *is a star graph.*

Proof Since $n > 3$ and G is connected, the number of edges m should be greater than or equal to 3. We will prove the result by using method of induction on the number of edges m of G. Clearly the result is true for $m = 3$. Let the result be true for all graphs G with less than m edges. And let G be a graph with m edges such that all the pairs of edges have a common end vertex. By deleting any edge e from G, we have a graph with $m-1$ edges. Clearly all the pairs of edges of $G-e$ are incident to a common vertex. Hence by induction assumption, $G-e$ is a star. Let v be the center vertex of the star so that the edges of $G-e$ be represented by $e_i = vv_i$ where $i = 1, 2, \cdots, m-1$. Since the edges e and e_1 of G are incident to a common vertex, either $e = vw$ or $e = v_1w$. In the first case G is a star and the proof is complete. And in the second case, there are two possibilities according as w belongs to $\{v_1, v_2, \cdots, v_{m-1}\}$ or not. If w belongs to the set, let $w = v_i$ where $i \in \{1, 2, \cdots, m-1\}$. Then the edges v_1w and and vv_{i+1} have no common end vertex. which ruled out the possibility. If w doesn't belong to the set, then the edges v_1w and vv_3 have no common end vertex. Hence by the induction assumption, the second case is ruled out. Hence the result follows. □

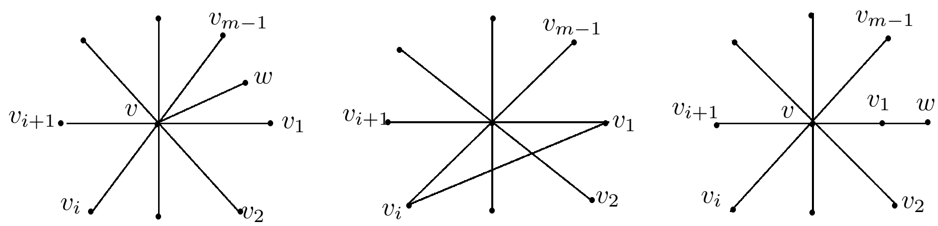

Figure 1 Figure showing different cases of Lemma 2.18

The line graph $L(G)$ of a graph G is the graph with vertex set the set of all edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are incident to a common vertex.

Theorem 2.19 *Let G be a connected graph of order $n > 3$. The number k of cliques of size $r > 1$ in the line graph of G is given by* $k = \sum_{i=1}^{n-r} i|N_r(G,i)|$.

Proof Let S be the collection of all r-tuples of vertices $(u_1, u_2, \cdots, u_r)$ of G which have at least one common neighbor in G. Also let the r-tuple $(u_1, u_2, \cdots, u_r)$ repeat as many times in S as its number of common neighbors. Then, $|S| = \sum_{i=1}^{n-2} i|N_r(G,i)|$. Let P be the collection of all cliques of size r in the line graph $L(G)$ of G. Let the vertices of $L(G)$ be denoted by uv where u, v are adjacent vertices of G. Define $\phi : S \to P$ as follows.

Let $u = (u_1, u_2, \cdots, u_r) \in S$ which repeats i-times in S. Let these i members be represented by $u_k = (u_1, u_2, \cdots, u_r)^{(k)}$ where $k = 1, 2, \cdots, i$. Then each $(u_1, u_2, \cdots, u_r)^{(k)}$ can be assigned to exactly one common neighbor w_k of $(u_1, u_2, \cdots, u_r)$ in G. It follows that all the pairs of vertices u_lw_k and u_mw_k where $l, m \in \{1, 2, \cdots, r\}$ and $l \neq m$ are adjacent vertices of $L(G)$ which forms a clique C_{uk} of size r in $L(G)$.

Now define $\phi : S \to P$ as $\phi((u_1, u_2, \cdots, u_r)^{(k)} = C_{uk}$. Clearly ϕ is one-one. We claim that $\phi : S \to P$ is onto. Let C be a clique of size r in the line graph $L(G)$ of G. Since any pair of vertices of C are adjacent in $L(G)$, all the pairs of edges in G which constitute the vertex set of C, have a common end vertex in G. Hence by Lemma 2.18, those edges form a star in G whose pendent vertices forms an r-tuple $(u_1, u_2, \cdots, u_r) \in S$ such that $\phi(u_1, u_2, \cdots, u_r) = C$. Thus ϕ is onto.

It follows that ϕ is a bijection from S to P and $|S| = |P|$. This completes the proof. □

Corollary 2.20 *Let G be a graph of order n. Then the number of edges of the line graph $L(G)$ of G equals* $\sum_{i=1}^{n-2} i|N(G,i)|$.

Proof The result follows from the fact that the 2-cliques of any graph are the edges of the graph. □

Theorem 2.21(Schwartz 1969 and Ghirlanda 1963) *A graph is isomorphic to its line graph if and only if it is regular of degree two.*

Corollary 2.22 *If a graph G is regular of degree two, then the number of edges of G equals* $\sum_{i=1}^{n-2} i|N(G,i)|$.

2.2 Simplicial Complexes of Graphs and Common Neighbor Sets

In this section, we first define the simplicial complex of a graph G and introduce the cluster of a vertex $v \in G$ as a simplicial complex of G. Then we incorporate the concept of generalized i-common neighbor set of a graph with the cluster of vertices in it, to deduce some interesting properties of generalized i-common neighbor sets.

Definition 2.23 *Let $G(V,E)$ be a graph and let Δ be a collection of subsets of V. The elements of Δ are called simplexes. Let τ be an element in Δ. Then the subsets of τ are called its faces. We say that Δ is a simplicial complex of G if for every τ in Δ, all its faces are in Δ.*

Let G be a simple finite graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. For each vertex v_i, the cluster of v_i is defined as

$$\mathrm{clr}(v_i) =: \{W \subset V : v_i \in \cap_{v\in W} N(v)\}.$$

Then each $\mathrm{clr}(v_i)$ where $i \in \{1, 2, \cdots, n\}$ is a simplicial complex of G. We may consider $clr(v_i)$ as a simplicial complex of G generated by the vertex v_i. Note that each simplex W of $clr(v_i)$ spans a subgraph of G which is a star graph with center vertex v_i. So these simplexes are called the stars of v_i denoted by $str(v_i)$. The facets of $clr(v_i)$ are the maximal stars in $clr(v_i)$.

Lemma 2.24 *Let v be a vertex of the graph G having degree d. Then the cluster of v contains $\binom{d}{r}$ number of $(r-1)$-simplexes.*

Proof Let S be the set of all neighbors of the vertex v such that $|S| = d$. Any subset S_1 of S with cardinality $r \leq d$ will act as a r-tuple of vertices with v as a common neighbor. There are exactly $\binom{d}{r}$ distinct subsets of S with cardinality r and these subsets are exactly the $(r-1)$-simplexes of the cluster of v. Hence the result follows. □

Theorem 2.25 *Let $G(V,E)$ be a simple graph and let $v \in V$. Let f_i, $i = 1, 2, \cdots, m$ be the facets of the simplicial complex $clr(v)$. If the facet f_i is of cardinality d_i, then $clr(v)$ contains* $\sum_{i=1}^{m}\sum_{r=1}^{d_i}\binom{d_i}{r}$ *distinct simplexes.*

Proof According to the definition of a simplicial complex, all the subsets of its facets must also be simplexes of the complex. If the facet f_i of $clr(v)$ is of cardinality d_i, there are $\binom{d_i}{r}$

simplexes of dimension r in $clr(v)$. Thus corresponding to each facet f_i, there are $\sum_{r=1}^{d_i}\binom{d_i}{r}$ distinct simplexes in $clr(v)$. As there are m facets, the result follows. □

Theorem 2.26 *If G is a graph having degree sequence $(d_1, d_2, \cdots, d_n)$, then we have the following:*

$$\sum_{i=1}^{n-r} i|N_r(G,i)| = \sum_{i=1}^{n}\binom{d_i}{r}.$$

Proof Let $clr(v_i)$, $i = 1, 2, \cdots, n$ be the simplicial complexes generated by the vertices $v_1, v_2, \cdots, v_n$ of the graph G. We will show that the expression on both sides of the equation equates the total number of $(r-1)$-simplexes of $clr(v_i)$ where $i = 1, 2, \cdots, n$.

By Lemma 2.24, the number of $(r-1)$-simplexes in $clr(v_i)$ is given by $\binom{d_i}{r}$ where d_i is the degree of the vertex v_i which generates $clr(v_i)$. Hence if all the simplicial complexes $clr(v_i)$, $i \in \{1, 2, \cdots, n\}$ are taken into account, there are altogether $\sum_{i=1}^{n}\binom{d_i}{r}$ number of $(r-1)$-simplexes.

Now, for a fixed $i \in \{1, 2, \cdots, n\}$, the $(r-1)$-simplexes of $clr(v_i)$ are exactly r-tuples of vertices with v_i as a common neighbor. Hence the total number of $(r-1)$-simplexes of $clr(v_i)$, $i = 1, 2, \cdots, n$ equals the number of r-tuples of vertices with at least one common neighbor where the r-tuple with i common neighbors has to be counted i times. From the definition of generalized i-common neighbor set of G, the number of such r-tuple of vertices is given by $\sum_{i=1}^{n-r} i|N_r(G,i)|$. This completes the proof. □

Theorem 2.27 *The generalized i-common neighbor set $N_r(G,i)$ is the set of all $(r-1)$-simplexes which belongs to the intersection of exactly i of the clusters of vertices of G.*

Proof Let W be a $(r-1)$-simplex which belongs to a simplicial complex $clr(v_j)$, for some $j \in \{1, 2, \cdots, n\}$. From the definition of $clr(v_j)$, it is clear that the members of W constitute a r-tuple of vertices of G having v_j as a common neighbor. Now fix an integer i such that $1 \leq i \leq n-2$. W belongs to exactly i of the $clr(v_j)$, if and only if the corresponding r-tuple of vertices has exactly i common neighbors. It follows that $W \in N_r(G,i)$. □

Remark 2.28 *We observe the following properties of the simplicial complexes $clr(v_i)$ generated by the vertices v_i of a simple graph G.*

For $i, j, k \in \{1, 2, \cdots, n\}$,

(1) *If a simplicial complex $clr(v_i)$ is generated by a vertex v_i, then, $\{v_i\} \notin clr(v_i)$;*

(2) *$clr(v_i)$ contains all possible unions of the 0-simplexes containing in it;*

(3) *If $\{v_i\} \in clr(v_j)$, then $\{v_j\} \in clr(v_i)$.*

The first statement follows from the fact that a vertex cannot be adjacent to itself as we are considering only simple graphs. The second and third statements directly follows from the definition of $clr(v_i)$.

The following theorem shows that these are the sufficient conditions for a collection of simplicial complexes $\{clr(v_i)\}, i \in \{1, 2, \cdots, n\}$ on a set of cardinality n to be generated by a set of vertices $V = \{v_1, v_2, \cdots, v_n\}$ of a simple graph G.

Theorem 2.29 *Let $V = \{v_1, v_2, \cdots, v_n\}$ be any set of n elements. If $clr(v_i)$, $i \in \{1, 2, \cdots, n\}$ are simplicial complexes on the set V satisfying the conditions (1),(2) and (3) stated in above remark, then there exists a simple graph G with vertex set V where $clr(v_i)$ is the simplicial complex generated by the vertex v_i of G.*

Proof Given a set of elements $V = \{v_1, v_2, \cdots, v_n\}$ and a collection of simplicial complexes $\{clr(v_i)\}, i \in \{1, 2, \cdots, n\}$ on the set V, construct a graph with vertex set V and edge set E where an edge $v_iv_j \in E$ if and only if $\{v_j\} \in clr(v_i)$.

By condition (1), $\{v_i\} \notin clr(v_i)$ which implies that G has no loops. Also by condition (3), if $\{v_i\} \in clr(v_j)$, then $\{v_j\} \in clr(v_i)$ which implies that the adjacency of vertices of the graph is well defined in the sense that whenever v_i adjacent to v_j, v_j is adjacent to v_i also.

Now we will prove that $\{clr(v_i)\}$ are the simplicial complexes generated by the vertices $\{v_i\}$ of the graph G. Let V_1 be a subset of V which belongs to $clr(v_i)$. Then $V_1 = \{v_{j_1}, v_{j_2}, \cdots, v_{j_m}\}$ where each of the vertices in the set are adjacent to a vertex $v_i \in V$ in G. Then $v_iv_{j_k} \in E$ and $\{v_{j_k}\} \in clr(v_i)$ for all $k \in \{1, 2, \cdots, m\}$. Hence by condition(3), all the subsets of V_1 are in $clr(v_i)$. It follows that $clr(v_i)$ is a simplicial complex on V. And by definition of edge set of G, it is generated by v_i. This completes the proof. □

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