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On Hyper Generalized Quasi Einstein Manifolds

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Abstract: In this paper its proved three theorems about global properties of hyper generalized quasi-Einstein manifolds.

Key Words: Non-flat Riemannian manifold, hyper generalized quasi Einstein manifold $(HGQE)_n$, compact orientable.

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§1. Introduction

The notion of quasi Einstein manifold was introduced in a paper [8] by M.C.Chaki and R.K.Maity. According to them a non-flat Riemannian manifold (M^n, g),$(n \geq 3)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{1}$$

and is not identically zero, where a, b are scalars $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, \xi_1) = A(X), \qquad \forall X \in TM, \tag{2}$$

where, ξ_1 is a unit vector field.

In such a case a, b are called the associated scalars. A is called the associated 1-form. Such an n-dimensional manifold is denoted by the symbol$(QE)_n$.

Again, U.C.De and G.C.Ghosh defined generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci-tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y), \tag{3}$$

where a, b, c are non-zero scalars and A, B are two 1-forms such that

$$g(X, \xi_1) = A(X) \quad \text{and} \quad g(X, \xi_2) = B(X) \tag{4}$$

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with ξ_1, ξ_2 unit vectors which are orthogonal, i.e,

$$g(\xi_1, \xi_2) = 0. \tag{5}$$

This type of manifold are denoted by $G(QE)_n$.

In [16], H.G. Nagaraja introduced the concept of $N(k)$-mixed quasi Einstein manifold and mixed quasi constant curvature.A non flat Riemannian manifold (M^n, g) is called a $N(k)$-mixed quasi Einstein manifold if its Ricci tensor of type $(0, 2)$ is non zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)B(Y) + cB(X)A(Y), \tag{6}$$

where a, b, c are smooth functions and A, B are non zero 1-forms such that

$$g(X, \xi_1) = A(X) \quad \text{and} \quad g(X, \xi_2) = B(X) \qquad \forall \quad X, \tag{7}$$

with ξ_1, ξ_2 the orthogonal unit vector fields. Such aa manifold is denoted by the symbol $N(k)-(MQE)_n$.

The notion of hyper-generalized quasi Einstein manifold has been introduced by A.A.Shaikh, C. Özgür and A.Patra[17] in 2011. An n-dimensional Riemannian manifold (M^n, g) $(n > 2)$ is called a hyper generalized quasi-Einstein manifold if its Ricci tensor of type $(0, 2)$ is non zero and satisfies the following condition

$$\begin{aligned} S(X, Y) \quad &= \quad ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \\ &\quad +d[A(X)D(Y) + A(Y)D(X)] \end{aligned} \tag{8}$$

for all $X, Y \in \chi(M)$, where a, b, c and d are real valued, non-zero scalars functions on (M^n, g). A, B and D are non zero 1-forms such that

$$g(X, \xi_1) = A(X), \quad g(X, \xi_2) = B(X) \quad and \quad g(X, \xi_3) = D(X), \tag{9}$$

wheer ξ_1, ξ_2, ξ_3 are three unit vector fields mutually orthogonal to each other at every point on M.Here a, b, c, d are called the associated scalars, A, B, D are called the associated main and auxiliary 1-forms. We denote this type of manifold $(HGQE)_n$.

§2. **Preliminaries**

From (8) and (9), we get

$$S(X, X) = a|X|^2 + b|g(X, \xi_1)|^2 + 2c|g(X, \xi_1)g(X, \xi_2)| + 2d|g(X, \xi_1)g(X, \xi_3)|, \quad \forall \quad X. \tag{10}$$

Let θ_1 be the angle between ξ_1 and any vector X; θ_2 be the angle between ξ_2 and any vector X; θ_3 be the angle between ξ_3 and any vector X. Then

$$\cos\theta_1 = \frac{g(X,\xi_1)}{\sqrt{g(\xi_1,\xi_1)}\sqrt{g(X,X)}} = \frac{g(X,\xi_1)}{\sqrt{g(X,X)}}$$

as $g(\xi_1,\xi_1)=1$), and

$$\cos\theta_2 = \frac{g(X,\xi_2)}{\sqrt{g(X,X)}} \quad \text{and} \quad \cos\theta_3 = \frac{g(X,\xi_3)}{\sqrt{g(X,X)}}.$$

If $b>0$, $c>0$, we have from (10)

$$\begin{aligned}(a+b+2c+2d)|X|^2 \quad &\geq \quad a|X|^2 + b|g(X,\xi_1)|^2 + 2c|g(X,\xi_1)g(X,\xi_2)| \\ &\quad +2d|g(X,\xi_1)g(X,\xi_3)| = S(X,X).\end{aligned} \tag{11}$$

Now, contracting (8) over X and Y, we get

$$r = na, \tag{12}$$

where r is the scalar curvature. Since ξ_1, ξ_2 and ξ_3 are orthogonal unit vector fields, therefore $g(\xi_1,\xi_1)=1$, $g(\xi_2,\xi_2)=1$, $g(\xi_3,\xi_3)=1$, $g(\xi_1,\xi_2)=0$, $g(\xi_1,\xi_3)=0$ and $g(\xi_2,\xi_3)=0$.

Again, putting $X=Y=\xi_1$ in (8) we get $S(\xi_1,\xi_1)=a+b$. Putting $X=Y=\xi_2$ in (8) we get $S(\xi_2,\xi_2)=a$. Putting $X=Y=\xi_3$ in (8) we get $S(\xi_3,\xi_3)=a$.

If X is a unit vector field, then $S(X,X)$ is the Ricci-curvature in the direction of X.

Notice that Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S, where

$$g(QX,Y) = S(X,Y) \ \ \forall \ X,Y \in TM. \tag{13}$$

Let l^2 denote the squares of the lengths of the Ricci-tensor S. Then

$$l^2 = \sum_{i=1}^{n} S(Qe_i,e_i), \tag{14}$$

where $\{e_i\}$, $i=1,2,\cdots,n$ is an orthonormal basis of the tangent space at a point of $(HGQE)_n$.

Now from (8) we get

$$\begin{aligned}S(Qe_i,e_i) \quad &= \quad ag(Qe_i,e_i) + bA(Qe_i)A(e_i) + c[A(Qe_i)B(e_i) + A(e_i)B(Qe_i)] \\ &\quad +d[A(Qe_i)D(e_i) + A(e_i)D(Qe_i)],\end{aligned}$$

i.e,

$$l^2 = (n-2)a^2 + (a+b)^2 + 2c^2 + 2d^2. \tag{15}$$

These result will be used in the sequel.

§3. Ricci Semi-symmetric $(HGQE)_n(n > 3)$

Chaki and Maity proved that $(QE)_n(n > 3)$ is Ricci Semi-symmetric if and only if

$$A(R(X,Y)Z = 0.$$

Let us suppose that $(HGQE)_n$ $(n > 3)$ is Ricci-Semi symmetric. Then

$$A(R(X,Y)Z = 0. \tag{16}$$

From (16) we get

$$A(Q(X)) = 0, \tag{17}$$

where Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. Then

$$g(QX,Y) = S(X,Y). \tag{18}$$

Then from (8) we get

$$A(Q(X)) = (a+b)A(X) + cB(X) + dD(X). \tag{19}$$

From (17) and (19) it follows that

$$(a+b)A(X) + cB(X) + dD(X) = 0. \tag{20}$$

Thus we can state the following.

Theorem 3.1 *If a $(HGQE)_n$ is Ricci Semi symmetric than (a+b)A(X)+cB(X)+dD(X)=0.*

§4. Sufficient Condition for a Compact Orientable $(HGQE)_n$(n≥3) Without Boundary to be Isometric to a Sphere

In this section we consider a compact, orientable $(HGQE)_n$ without boundary having constant associated scalars a, b, c and d. Then from (11) and (15), it follows that the scalar curvature is constant and so also is the length of the Ricci-tensor.

We further suppose that $(HGQE)_n$ under consideration admits a non-isometric conformal motion generated by a vector field X. Since l^2 is constant, it follows that

$$\pounds_X l^2 = 0, \tag{21}$$

where $\pounds_X$ denotes Lie differentiation with respect to X.

Now, it is known ([2], [4], [5], [9], [12], [13], [14], [15]) that if a compact Riemannian manifold M of dimension $n > 2$ with constant scalar curvature admits an infinitesimal non-isometric conformal transformation X such that $\pounds_X l^2 = 0$ then M is isometric to a sphere. But a sphere is Einstein so that b, c and d vanish which is a contradiction.This leads to the following theorem.

Theorem 4.1 *A compact orientable hyper generalized quasi Einstein manifold* $(HGQE)_n$ $(n \geq 3)$ *without boundary does not admit a non-isometric conformal vector field.*

§5. Killing Vector Field in a Compact Orientable $(HGQE)_n(n \geq 3)$ Without Boundary

In this section, we consider a compact, orientable $(HGQE)_n(n \geq 3)$ without boundary with a, b, c and d as associated scalars.

It is known [4] that in such a manifold M, the following relation holds

$$\int_M [S(X,X) - |\nabla X|^2 - (divX)^2]dv \leq 0 \qquad \forall X. \tag{22}$$

If X is a killing vector field, then $divX = 0$ ([4]). Hence (22) takes the form

$$\int_M [S(X,X) - |\nabla X|^2]dv = 0. \tag{23}$$

Let $b > 0$, $c > 0$, $d > 0$. Then by (11)

$$(a + b + 2c + 2d)|X|^2 \geq S(X,X). \tag{24}$$

Therefore,

$$(a + b + 2c + 2d)|X|^2 - |\nabla X|^2 \geq S(X,X) - |\nabla X|^2. \tag{25}$$

Consequently,

$$\int_M [(a + b + 2c + 2d)|X|^2 - |\nabla X|^2]dv \geq \int_M [S(X,X) - |\nabla X|^2]dv, \tag{26}$$

and by (23)

$$\int_M [(a + b + 2c + 2d)|X|^2 - |\nabla X|^2]dv \geq 0. \tag{27}$$

If $a + b + 2c + 2d < 0$, then

$$\int_M [(a + b + 2c + 2d)|X|^2 - |\nabla X|^2]dv = 0. \tag{28}$$

Therefore, $X = 0$. This leads to the following.

Theorem 5.1 *If in a compact orientable* $(HGQE)_n(n \geq 3)$ *without boundary and the associated scalars are such that* $b > 0, c > 0, d > 0$ *and* $a + b + 2c + 2d < 0$*, then there exists no non-zero killing vector field in this manifold.*

Corollary 5.1 *If in a compact orientable* $(HGQE)_n(n \geq 3)$ *without boundary, and each of the associated scalars* a, b, c, d*, is greater than zero, then any harmonic vector field* X *in the* $(HGQE)_n$ *is parallel and orthogonal to one of the generators of the manifold which makes greatest angle with the vector* X*.*

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