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Steiner Domination Number of Splitting and Degree Splitting Graphs

Samir K. Vaidya

(Department of Mathematics, Saurashtra University Rajkot - 360 005, Gujarat, India)

Sejal H. Karkar

(Government Engineering College, Rajkot - 360 005, Gujarat, India)

Email: samirkvaidya@yahoo.co.in, sdpansuria@gmail.com

Abstract: A tree T contained in graph G is a Steiner tree with respect to $W \subseteq V(G)$ if T is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all the vertices of G which lie on some Steiner tree with respect to W. The set W is a Steiner set for G if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of G is the Steiner number of G, denoted as $s(G)$. The set W is called Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality among such sets is a Steiner domination number, denoted as $\gamma_s(G)$. We investigate Steiner domination number of some splitting and degree splitting graphs.

Key Words: Steiner distance, Steiner set, Steiner number, domination number, Steiner domination number.

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§1. Introduction

We consider simple, finite, connected and undirected graph G with vertex set V and edge set E. For the standard graph theoretic terminology and notation we follow Chatrand and Lesniak [2] while the terms related to the theory of domination are used in the sense of Haynes et al. [6].

Definition 1.1 *The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest $u - v$ path in G.*

Definition 1.2 *The Steiner distance $sd(W)$ of a subset W of vertices of a connected graph G is the minimum number of edges in a connected subgraph of G that contains W. If H is a subgraph of minimum size that contains a set W, then H is necessarily a tree, called a Steiner tree for W or a Steiner W-tree.*

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Chartrand et al. have introduced a generalization of distance in [3]. The sharp upper and lower bounds for the Steiner k-diameter of G and $\overline{G}$ are given by Mao [9] while the same author have identified some graph classes attaining these bounds. Let n be an integer such that $2 \leq n \leq |V(G)|$, then the n diameter of G, $diam_n(G)$, is defined to be the maximum Steiner distance of any n-subset(subset with n elements) of vertices of G. If G be any graph of order p with minimum degree $\delta \geq 2$ and $2 \leq n \leq p$ then $diam_n(G) \leq \dfrac{p}{\delta+1} + 2n - 5$, is proved by Ali et al. [1].

Definition 1.3 *The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $S(W) = V(G)$, then W is called a Steiner set for G. A Steiner set of minimum cardinality is a minimum Steiner set and this cardinality is the Steiner number $s(G)$.*

The concept of Steiner number was introduced by Chartrand and Zhang [4]. In the same paper authors have proved many results on this newly defined concept. This concept was further studied by Santhakumaran and John [8]. For the graph G of Figure 1, there are three Steiner trees related to $W = \{w_1, w_2\}$ which are shown in the same figure. Since $S(W) = V(G)$, W is a Steiner set of G.

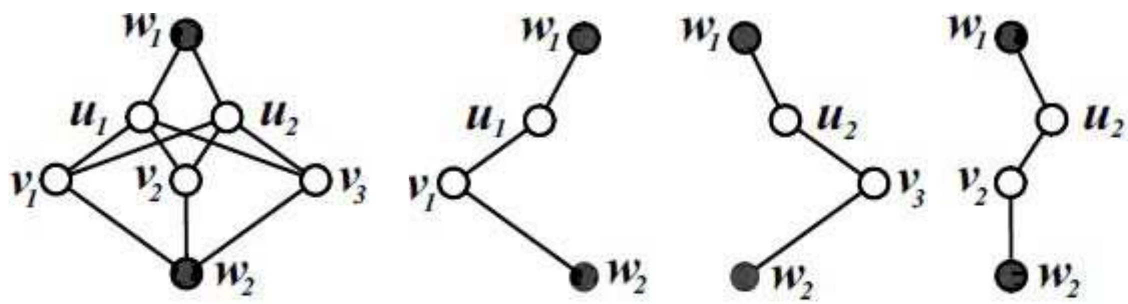

Figure 1 The graph G and its Steiner trees

Definition 1.4 *A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is a minimal dominating set if no proper subset $S' \subset S$ is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in graph G.*

Definition 1.5 *Let G be a connected graph with vertex set $V(G)$. A set of vertices W in G is called a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is called its Steiner domination number, denoted by $\gamma_s(G)$.*

The concept of Steiner domination number was introduced by John et al. [7]. It is very interesting to investigate Steiner domination number of graph or graph families as it is known only for handful number of graphs. Vaidya and Mehta [11] have investigated the Steiner domination number of W_n, H_n and Fl_n and the same authors [12] have established some characterizations for Steiner domination in graphs while Steiner domination number for $S'(P_n)$, $S'(C_n)$, $M(P_n)$,

$M(C_n)$ and F_n are obtained by Vaidya and Karkar [10].

For the graph G of Figure 1, $W = \{w_1, w_2\}$ is a Steiner dominating set of minimum cardinality. Therefore, $\gamma_s(G) = 2$.

Definition 1.6 *A vertex v is an extreme vertex of a graph G if the subgraph induced by neighbors of v is complete.*

Definition 1.7([5]) *A systematic visit of each vertex of a tree is called a tree traversal.*

Definition 1.8 *The bistar $B_{m,n}$ is the graph obtained by joining the center(apex) vertices of $K_{1,m}$ and $K_{1,n}$ by an edge.*

Definition 1.9 *Let G be a graph with $V(G) = S_1 \cup S_2 \cup S_3 \cup \cdots S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) \setminus \bigcup_{i=1}^{t} S_i$. The degree splitting of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, w_3, \cdots, w_t$ and joining w_i to each vertex of S_i for $1 \leqslant i \leqslant t$. Note that if $V(G) = \bigcup_{i=1}^{t} S_i$ then $T = \emptyset$.*

Definition 1.10 *For a graph G the splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.*

Definition 1.11 *A friendship graph F_n is a one point union of n copies of cycle C_3.*

§2. Main Results

Observation 2.1 $\gamma(B_{m,n}) = m + n$.

Theorem 2.2 $\gamma_s(S'(B_{m,n})) = m + n + 2$.

Proof Let $u, u_1, u_2, \cdots u_m, v, v_1, v_2, \cdots v_n$ be $m+n+2$ vertices of $B_{m,n}$ and $u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots, v'_n$ be the corresponding vertices which are added to obtain $S'(B_{m,n})$. Then $V(S'(B_{m,n})) = \{u, u_1, u_2, \cdots, u_m, v_1, v_2, \cdots v_n, v, u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n\}$. Now $u'_1, u'_2, \cdots u'_m, v'_1, v'_2, \cdots v'_n$ are extreme vertices as the subgraph induced by their neighbors is complete, namely, the complete graph K_1. Therefore, they must be in Steiner dominating set W. If $u'_1, u'_2, \cdots u'_m, v'_1, v'_2, \cdots v'_n \in W$ then $u'_1, u'_2, \cdots u'_m, v'_1, v'_2, \cdots v'_n, u, v \in S(W)$. Now there some trees between u' and v' which include remaining vertices $u_1, u_2, \cdots u_m, v_1, v_2, \cdots v_n$. So if $u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n \in W$ then there are four Steiner W-trees which include all the vertices of the graph. That is, if $u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n \in W$ then $u_1, u_2, \cdots u_m, v_1, v_2, \cdots v_n, u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n \in S(W)$. Therefore, $W = \{u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n\}$ becomes a Steiner set of minimum cardinality $m + n + 2$ and it is also a dominating set. Hence

$$\gamma_s(S'(B_{m,n})) = m + n + 2. \qquad \square$$

Theorem 2.3 $\gamma_s(DS(B_{m,n})) = 2$.

Proof Let $u, u_1, u_2, \cdots u_m, v, v_1, v_2, \cdots v_n$ be $m + n + 2$ vertices of $B_{m,n}$ and x_1, x_2 be the corresponding vertices which are added in order to obtain $DS(B_{m,n})$. Then, $V(DS(B_{m,n})) = \{u, u_1, u_2, \cdots u_m, v, v_1, v_2, \cdots v_n, x_1, x_2\}$. Now if G is a connected graph of order $n \geq 2$ then $2 \leq S(G) \leq n$. Without loss of generality let $x_1, x_2 \in W$ then there are four Steiner W-tree traversal between x_1 and x_2 which together include all the vertices of $DS(B_{m,n})$. Therefore, $W = \{x_1, x_2\}$ becomes a Steiner set of minimum cardinality and it is also a dominating set. Therefore, $W = \{x_1, x_2\}$ becomes a Steiner dominating set of minimum cardinality. Hence

$$\gamma_s(DS(B_{m,n})) = 2. \qquad \square$$

Lemma 2.4 *$S(DS(P_n)) = n - 5$, $n \geq 7$.*

Proof Consider P_n with $V(P_n) = \{v_1, v_2, \cdots v_n\}$ with partition $V_1 = \{v_2, v_3, \cdots v_{n-1}\}$ and $V_2 = \{v_1, v_n\}$. Now in order to obtain $DS(P_n)$ from P_n we add x_1 and x_2 corresponding to V_1 and V_2. Thus, $V(DS(P_n)) = \{x_1, x_2, v_1, v_2, \cdots v_n\}$. Let $x_1, v_4 \in W$ then there are some Steiner W-trees which include the vertices $x_1, v_1, v_2, v_3, v_4, x_2$. So, if $x_1, v_4 \in W$ then $x_1, v_1, v_2, v_3, v_4, x_2 \in S(W)$. Let $x_1, v_4, v_{n-3} \in W$ then $x_1, v_1, v_2, v_3, v_4, x_2, v_{n-3}, v_{n-2}, v_{n-1}, v_n \in S(W)$. Then, there does not exist tree traversal containing x_1, v_4, v_{n-3} which includes $v_5, v_6, \cdots, v_{n-4}$. The vertices $v_5, v_6, \cdots, v_{n-4}$ must be included in W to obtain Steiner tree of minimum size which include $v_5, v_6, \cdots, v_{n-4}$. Therefore, if $x_1, v_4, v_5, \cdots v_{n-4}, v_{n-3} \in W$. Then there are following four Steiner W- trees as listed below:

(1) $x_1 v_1 v_2 v_3 \cdots v_{n-4} v_{n-3}$,

(2) $x_1 v_1 v_2 x_2 v_4 v_5 v_6 \cdots v_{n-3}$,

(3) $x_1 v_n v_{n-1} v_{n-2} v_{n-3} \cdots v_5 v_4$,

(4) $x_1 v_n v_{n-1} x_2 v_{n-2} v_{n-3} v_{n-4} \cdots v_5, v_4$,

which include all the vertices of the graph. Thus $W = \{x_1, v_4, v_5, \cdots v_{n-4}, v_{n-3}\}$ becomes a Steiner set of minimum size which include $n - 6$ vertices of P_n and a vertex x_1. Hence

$$S(DS(P_n)) = n - 5. \qquad \square$$

Theorem 2.5 *$\gamma_s(DS(P_n)) = n - 3$, $n \geq 7$.*

Proof From the Theorem 2.4 $W = \{x_1, v_4, v_5, \cdots v_{n-4}, v_{n-3}\}$ is a Steiner set of minimum cardinality. But it is not a dominating set as v_2 and v_{n-1} are not dominated by any of the vertices. Therefore, these two vertices must be in Steiner dominating set W. So, $\{x_1, v_2, v_4, v_5, \cdots v_{n-4}, v_{n-3}, v_{n-1}\}$ is a Steiner dominating set of minimum cardinality. Hence

$$\gamma_s(DS(P_n)) = n - 3. \qquad \square$$

Proposition 2.6([7]) *$\gamma_s(K_{m,n}) = min\{m, n\}$ if $m, n \geq 2$.*

Theorem 2.7 *$\gamma_s(S'(K_{m,n})) = m + n$.*

Proof Let $v_1, v_2, \cdots v_m, u_1, u_2, \cdots, u_n$ be $m+n$ vertices of $K_{m,n}$. Now $v'_1, v'_2, \cdots v'_m, u'_1, u'_2, \cdots, u'_n$ be the corresponding vertices which are added in order to obtain $S'(K_{m,n})$ with partitions $W = \{v'_1, v'_2, \cdots v'_m, u'_1, u'_2, \cdots u'_n\}$ and $X = \{v_1, v_2, \cdots v_m, u_1, u_2, \cdots u_n\}$. It is very clear that W is a Steiner set as there are $\max\{m, n\}$ number of Steiner trees which include all the vertices of the graph. Here W dominates all the vertices of the graph. Therefore, it is also a dominating set. Thus, W is a Steiner dominating set. We claim that W is a Steiner dominating set with minimum cardinality. If possible let U be any Steiner set such that $|U| < |W|$ and $U \subset W$. Then, there exists a vertex $v'_i \in W$ such that $v'_i \notin U$. But as the vertices of W are mutually non adjacent, the Steiner U-tree containing v'_j and v'_k $(j \neq i,\ k \neq i,\ 1 \leq j, k \leq n)$ will not contain v'_i. Therefore, U is not Steiner set. If $U \subset X$ then some vertices of W and some vertices of X which are not included in U are not in any Steiner U- trees. Therefore, U is not Steiner set. Let $U \subset W \cup X$ such that U contain at least one vertex from each of W and X then some vertices of W and X do not lie on any Steiner U-tree. Thus, U is not a Steiner set. So, W is a Steiner dominating set of minimum cardinality $m + n$. Hence

$$\gamma_s(S'(K_{m,n})) = m + n. \qquad \square$$

Proposition 2.8([4]) *Let G be a connected graph of order $p \geq 2$. Then $\gamma_s(G) = 2$ if and only if there exists a Steiner dominating set $S = \{u, v\}$ of G such that $d(u, v) \leq 3$.*

Theorem 2.9 $\gamma_s(DS(K_{m,n})) = 2$, $m \neq n$, $m, n \geq 2$.

Proof Let $v_1, v_2, \cdots v_m, u_1, u_2, \cdots u_n$ be $m + n$ vertices of $K_{m,n}$ with partitions $W = \{v_1, v_2, \cdots v_m\}$ and $X = \{u_1, u_2, \cdots u_n\}$. In order to construct $DS(K_{m,n})$ we add w_1 and w_2. If we consider w_1 and w_2 in Steiner set W then $S(W) = V(G)$ and W is also a dominating set. Therefore W becomes a Steiner dominating set and $d(w_1, w_2) = 3$. Hence by Proposition 2.8,

$$\gamma_s(DS(K_{m,n})) = 2. \qquad \square$$

Proposition 2.10([7]) *Each extreme vertex of a connected graph G belongs to every Steiner dominating set of G.*

Theorem 2.11 $\gamma_s(S'(F_n)) = 2n + 1$.

Proof Let $v_0, v_1, v_2, \ldots v_n, v_{n+1}, \cdots v_{2n}$ be the $2n + 1$ vertices of F_n where v_0 is the apex vertex. Now $v'_0, v'_1, v'_2, \ldots v'_n, v'_{n+1}, \cdots v'_{2n}$ be the vertices which are added to obtain $S'(F_n)$. The vertices $v'_0, v'_1, v'_2, \ldots v'_n, v'_{n+1}, \cdots v'_{2n}$ must be in Steiner dominating set W as they are extreme vertices. But $W = \{v'_0, v'_1, v'_2, \ldots v'_n, v'_{n+1}, \cdots v'_{2n}\}$ is not a Steiner dominating set as it is neither a Steiner set nor a dominating set. Therefore, we must include some more vertices to obtain a Steiner dominating set. Let $v_0 \in W$ then $S(W) = V(S'(F_n))$ and

$$W = \{v'_0, v'_1, v'_2, \cdots v'_n, v'_{n+1}, \cdots v'_{2n}\}$$

is a dominating set of minimum cardinality. Hence

$$\gamma_s(S'(F_n)) = 2n + 1. \qquad \square$$

§3. Concluding Remarks

The Steiner domination in graphs is one of the interesting domination models. It is always challenging to investigate Steiner domination number of a graph. We have obtained Steiner domination number of larger graphs which are obtained by means of various graph operations.

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References

[1] P. Ali, P. Dankelmann and S. Mukwembi, Upper bounds on the Steiner diameter of a graph, *Discrete Appl. Math.*, 160(2012), 1845-1850.

[2] G. Chartrand and L. Lesniak, *Graphs & Diagraphs*, 4/e, Chapman and Hall/CRC press, 2005.

[3] G. Chartrand, O. R. Oellermann, S. L. Tian and H. B. Zou, Steiner distance in graphs, *Casopis Pro Pestovani Matematiky*, 114(1989), 399-410.

[4] G. Chartrand and P. Zhang, The Steiner number of a graph, *Discrete Mathematics*, 242(2002), 41-54.

[5] J. Gross and J. Yellen, *Graph Theory and Its Applications*, CRC Press, 1999.

[6] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[7] J. John, G. Edwin and P. Sudhahar, The Steiner domination number of a graph, *International Journal of Mathematics and Computer Application Research*, 3(2013), 37-42.

[8] A. P. Santhakumaran and J. John, The forcing Steiner number of a graph, *Discussiones Mathematicae Graph Theory*, 31(2011), 171-181.

[9] Y. Mao, The Steiner diameter of a graph, *arXiv*: 1509.02801 [math.CO], (2015).

[10] S. K. Vaidya and S. H. Karkar, Steiner domination number of some graphs, *International Journal of Mathematics and Scientific Computing*, 5(2015), 1-3.

[11] S. K. Vaidya and R. N. Mehta, Steiner domination number of some wheel related graphs, *International Journal of Mathematics and Soft Computing*, 5(2015), 15-19.

[12] S. K. Vaidya and R. N. Mehta, On Steiner domination in graphs, *Malaya Journal of Matematik*, 6(2018), 381-384.