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Various Domination Energies in Graphs

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Abstract: Representing a subset of vertices in a graph by means of a matrix was introduced by E. Sampath Kumar. Let $G(V,E)$ be a graph and $S \subseteq V$ be a set of vertices. We can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if, $v_i \in S$. In this paper we study the set S being dominating set and corresponding domination energy of some class of graphs.

Key Words: Adjacency matrix, Smarandachely k-dominating set, eigenvalues, energy of graph, distance energy, Laplacian energy.

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§1. Introduction

A set $D \subseteq V$ of G is said to be a Smarandachely k-dominating set if each vertex of G is dominated by at least k vertices of S and the Smarandachely k-domination number $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the domination number of G and denoted by $\gamma(G)$ in general.

The concept of graph energy arose in theoretical chemistry where certain numerical quantities like the heat of formation of a hydrocarbon are related to total π electron energy that can be calculated as the energy of corresponding molecular graph. The molecular graph is a representation of the molecular structure of a hydrocarbon whose vertices are the position of carbon atoms and two vertices are adjacent if there is a bond connecting them.

Eigen values and eigenvectors provide insight into the geometry of the associated linear transformation. The energy of a graph is the sum of the absolute values of the Eigen values of its adjacency matrix. From the pioneering work of Coulson []1 there exists a continuous interest towards the general mathematical properties of the total π electron energy ε as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of ε on molecular structure. The properties of $\varepsilon(G)$ are discussed in detail in [2, 3, 4].

The importance of Eigen values is not only used in theoretical chemistry but also in analyzing structures. Car designers analyze Eigen values in order to damp out the noise to reduce

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the vibration of the car due to music. Eigen values can be used to test for cracks or deformities in a solid. Oil companies frequently use Eigen value analysis to explore land for oil. Eigen values are also used to discover new and better designs for the future.

§2. Definitions and Notations

Representation of a subset of vertices of a graph by means of a matrix was first introduced by E.Sampath Kumar [5]. Let $G(V,E)$ be a graph and $S \subseteq V$ be a set of vertices. We can represent the set S by means of a matrix as follows:

In the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S denoted by $A_S(G)$. The energy $E(G)$ obtained from the matrix $A_S(G)$ is called the set energy denoted by $E_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix as domination matrix denoted by $A_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the domination matrix $A_\gamma(G)$ is defined as domination energy denoted by $E_\gamma(G)$.

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The domination matrix of G is defined to be the square matrix $A_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the domination matrix denoted by $\kappa_1, \kappa_2, \kappa_3, \cdots, \kappa_n$ are said to be the A_γ Eigen values of G. Since the A_γ matrix is symmetric, its Eigen values are real and can be ordered $\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \cdots \geqslant \kappa_n$. Therefore, the domination energy

$$E_\gamma = E_\gamma(G) = \sum_{i=1}^{n} |\kappa_i|. \tag{1}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy ([2]).

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{2}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$ are the Eigen values of the adjacency matrix $A(G)$. Recall that in the last few years, the graph energy $E(G)$ and domination energy [9,10] or covering energy ([6]) has been extensively studied in the mathematics ([6,7]) and mathematic-chemical literature ([8,12]).

Definition 2.1(Minimal domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The domination energy $E_\gamma(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{\gamma-\min}(G)$.*

Definition 2.2(Maximal domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The domination energy $E_\gamma(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{\gamma-\max}(G)$.*

Similarly to domination energy of graph G, *distance domination energy, Laplacian domi-*

nation energy and Laplacian distance domination energy can also be defined as follows.

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The *distance matrix* of G, denoted by $D(G)$ is defined to be the square matrix $D(G) = [d_{ij}]$, where d_{ij} is the shortest distance between the vertex v_i and v_j in G. The Eigen values of the distance matrix denoted by $\mu_1, \mu_2, \mu_3, \cdots, \mu_n$ are said to be the D Eigen values of G. Since the $D(G)$ matrix is symmetric, its Eigen values are real and can be ordered $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots \geqslant \mu_n$. Therefore, the distance energy

$$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i|. \tag{3}$$

In the distance matrix $D(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the distance matrix can be considered as the *distance matrix of the set* S denoted by $D_S(G)$. The energy $E(G)$ obtained from the matrix $D_S(G)$ is called the *distance set energy* denoted by $D_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix is *distance domination matrix* denoted by $D_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the distance domination matrix $D_\gamma(G)$ is defined as *distance domination energy* denoted by $E_{D\gamma}(G)$.

The distance domination matrix of G is defined to be the square matrix $D_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the distance domination matrix denoted by $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n$ are said to be the D_γ Eigen values of G. Since the $D_\gamma(G)$ matrix is symmetric, its D-Eigen values are real and can be ordered as $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \cdots \geqslant \sigma_n$. Therefore, the distance domination energy

$$E_{D\gamma} = E_{D\gamma}(G) = \sum_{i=1}^{n} |\sigma_i|. \tag{4}$$

Definition 2.3(Minimal distance domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The distance domination energy $E_{D\gamma}(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{D\gamma-\min}(G)$.*

Definition 2.4(Maximal distance domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The distance domination energy $E_{D\gamma}(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{D\gamma-\max}(G)$.*

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The *Laplacian matrix* of G is denoted by $L(G)$ is defined to be the square matrix $L(G) = d(G) - A(G)$, where $A(G)$ and $d(G)$ are the adjacency matrix and diagonal matrix with vertex degree of G on the principal diagonal element respectively. The Eigen values of the Laplacian matrix denoted by $\psi_1, \psi_2, \psi_3, \cdots, \psi_n$ are said to be the L Eigen values of G. Since the $L(G)$ matrix is symmetric, its Eigen values are real and can be ordered $\psi_1 \geqslant \psi_2 \geqslant \psi_3 \geqslant \cdots \geqslant \psi_n$. Therefore, the Laplacian energy

$$E_L = E_L(G) = \sum_{i=1}^{n} |\psi_i|. \tag{5}$$

The energy $E_{L\gamma}(G)$ obtained from the matrix $L_S(G) = d(G) - A_S(G)$ is called the *Laplacian set energy* denoted by $L_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix is *Laplacian domination matrix* denoted by $L_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the Laplacian domination matrix $L_\gamma(G)$ is defined as *Laplacian domination energy* denoted by $E_{L\gamma}(G)$.

The Laplacian domination matrix of G is defined to be the square matrix $L_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the Laplacian domination matrix denoted by $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ are said to be the L_γ Eigen values of G. Since the $L_\gamma(G)$ matrix is symmetric, its L-Eigen values are real and can be ordered as $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \cdots \geqslant \alpha_n$. Therefore, the Laplacian domination energy

$$E_{L\gamma} = E_{L\gamma}(G) = \sum_{i=1}^{n} |\alpha_i|. \tag{6}$$

Definition 2.5(Minimal lapalcian domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The Laplacian domination energy $E_{L\gamma}(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{L\gamma-\min}(G)$.*

Definition 2.6(Maximal lapalcian domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The Laplacian domination energy $E_{L\gamma}(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{L\gamma-\max}(G)$.*

The energy $E_{LD\gamma}(G)$ obtained from the matrix $LD_S(G) = d(G) - D_S(G)$ is called the *Laplacian distance set energy* denoted by $LD_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix is *Laplacian distance domination matrix* denoted by $LD_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the Laplacian distance domination matrix $LD_\gamma(G)$ is defined as *Laplacian distance domination energy* denoted by $E_{LD\gamma}(G)$.

The Laplacian distance domination matrix of G is defined to be the square matrix $LD_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the Laplacian distance domination matrix denoted by $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ are said to be the LD_γ Eigen values of G. Since the $LD_\gamma(G)$ matrix is symmetric, its L-Eigen values are real and can be ordered as $\beta_1 \geqslant \beta_2 \geqslant \beta_3 \geqslant \cdots \geqslant \beta_n$. Therefore, the Laplacian distance domination energy

$$E_{LD\gamma} = E_{LD\gamma}(G) = \sum_{i=1}^{n} |\beta_i|. \tag{7}$$

Definition 2.7(Minimal Lapalcian distance domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The Laplacian dis-*

tance domination energy $E_{LD\gamma}(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{LD\gamma-\min}(G)$.

Definition 2.8(Maximal Lapalcian distance domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The Laplacian distance domination energy $E_{LD\gamma}(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{LD\gamma-\max}(G)$.*

§3. Various Domination Energies

Definition 3.1 *A book graph (B_m) consists of m quadrilaterals sharing a common edge. That is, it is a Cartesian product S_{m+1} and P_2, where S_m is a star graph and P_2 is the path graph on two nodes. Some book graphs are shown in Figure* 1.

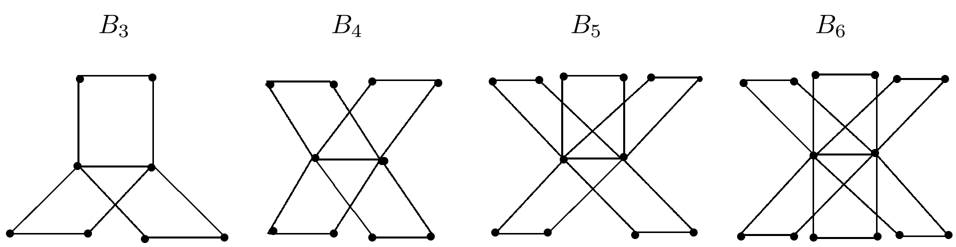

Figure 1 Book graph B_m, $3 \leqslant m \leqslant 6$

Theorem 3.1 *For $m \geqslant 3$, the minimum dominating energy of a book graph (B_m) is*

$$2\left(\sqrt{4m+1}+m-1\right).$$

Proof Calculation enables one to find the characteristic polynomial of B_m for $m \geqslant 3$ directly.

For $m = 3$, B_3 is a book graph with 8 vertices. The minimum dominating set is $S = \{v_1, v_2\}$.

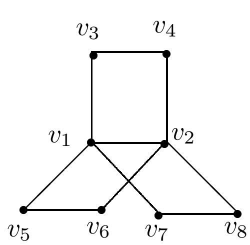

Figure 2 Book graph B_3

Calculation shows that the domination matrix and the characteristic polynomial of B_3 are respectively given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^8-2\kappa^7-9\kappa^6+12\kappa^5+18\kappa^4-18\kappa^3-13\kappa^2+8\kappa+3 = (\kappa-1)^2(\kappa+1)^2(\kappa^2-3\kappa-1)(\kappa^2+\kappa-3)$.

And calculation shows that the domination matrix and the characteristic polynomial of B_4 are respectively given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^{10}-2\kappa^9-12\kappa^8+16\kappa^7+38\kappa^6-36\kappa^5-52\kappa^4+32\kappa^3+33\kappa^2-10\kappa-8 = (\kappa-1)^3(\kappa+1)^3(\kappa^2-3\kappa-2)(\kappa^2+\kappa-4)$.

Similarly, the domination matrix and the characteristic polynomial of B_5 are respectively given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $(\kappa-1)^4(\kappa+1)^4(\kappa^2-3\kappa-3)(\kappa^2+\kappa-5)$, respectively.

And the characteristic polynomial of B_6 is given by

$$(\kappa-1)^5(\kappa+1)^5(\kappa^2-3\kappa-4)(\kappa^2+\kappa-6)$$

Generally, the characteristic polynomial of B_m using domination adjacency matrix is

$$(\kappa-1)^{m-1}(\kappa+1)^{m-1}(\kappa^2-3\kappa-(m-2))(\kappa^2+\kappa-m).$$

Solving the equation we get

$(\kappa-1)^{m-1}=0$, or $(\kappa+1)^{m-1}=0$, or $(\kappa^2-3\kappa-(m-2))=0$ or $(\kappa^2+\kappa-m)=0$. So $\kappa=1,1,1,\cdots,1$ ($(m-1)$times), or $\kappa=-1,-1,-1,\cdots,-1$($(m-1)$times).

By $(\kappa^2-3\kappa-(m-2))=0$, we get

$$\kappa_1=\frac{1}{2}\left(3-\sqrt{4m+1}\right) \text{ and}$$
$$\kappa_2=\frac{1}{2}\left(3+\sqrt{4m+1}\right) \quad \text{here } m\geqslant 3.$$

By $(\kappa^2+\kappa-m)=0$ we know that

$$\kappa_3=\frac{1}{2}\left(-1-\sqrt{4m+1}\right) \text{ and}$$
$$\kappa_4=\frac{1}{2}\left(-1+\sqrt{4m+1}\right)$$

Hence,

$$\begin{aligned}E_{\gamma-\min}=E_{\gamma-\min}(G)&=\sum_{i=1}^{n}|\kappa_i|\\&=(m-1)+(m-1)+\left|\frac{1}{2}\left(3-\sqrt{4m+1}\right)\right|\\&\quad+\left|\frac{1}{2}\left(3+\sqrt{4m+1}\right)\right|+\left|\frac{1}{2}\left(-1-\sqrt{4m+1}\right)\right|\\&\quad+\left|\frac{1}{2}\left(-1+\sqrt{4m+1}\right)\right|.\end{aligned}$$

Therefore,

$$E_{\gamma-\min}=E_{\gamma-\min}(B_m)=2\left(\sqrt{4m+1}+m-1\right).$$

This completes the proof. □

Theorem 3.2 *For $m\geqslant 3$, the minimum distance domination energy of a book graph (B_m) is $4(m-1)+\sqrt{25m^2-24m+36}+\sqrt{m}\sqrt{m+4}$.*

Proof Calculation enables one to find the characteristic polynomial of B_m for $m\geqslant 3$ directly.

For $m = 3$, B_3 is a book graph with 8 vertices. The minimum dominating set is $S = \{v_1, v_2\}$. Calculation shows that the distance domination matrix and the characteristic polynomial of B_3 are respectively given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 1 & 0 & 3 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 2 & 1 & 0 & 3 & 2 \\ 1 & 2 & 2 & 3 & 2 & 3 & 0 & 1 \\ 2 & 1 & 3 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^8 - 2\sigma^7 - 111\sigma^6 - 512\sigma^5 - 545\sigma^4 + 504\sigma^3 + 240\sigma^2 = \sigma^2 (\sigma+4)^2 \left(\sigma^2 - 13\sigma - 5\right)\left(\sigma^2 + 3\sigma - 3\right)$.

Similarly, calculation shows that the distance domination matrix and the characteristic polynomial of B_4 are respectively given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 & 3 & 2 & 3 & 2 & 3 \\ 2 & 1 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 3 & 2 & 1 & 0 & 3 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 2 & 3 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 2 & 3 & 2 & 1 & 0 & 3 & 2 \\ 1 & 2 & 2 & 3 & 2 & 3 & 2 & 3 & 0 & 1 \\ 2 & 1 & 3 & 2 & 3 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^{10} - 2\sigma^9 - 200\sigma^8 - 1512\sigma^7 - 4048\sigma^6 - 2240\sigma^5 + 4352\sigma^4 + 1024\sigma^3 = \sigma^3 (\sigma+4)^3 \left(\sigma^2 - 18\sigma - 4\right)\left(\sigma^2 + 4\sigma - 4\right)$.

And the characteristic polynomial of B_5 and B_6 are respectively given by

$$\sigma^4 (\sigma+4)^4 \left(\sigma^2 - 23\sigma - 3\right)\left(\sigma^2 + 5\sigma - 5\right),$$
$$\sigma^5 (\sigma+4)^5 \left(\sigma^2 - 28\sigma - 2\right)\left(\sigma^2 + 6\sigma - 6\right).$$

Generally, the characteristic polynomial of B_m using the distance domination matrix is

$$\sigma^{m-1} (\sigma+4)^{m-1} \left[\sigma^2 - (5m-2)\sigma + (m-8)\right]\left(\sigma^2 + m\sigma - m\right) = 0.$$

Solving the equation we get

$\sigma^{m-1} = 0$, or $(\sigma+4)^{m-1} = 0$, or $(\sigma^2 - (5m-2)\sigma + (m-8)) = 0$, or $(\sigma^2 + m\sigma - m) = 0$. So $\sigma = 0, 0, 0, \cdots, 0$ ($(m-1)$times), or $\sigma = -4, -4, -4, \cdots, -4$ ($(m-1)$times), and $(\sigma^2 - (5m-2)\sigma + (m-8)) = 0$,

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}\left(5m - 2 - \sqrt{25m^2 - 24m + 36}\right) \text{ and} \\
\sigma_2 &= \frac{1}{2}\left(5m - 2 + \sqrt{25m^2 - 24m + 36}\right) \qquad \text{here } m \geqslant 3, \\
(\sigma^2 + m\sigma - m) &= 0, \\
\sigma_3 &= \frac{1}{2}\left(-m - \sqrt{m}\sqrt{m+4}\right) \text{ and} \\
\sigma_4 &= \frac{1}{2}\left(-m + \sqrt{m}\sqrt{m+4}\right) \\
E_{D\gamma-\min} &= E_{D\gamma-\min}(G) \\
&= \sum_{i=1}^{n} |\sigma_i| \\
&= 4(m-1) + \left|\frac{1}{2}\left(2\sqrt{25m^2 - 24m + 36}\right)\right| + \left|\frac{1}{2}\left(2\sqrt{m}\sqrt{m+4}\right)\right|.
\end{aligned}$$

Therefore,

$$E_{D\gamma-\min} = E_{D\gamma-\min}(G) = 4(m-1) + \sqrt{25m^2 - 24m + 36} + \sqrt{m}\sqrt{m+4}.$$

This completes the proof. □

Theorem 3.3 *For $m \geqslant 3$, the minimum Laplacian domination energy of a book graph (B_m) is $5m + \sqrt{m^2 + 4}$.*

Proof Calculation enables one to find the characteristic polynomial of B_m for $m \geqslant 3$ directly.

For $m = 3$, B_3 is a book graph with 8 vertices. The minimum dominating set is $S = \{v_1, v_2\}$. The Laplacian domination matrix and the characteristic polynomial of B_3 are respectively calculated by

$$L_\gamma(G) = \begin{bmatrix}
3 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & 3 & 0 & -1 & 0 & -1 & 0 & -1 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}$$

and $\alpha^8-18\alpha^7+131\alpha^6-496\alpha^5+1038\alpha^4-1154\alpha^3+543\alpha^2+36\alpha-81=(\alpha-1)^2(\alpha-3)^2(\alpha^2-7\alpha+9)(\alpha^2-3\alpha-1)$.

Similarly, the Laplacian domination matrix and the characteristic polynomial of B_4 are respectively given by

$$L_\gamma(G)=\begin{bmatrix} 4 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0\\ -1 & 4 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1\\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

and $\alpha^{10}-24\alpha^9+243\alpha^8-1360\alpha^7+4618\alpha^6-9792\alpha^5+12774\alpha^4-9520\alpha^3+3141\alpha^2+216\alpha-297=(\alpha-1)^3(\alpha-3)^3(\alpha^2-8\alpha+11)(\alpha^2-4\alpha-1)$

And the characteristic polynomial of B_5 and B_6 is given by $(\alpha-1)^4(\alpha-3)^4(\alpha^2-9\alpha+13)(\alpha^2-5\alpha-1)$, $(\alpha-1)^5(\alpha-3)^5(\alpha^2-10\alpha+15)(\alpha^2-6\alpha-1)$, respectively.

Generally, the characteristic polynomial of B_m using the Laplacian domination matrix is

$$(\alpha-1)^{m-1}(\alpha-3)^{m-1}(\alpha^2-(m+4)\alpha+(2m+3))(\alpha^2-m\alpha-1)=0.$$

solving the equation we get

$(\alpha-1)^{m-1}=0$, or $(\alpha-3)^{m-1}=0$, or $(\alpha^2-(m+2)\alpha+(2m+3))=0$, or $(\alpha^2-m\alpha-1)=0$. So $\alpha=1,1,1,\cdots,1$ ($(m-1)$times), or $\alpha=3,3,3,\cdots,3$ ($(m-1)$times), and $(\alpha^2-(m+2)\alpha+(2m+3))=0$,

$$\begin{aligned} \alpha_1&=\frac{1}{2}\left(m+4-\sqrt{m^2+28}\right) \text{ and}\\ \alpha_2&=\frac{1}{2}\left(m+4+\sqrt{m^2+28}\right) \qquad \text{here } m\geqslant 3,\\ (\alpha^2-m\alpha-1)&=0,\\ \alpha_3&=\frac{1}{2}\left(m-\sqrt{m^2+4}\right) \text{ and}\\ \alpha_4&=\frac{1}{2}\left(m+\sqrt{m^2+4}\right), \end{aligned}$$

$$E_{L\gamma-\min} = E_{L\gamma-\min}(G) = \sum_{i=1}^{n} |\alpha_i|$$
$$= (m-1) + 3(m-1) + \left|\frac{1}{2}\left(2\sqrt{m^2+4}\right)\right| + \left|\frac{1}{2}\left(2(m+4)\right)\right|$$

Therefore, $E_{L\gamma-\min} = E_{L\gamma-\min}(G) = 5m + \sqrt{m^2+4}$. This completes the proof. □

Theorem 3.4 *For $m \geqslant 3$, the minimum Laplacian distance domination energy of a Book Graph (B_m) is $10m - 5 + \sqrt{36m^2 - 48m + 49}$.*

Proof The characteristic polynomial of B_m for $m \geqslant 3$ can be found directly.

For $m = 3$, B_3 is a book graph with 8 vertices. The minimum dominating set is $S = \{v_1, v_2\}$. The Laplacian distance domination matrix and the characteristic polynomial of B_3 are respectively calculated by

$$LD_\gamma(G) = \begin{bmatrix} 3 & -1 & -1 & -2 & -1 & -2 & -1 & -2 \\ -1 & 3 & -2 & -1 & -2 & -1 & -2 & -1 \\ -1 & -2 & 2 & -1 & -2 & -3 & -2 & -3 \\ -2 & -1 & -1 & 2 & -3 & -2 & -3 & -2 \\ -1 & -2 & -2 & -3 & 2 & -1 & -2 & -3 \\ -2 & -1 & -3 & -2 & -1 & 2 & -3 & -2 \\ -1 & -2 & -2 & -3 & -2 & -3 & 2 & -1 \\ -2 & -1 & -3 & -2 & -3 & -1 & -1 & 2 \end{bmatrix}$$

and $\beta^8 - 18\beta^7 + 29\beta^6 + 1612\beta^5 - 16629\beta^4 + 75536\beta^3 - 181032\beta^2 + 222336\beta - 110160 = (\beta-2)^2(\beta-6)^2(\beta^2 - 9\beta + 17)(\beta^2 + 7\beta - 45)$.

Similarly, calculation shows that the Laplacian distance domination matrix and the characteristic polynomial of B_4 are respectively given by

$$LD_\gamma(G) = \begin{bmatrix} 4 & -1 & -1 & -2 & -1 & -2 & -1 & -2 & -1 & -2 \\ -1 & 4 & -2 & -1 & -2 & -1 & -2 & -1 & -2 & -1 \\ -1 & -2 & 2 & -1 & -2 & -3 & -2 & -3 & -2 & -3 \\ -2 & -1 & -1 & 2 & -3 & -2 & -3 & -2 & -3 & -2 \\ -1 & -2 & -2 & -3 & 2 & -1 & -2 & -3 & -2 & -3 \\ -2 & -1 & -3 & -2 & -1 & 2 & -3 & -2 & -3 & -2 \\ -1 & -2 & -2 & -3 & -2 & -3 & 2 & -1 & -2 & -3 \\ -2 & -1 & -3 & -2 & -3 & -2 & -1 & 2 & -3 & -2 \\ -1 & -2 & -2 & -3 & -2 & -3 & -2 & -3 & 2 & -1 \\ -2 & -1 & -3 & -2 & -3 & -2 & -3 & -2 & -1 & 2 \end{bmatrix}$$

$\beta^{10} - 24\beta^9 + 55\beta^8 + 4208\beta^7 - 66192\beta^6 + 494272\beta^5 - 2178656\beta^4 + 5934336\beta^3 - 9801216\beta^2 + 8985600\beta - 3504384 = (\beta-2)^3(\beta-6)^3(\beta^2 - 11\beta + 26)(\beta^2 + 11\beta - 78)$.

The characteristic polynomial of B_5 is given by $(\beta-2)^4(\beta-6)^4(\beta^2-13\beta+37)(\beta^2+15\beta-121)$, and the characteristic polynomial of B_6 is given by $(\beta-2)^5(\beta-6)^5(\beta^2-15\beta+50)(\beta^2+19\beta-174)$.

Generally, the characteristic polynomial of B_m using the Laplacian distance domination matrix is

$$(\beta-2)^{m-1}(\beta-6)^{m-1}(\beta^2-(2m+3)\beta+(m+1)^2+1)(\beta^2+(4m-5)\beta-(5m^2-2m+6)).$$

Solving the equation we get $(\beta-2)^{m-1}=0$, or $(\beta-6)^{m-1}=0$, or $\beta^2-(2m+3)\beta+(m+1)^2+1=0$, or $\beta^2+(4m-5)\beta-(5m^2-2m+6)=0$. So $\beta = 2,2,2,\cdots,2$ ($(m-1)$times), or $\beta = 6,6,6,\cdots,6$ ($(m-1)$times), and $\beta^2-(2m+3)\beta+(m+1)^2+1=0$,

$$\begin{aligned}
\beta_1 &= \frac{1}{2}\left(2m+3-\sqrt{4m+1}\right) \text{ and}\\
\beta_2 &= \frac{1}{2}\left(2m+3+\sqrt{4m+1}\right) \quad \text{here } m \geqslant 3,\\
&\beta^2+(4m-5)\beta-(5m^2-2m+6)=0,\\
\beta_3 &= \frac{1}{2}\left(-\sqrt{36m^2-48m+49}-4m+5\right) \text{ and}\\
\beta_4 &= \frac{1}{2}\left(\sqrt{36m^2-48m+49}-4m+5\right),\\
E_{LD\gamma-\min} &= E_{LD\gamma-\min}(G)\\
&= \sum_{i=1}^{n}|\beta_i| = 8(m-1) + \left|\frac{1}{2}(4m+6)\right| + \left|\frac{1}{2}\left(2\sqrt{36m^2-48m+49}\right)\right|.
\end{aligned}$$

Whence, $E_{LD\gamma-\min} = E_{LD\gamma-\min}(G) = 10m-5+\sqrt{36m^2-48m+49}$. This completes the proof. □

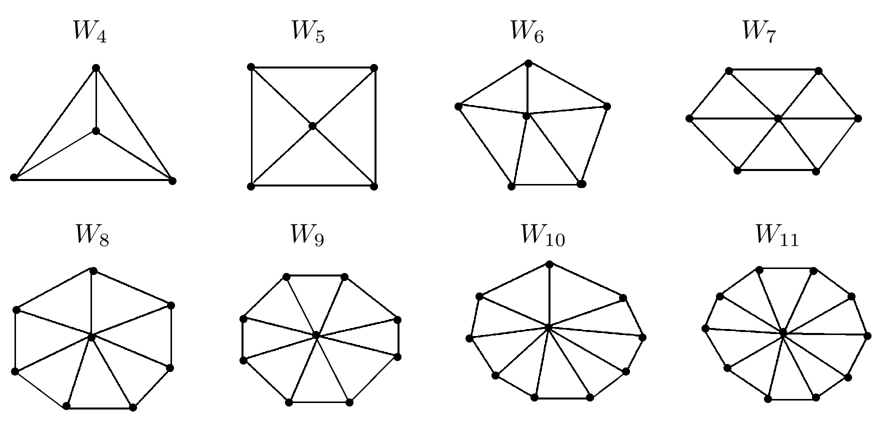

Figure 3 Wheel graph W_n, $4 \leqslant n \leqslant 11$

Definition 3.2 *A wheel graph W_n of order n, sometimes simply called an n-wheel, is a graph* *that contains a cycle of order $n-1$, and for which every graph vertex in the cycle is connected to one other graph vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. The wheel W_n can be defined as the graph K_1+C_{n-1}, where K_1 is the singleton graph and C_n is the cycle graph. Some wheel graphs are shown in Figure* 3.

Theorem 3.5 *For $n \geqslant 4$, the minimum dominating energy of a wheel graph (W_n) is $> \sqrt{4n-3}$.*

Proof We can find the characteristic polynomial of W_n for $n \geqslant 4$ by calculation directly.

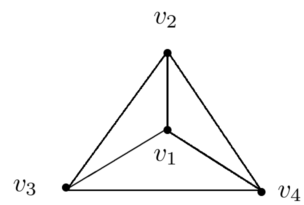

Figure 4 W_4

For $n=4$, W_4 is a wheel graph with 4 vertices. The minimum dominating sets are $S=\{v_1\}$ or $S=\{v_2\}$ or $S=\{v_3\}$.

For $S=\{v_1\}$ the domination matrix and the characteristic polynomial of W_4 are respectively calculated by

$$A_\gamma(G)=\begin{bmatrix}1&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}$$

and $\kappa^4-\kappa^3-6\kappa^2-5\kappa-1=(\kappa^2-3\kappa-1)(\kappa^2+2\kappa+1)$. The characteristic polynomial is found to be same when $S=\{v_2\}$ or $S=\{v_3\}$.

For $n=5$, W_5 is a wheel graph with 5 vertices. The minimum dominating sets is $S=\{v_1\}$. Calculation shows that the domination matrix and the characteristic polynomial of W_5 are respectively given by

$$A_\gamma(G)=\begin{bmatrix}1&1&1&1&1\\1&0&1&1&0\\1&1&0&0&1\\1&1&0&0&1\\1&0&1&1&0\end{bmatrix}$$

and $\kappa^5-\kappa^4-8\kappa^3-4\kappa^2=(\kappa^2-3\kappa-2)(\kappa^3+2\kappa^2)$.

Similarly the characteristic polynomial of W_6, W_7 and W_8 are given by $(\kappa^2-3\kappa-3)(\kappa^2+\kappa-1)^2$, $(\kappa^2-3\kappa-4)(\kappa-1)^2(\kappa+1)^2(\kappa+2)$ and $(\kappa^2-3\kappa-5)(\kappa^3+\kappa^2-2\kappa-\kappa)^2$, respectively.

Generally, the characteristic polynomial of W_n for $n \geqslant 4$ using domination matrix is

$$\left[\kappa^2-3\kappa-(n-3)\right]P(\kappa).$$

Solving the equation $(\kappa^2 - 3\kappa - (n-3)) = 0$ we get $\kappa_1 = \frac{1}{2}\left(3 - \sqrt{4n-3}\right)$ and $\kappa_2 = \frac{1}{2}\left(3 + \sqrt{4n-3}\right)$. $E_{\gamma-\min} = E_{\gamma-\min}(G) > \sum\limits_{i=1}^{2} |\kappa_i|$, $E_{\gamma-\min}(G) > \sqrt{4n-3}$. This completes the proof. □

Theorem 3.6 *For $n \geqslant 4$, the minimum distance dominating energy of a wheel graph (W_n) is* $> \sqrt{4n^2 - 24n + 45}$.

Proof The characteristic polynomial of W_n for $n \geqslant 4$ can be obtained by calculation directly.

For $n = 4$, W_4 is a wheel graph with 4 vertices. The minimum dominating sets are $S = \{v_1\}$ or $S = \{v_2\}$ or $S = \{v_3\}$. For $S = \{v_1\}$, Calculation shows that the distance domination matrix and the characteristic polynomial of W_4 are respectively given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\sigma^4 - \sigma^3 - 6\sigma^2 - 5\sigma - 1 = (\sigma^2 - 3\sigma - 1)(\sigma^2 + 2\sigma + 1)$.

The characteristic polynomial is found to be same when $S = \{v_2\}$ or $S = \{v_3\}$.

For $n = 5$, W_5 is a wheel graph with 5 vertices. The minimum dominating sets is $S = \{v_1\}$. Calculation shows that the distance domination matrix and the characteristic polynomial of W_5 are respectively given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix}$$

and $\sigma^5 - \sigma^4 - 16\sigma^3 - 20\sigma^2 = \sigma(\sigma^2 - 5\sigma + 0)(\sigma + 2)^2$.

Similarly, the characteristic polynomial of W_6, W_7 and W_8 are given by $(\sigma^2 - 7\sigma + 1)(\sigma^2 + 3\sigma + 1)^2, \sigma(\sigma^2 - 9\sigma + 2)(\sigma + 1)^2(\sigma + 3)^2$ and $(\sigma^2 - 11\sigma + 3)(\sigma^3 + 5\sigma^2 + 6\sigma + 1)^2$, respectively.

Generally, the characteristic polynomial of W_n for $n \geqslant 4$ using distance domination matrix is

$$\left[\sigma^2 - (2n-5)\sigma + (n-5)\right] P(\sigma).$$

Solving the equation $(\sigma^2 - (2n-5)\sigma + (n-5)) = 0$ we get

$$\begin{aligned} \sigma_1 &= \frac{1}{2}\left(2n - 5 - \sqrt{4n^2 - 24n + 45}\right), \\ \sigma_2 &= \frac{1}{2}\left(2n - 5 + \sqrt{4n^2 - 24n + 45}\right) \end{aligned}$$

and $E_{D\gamma-\min} = E_{D\gamma-\min}(G) > \sum_{i=1}^{2} |\sigma_i|$, $E_{D\gamma-\min}(G) > \sqrt{4n^2 - 24n + 45}$. Hence, we complete the proof. □

Theorem 3.7 *For $n \geqslant 4$, the minimum Laplacian domination energy of a wheel graph (W_n) is $> \sqrt{n^2 - 2n + 5}$.*

Proof Calculation enables one to find the characteristic polynomial of W_n for $n \geqslant 4$ directly.

For $n = 4$, W_4 is a wheel graph with 4 vertices. The minimum dominating sets are $S = \{v_1\}$ or $S = \{v_2\}$ or $S = \{v_3\}$. For $S = \{v_1\}$, Calculation shows that the Laplacian domination matrix and the characteristic polynomial of W_4 are respectively given by

$$L_\gamma(G) = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

and $\alpha^4 - 11\alpha^3 + 39\alpha^2 - 40\alpha - 16 = (\alpha^2 - 3\alpha - 1)(\alpha - 4)^2$.

The characteristic polynomial is found to be same when $S = \{v_2\}$ or $S = \{v_3\}$.

For $n = 5$, W_5 is a wheel graph with 5 vertices. The minimum dominating sets is $S = \{v_1\}$. The Laplacian domination matrix and the characteristic polynomial of W_5 are respectively calculated by

$$L_\gamma(G) = \begin{bmatrix} 3 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

and $\alpha^5 - 15\alpha^4 + 82\alpha^3 - 190\alpha^2 + 141\alpha + 45 = (\alpha^2 - 4\alpha - 1)(\alpha - 3)^2(\alpha - 5)$.

Similarly, the characteristic polynomial of W_6, W_7 and W_8 are given by $(\alpha^2 - 5\alpha - 1)\left(\alpha^2 - 7\alpha + 11\right)^2$, $(\alpha^2 - 6\alpha - 1)(\alpha - 2)^2(\alpha - 4)^2(\alpha - 5)$ and $(\alpha^2 - 7\alpha - 1)\left(\alpha^3 - 10\alpha^2 + 31\alpha - 29\right)^2$, respectively.

Generally, the characteristic polynomial of W_n for $n \geqslant 4$ using Laplacian domination matrix is

$$\left[\alpha^2 - (n-1)\alpha - 1\right] P(\alpha).$$

Solving the equation $(\alpha^2 - (n-1)\alpha - 1) = 0$ we get

$$\alpha_1 = \frac{1}{2}\left(n - 1 - \sqrt{n^2 - 2n + 5}\right)$$

and

$$\alpha_2 = \frac{1}{2}\left(n - 1 + \sqrt{n^2 - 2n + 5}\right),$$

$$E_{L\gamma-\min} = E_{L\gamma-\min}(G) > \sum_{i=1}^{2} |\alpha_i| = \sqrt{n^2 - 2n + 5}.$$

Hence the proof is completed. □

Theorem 3.8 *For $n \geqslant 4$, the minimum Laplacian distance dominating energy of a wheel graph (W_n) is $> \sqrt{9n^2 - 62n + 117}$.*

Proof The characteristic polynomial of W_n for $n \geqslant 4$ can be obtained by calculation directly.

For $n = 4$, W_4 is a wheel graph with 4 vertices. The minimum dominating sets are $S = \{v_1\}$ or $S = \{v_2\}$ or $S = \{v_3\}$. For $S = \{v_1\}$, Calculation shows that the Laplacian distance domination matrix and the characteristic polynomial of W_4 are respectively given by

$$LD_\gamma(G) = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

and $\beta^4 - 11\beta^3 + 39\beta^2 - 40\beta - 16 = (\beta^2 - 3\beta - 1)(\beta - 4)^2$.

The characteristic polynomial is found to be same when $S = \{v_2\}$ or $S = \{v_3\}$.

For $n = 5$, W_5 is a wheel graph with 5 vertices. The minimum dominating sets is $S = \{v_1\}$. The Laplacian distance domination matrix and the characteristic polynomial of W_5 are respectively given by

$$LD_\gamma(G) = \begin{bmatrix} 3 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -2 \\ -1 & -1 & 3 & -2 & -1 \\ -1 & -1 & -2 & 3 & -1 \\ -1 & -2 & -1 & -1 & 3 \end{bmatrix}$$

and

$$\beta^5 - 15\beta^4 + 74\beta^3 - 94\beta^2 - 235\beta + 525 = (\beta^2 - 2\beta - 7)(\beta - 5)^2(\beta - 3).$$

Similarly, the characteristic polynomial of W_6, W_7 and W_8 are given respectively by

$$(\beta^2 - \beta - 17)\left(\beta^2 - 9\beta + 19\right)^2,$$
$$(\beta^2 + 0\beta - 31)(\beta - 6)^2(\beta - 4)^2(\beta - 3)$$

and

$$(\beta^2 + \beta - 49)\left(\beta^3 - 14\beta^2 + 63\beta - 91\right)^2.$$

Generally, the characteristic polynomial of W_n for $n \geqslant 4$ using Laplacian distance domination matrix is

$$\left[\beta^2 + (n - 7)\beta - (2n^2 - 12n + 17)\right] p(\beta).$$

Solving the equation $\beta^2 + (n-7)\beta - (2n^2 - 12n + 17) = 0$ we get

$$\beta_1 = \frac{1}{2}\left(-\sqrt{9n^2 - 62n + 117} - n + 7\right),$$
$$\beta_2 = \frac{1}{2}\left(\sqrt{9n^2 - 62n + 117} - n + 7\right)$$

and

$$E_{LD\gamma-\min} = E_{LD\gamma-\min}(G) > \sum_{i=1}^{2} |\beta_i| = \sqrt{9n^2 - 62n + 117}.$$

Hence the proof is completed. □

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