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Isotropic Curves and Their Characterizations in Complex Space $\mathbb{C}^4$

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Abstract: In this study, we investigate the classical differential geometry of isotropic curves in the complex space $\mathbb{C}^4$. We examine the constant breadth of isotropic curves and obtain some results regarding these isotropic curves. We express some characterizations of these curves via the É. Cartan derivative formula. We also indicate that the isotropic vector of these curves and pseudo curvature satisfy a third order vector differential equation with variable coefficients. We study this differential equation in some special cases. We dene evolute and involute of the isotropic curve and express some characterizations of these curves in terms of É. Cartan equations. The isotropic rectifying curve and isotropic helix are characterized in $\mathbb{C}^4$. Finally, we present the conditions for an isotropic curve to be an isotropic helix.

Key Words: Complex spaces, isotropic helix, isotropic curve of constant breath, Bertrand curves, iotropic rectifying curves.

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§1. Introduction

At the beginning of the nineteenth century, V. Pancelet's isotropic curve opened a door for a number of new concepts. The imaginary curve in the complex space was pioneered by Cartan. He defined his moving frame and the Cartan equations in $\mathbb{C}^3$. Altınışık extended the Cartan apparatus of isotropic curves to $\mathbb{C}^4$. Furthermore, isotopic Bertrand curves and isotropic helices in $\mathbb{C}^3$ were characterized, [9], [10], [16]. Also, the concept of a slant helix in the complex space in $\mathbb{C}^4$ was offered by Yılmaz [13].

Curves of constant breadth were introduced by Euler [3]. The curves have been studied in different spaces by researchers. For instance, Izumiya and Takeuchi defined slant helices [5]. Ali and Lopez gave some characterizations of slant helices in Minkowski 3-space [1]. Yılmaz

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studied spherical indicatrices of curves in Euclidean 4-space and Lorenzian 4-space [14], [15]. In [7], Mağden and Yılmaz extended the well known properties of constant breadth of the curves in four dimensional Galilean space $\mathbb{G}^4$.

Many researchers have studied involute-evolute curves in other spaces. The Frenet apparatus of involute-evolute curves couple in the space $\mathbb{E}^3$ and $\mathbb{E}^4$ is given [4], [8]. In [12], Turgut and Yılmaz studied involute-evolute curve couple in Minkowksi space-time. Şemin mentioned involute-evolute isotropic curve in [11]. In Euclidean 4-space, rectifying curves are introduced by İlarslan and Nesoviç in [6] as space curves whose position vector always lies in its rectifying plane, spanned by tangent, the first binormal and second binormal vector fields T, B_1 and B_2.The position vector of a rectifying curve α in $\mathbb{E}^4$ according to chosen origin satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \varphi(s)B_1(s) + \mu(s)B_2(s),$$

where λ, φ and μ are some differentiable functions of the pseudo arc-length parameter s.

Thus, the main goal of this paper is to define some isotropic curves in the four dimensional complex space $\mathbb{C}^4$. In the present paper, we first study isotropic curves of constant breadth and the involute-evolute of the curve in $\mathbb{C}^4$. Then we introduce the Bertrand curve and present some characterizations of the mentioned curves in terms of É. Cartan equations. Also, we give a new characterization of the isotropic helix. Throughout this study some complex curves are characterized in the complex space $\mathbb{C}^4$.

§2. **Preliminaries**

To meet the requirements in the next sections, the basic elements of the theory of imaginary curves in the space $\mathbb{C}^4$ are briefly presented (a more complete elementary operation can be found in [11]).

Let x_p be a complex analytic function of a complex variable t. Then the vector function

$$\mathbf{x}(t) = \sum_{p=1}^{4} x_p(t)\mathbf{k}_p,$$

is called an imaginary curve, where $t = t_1 + it_2$, $\mathbf{x} : \mathbb{C} \to \mathbb{C}^4$ and $\mathbf{k}_p$ are standard basis unit vectors of $\mathbb{E}^4$, $i^2 = -1$. An arbitrary vector $\mathbf{x} \in \mathbb{C}^4$, is called an isotropic vector if and only if $\mathbf{x}^2 = 0$, $(\mathbf{x} \neq \mathbf{0})$. In this space, the curves for which the square of the distance between any two points equal to zero, are called minimal or isotropic curves [11]. Let s denote pseudo arc-length (for details, see [10] or [11]). Then, a curve is an isotropic curve if and only if

$$ds^2 = d\mathbf{x}^2 = 0.$$

The complex four dimensional space $\mathbb{C}^4$, is the real vector space $\mathbb{E}^4$ endowed with the standard flat Euclidean metric given by

$$g = dx_1^2 + 2dx_1dx_3 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is the complex coordinate system of $\mathbb{C}^4$.

The É. Cartan frame moving along the isotropic curve $\mathbf{x}$ in the space $\mathbb{C}^4$ is denoted by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. This frame is defined ([11]) as

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{x}' \\ \mathbf{e}_2 &= i\mathbf{x}'' \\ \mathbf{e}_3 &= -\frac{\beta}{2}\mathbf{x}' + \mathbf{x}''' \\ \mathbf{e}_4 &= \mu(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) \end{aligned} \tag{2.1}$$

where $\beta = (\mathbf{x}''')^2$, μ is taken as ± 1. If μ is taken as $+1$, the determinant of matrix $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4]$, the É. Cartan frame becomes positively oriented. Here, the triple vector product is cross product expressed as in [2]. The inner products of these frame vectors are given by

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & \text{if } i + j \equiv 1, 2, 3 \text{mod } 4 \\ 1 & \text{if } i + j = 4 \\ -1 & \text{if } i + j = 8 \end{cases}$$

where the vectors $\mathbf{e}_1$ and $\mathbf{e}_3$ are isotropic vectors; $\mathbf{e}_2$ is real and $\mathbf{e}_4$ is a complex vector. É. Cartan derivative formulas can be expressed as follows:

$$\begin{aligned} \mathbf{e}_1' &= -i\mathbf{e}_2 \\ \mathbf{e}_2' &= ik\mathbf{e}_1 + i\mathbf{e}_3 \\ \mathbf{e}_3' &= -ik\mathbf{e}_2 \qquad (2.2) \\ \mathbf{e}_4' &= -\xi(k'' + \xi k)\mathbf{e}_1 - \xi k\mathbf{e}_3 + \frac{\xi'}{\xi}\mathbf{e}_4 \end{aligned}$$

where $k(s) = \frac{1}{2}\beta(s)$ is the pseudo curvature of the isotropic curve in the class C^5 and $\xi(s) = \pm\frac{1}{\sqrt{\beta^2(s)+\gamma(s)}}$, where $\gamma(s) = (\mathbf{x}^{(iv)})^2$, the derivative being taken with respect to the pseudo arc-length s. In the rest of the paper, we shall suppose pseudo curvature is non-vanishing except in the case of an isotropic cubic.

An isotropic hypersphere with centre $\mathbf{m}$ and radius $r > 0$ in $\mathbb{C}^4$ is defined as

$$S^3 = \{\mathbf{p} = (p_1, p_2, p_3, p_4) \in \mathbb{C}^4 : (\mathbf{p} - \mathbf{m})^2 = r^2\}.$$

Definition 2.1 *An isotropic curve $\boldsymbol{x} = \boldsymbol{x}(s)$ in $\mathbb{C}^4$ is called an isotropic cubic if the pseudo curvature $k(s) = 0$, where s is the pseudo arc-length parameter of the curve.*

Definition 2.2 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be a complex curve in $\mathbb{C}^4$. If the pseudo curvature of the curve is constant, then $\boldsymbol{x}(s)$ is called a pseudo helix or isotropic helix in $\mathbb{C}^4$.*

Definition 2.3 *An isotropic curve $\boldsymbol{x} = \boldsymbol{x}(s)$ in $\mathbb{C}^4$ is called an isotropic helix if inner product of its tangent vector $\mathbf{e}_1$ is constant with some fixed isotropic vector $\mathbf{v}$, that is, $\mathbf{e}_1.\mathbf{v} = constant$.*

Definition 2.4 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be an isotropic curve in $\mathbb{C}^4$. If there exists another isotropic curve $\boldsymbol{x}^* = \boldsymbol{x}^*(s)$ in $\mathbb{C}^4$ such that principal normal vector field $\boldsymbol{x}^*$ coincides with that normal vector field of $\boldsymbol{x}$, then $\boldsymbol{x}$ is called a Bertrand curve and $\boldsymbol{x}^*$ is called the Bertrand mate of $\boldsymbol{x}$ and vice versa, where $\boldsymbol{x}(s)$ and $\boldsymbol{x}^*(s)$ are opposite points of the curve.*

Definition 2.5 *Let φ and δ be two unit speed complex curves in $\mathbb{C}^4$. If the tangent vector of the curve φ at the point $\varphi(s_0)$ is orthogonal to the tangent vector of the curve δ at the $\delta(s_0)$ then curve δ is called the involute of the curve φ as follows:*

$$g(\boldsymbol{e}_{1\varphi}, \boldsymbol{e}_{1\delta}) = 0,$$

where $\{\boldsymbol{e}_{1\varphi}, \boldsymbol{e}_{2\varphi}, \boldsymbol{e}_{3\varphi}, \boldsymbol{e}_{4\varphi}\}$ and $\{\boldsymbol{e}_{1\delta}, \boldsymbol{e}_{2\delta}, \boldsymbol{e}_{3\delta}, \boldsymbol{e}_{4\delta}\}$ are Frenet frames of φ and δ, respectively. Also, the curve φ is called the evolute of the curve δ. This definition suffices to define this curve mate as $\delta = \varphi + \lambda \boldsymbol{e}_{1\varphi}$.

Definition 2.6 *Let α be a complex curve in $\mathbb{C}^4$. A rectifying curve is defined in $\mathbb{C}^4$ as an α isotropic curve whose position vector always lies in orthogonal complement $\boldsymbol{e}_2^{\perp}$ of its principal normal vector field $\boldsymbol{e}_2$.*

§3. Isotropic Curves of Constant Breadth and Their Characterizations

Let $\mathbf{x}(s)$ and $\mathbf{x}^*(s)$ be isotropic curves in $\mathbb{C}^4$. If the tangent isotropic vector $\mathbf{e}_1$ of $\mathbf{x}(s)$ coincides with the tangent isotropic vector $\mathbf{e}_1^*$ of $\mathbf{x}^*(s)$ opposite directions at the corresponding points and the distance between these points is always constant, then $\mathbf{x}(s)$ is a constant breadth of the isotropic curve. Suppose that $\mathbf{x}(s)$ and $\mathbf{x}^*(s)$ are isotropic curves of constant breadth. Then $\mathbf{e}_1^*$ can be expressed by

$$\mathbf{e}_1 = -\mathbf{e}_1^*$$

where $\mathbf{e}_1$ and $\mathbf{e}_1^*$ are inverse direction and parallel vectors.

Let $\mathbf{x}(s)$ and $\mathbf{x}^*(s)$ be isotropic curves of constant breadth in $\mathbb{C}^4$. Taking into account the Cartan equations, it can be decomposed by

$$\mathbf{X}^*(s) = \mathbf{X}(s) + m_1(s)\mathbf{e}_1 + m_2(s)\mathbf{e}_2 + m_3(s)\mathbf{e}_3 + m_4(s)\mathbf{e}_4, (0 \leq s \leq 1), \tag{3.1}$$

where $\mathbf{X}(s)$ and $\mathbf{X}^*(s)$ are opposite points and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ denote the É. Cartan frame in $\mathbb{C}^4$.

Differentiating the equation (3.1) with respect to s, we get

$$\begin{aligned} \frac{d\mathbf{X}^*}{ds} &= \frac{d\mathbf{X}^*}{ds^*} \cdot \frac{ds^*}{ds} = \mathbf{e}_1^* \frac{ds^*}{ds} \\ &= (\frac{dm_1}{ds} + m_2 ik + m_4\eta_1)\mathbf{e}_1 + (-m_1 i + \frac{dm_2}{ds} - m_3 ik)\mathbf{e}_2 \\ &\quad + (m_2 i + \frac{dm_3}{ds} + m_4\eta_2)\mathbf{e}_3 + (\frac{dm_4}{ds} + m_4\eta_3)\mathbf{e}_4, \end{aligned} \tag{3.2}$$

where $\eta_1(s) = -\xi(k'' + \xi k), \eta_2(s) = -\xi k, \eta_3(s) = \frac{\xi'}{\xi}$ and $k = \frac{1}{2}\beta$ is a pseudo curvature of the isotropic curve in the class C^5. Since $\mathbf{e}_1^* = -\mathbf{e}_1$, we obtain

$$\begin{aligned}
&1 + \frac{dm_1}{ds} + m_2 ik + m_4\eta_1 = -\frac{d\mathbf{s}^*}{ds} \\
&-m_1 i + \frac{dm_2}{ds} - m_3 ik = 0 \\
&m_2 i + \frac{dm_3}{ds} - m_4\eta_2 = 0 \\
&\frac{dm_4}{ds} + m_4\eta_3 = 0.
\end{aligned} \tag{3.3}$$

Putting $f(s) = -1 - \frac{ds^*}{ds}$, in the equation (3.3), it can be written as

$$\begin{aligned}
\frac{dm_1}{ds} &= -m_2 ik - m_4\eta_1 + f(s) \\
\frac{dm_2}{ds} &= m_1 i + m_3 ik \\
\frac{dm_3}{ds} &= -m_2 i - m_4\eta_2 \\
\frac{dm_3}{ds} &= -m_4\eta_3.
\end{aligned} \tag{3.4}$$

By virtue of the equation $(3.4)_4$ (i.e. the fourth expression of the equation (3.4)) we have $m_4 = c$ is constant. Rearranging the equation (3.4) we get

$$\begin{aligned}
\frac{dm_1}{ds} &= -m_2 ik - c(k'' + \xi k) + f(s) \\
\frac{dm_2}{ds} &= m_1 i + m_3 ik \\
\frac{dm_3}{ds} &= -m_2 i - ck.
\end{aligned} \tag{3.5}$$

The following corollary is a consequence of the equations (3.4) and (3.5).

Corollary 3.1 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be an isotropic cubic. The isotropic position vector of $\boldsymbol{x}$ with respect to É. Cartan frame can be formed by the equations (3.5) and can be obtained as*

$$\begin{aligned}
\mathbf{x}(s) \quad = \quad & \mathbf{x}^*(s) + \left(\int f(s)ds + k_1(s)\right)\mathbf{e}_1 \\
& + \left(\left[\int \left(\int f(s)ds\right) + k_1(s)ds\right] + k_2(s)\right)\mathbf{e}_2 \\
& + \left(\int ((\int f(s)ds)ds) + k_1(s)\frac{s^2}{2}ds + ik_2(s) + k_3(s)\right)\mathbf{e}_3 + c\mathbf{e}_4.
\end{aligned}$$

Proof Let $\mathbf{x} = \mathbf{x}(s)$ be an isotropic cubic. Then, $k = 0$ from Definition 2.1. From equation $(3.5)_1$ we get $\frac{dm_1}{ds} = f(s)$. Integrating this expression we have,

$$m_1 = \int f(s)ds + k_1,$$

where k_1 is a complex constant, from equations (3.4),$(3.5)_2$ and $(3.5)_3$,

$$\begin{aligned} m_2 &= i(\int(\int f(s)ds) + k_1(s)ds) + k_2(s) \\ m_3 &= \int((\int(\int f(s)ds) + k_1(s)\frac{s^2}{2})ds) + ik_2(s) + k_3(s) \end{aligned}$$

and $m_4 = c$ is constant. After m_1, m_2, m_3 and m_4 are substituted into the isotropic position vector $\mathbf{x} = \mathbf{x}(s)$, the proof is completed. □

Theorem 3.1 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be complex curve of constant breadth with pseudo arc-length in $\mathbb{C}^4$. If $\boldsymbol{x} = \boldsymbol{x}(s)$ lies fully in the $\boldsymbol{e}_3\boldsymbol{e}_4$ subspace, then $\boldsymbol{x} = \boldsymbol{x}(s)$ is an isotropic helix.*

Proof Let $\mathbf{x} = \mathbf{x}(s)$ be the pseudo arc-length parameter of constant breadth of complex curve in $\mathbb{C}^4$. From equations (3.5), if we take $m_1 = m_3 = 0$, then we have $m_2 = c_1$ (where c_1 is a constant). Using this expression in the third equation of (3.5), we obtain $k = \frac{c_1}{c}i$ is constant. From Definition 2.3), it is clear that the curve $\mathbf{x} = \mathbf{x}(s)$ is an isotropic helix. □

Theorem 3.2 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be complex curve of constant breadth with pseudo arc-length in $\mathbb{C}^4$. There is no constant breadth of isotropic curve that lies fully in the $\boldsymbol{e}_1\boldsymbol{e}_2$ subspace.*

Proof Let $\mathbf{x} = \mathbf{x}(s)$ be the pseudo arc-length parameter of constant breadth of complex curve in $\mathbb{C}^4$. If we take $m_3 = m_4 = 0$ in equation (3.5), we get $m_1 = 0$ and $m_2 = cki$. So $\mathbf{x} = \mathbf{x}(s)$ lies fully in the $\mathbf{e}_1\mathbf{e}_2$ subspace. □

Theorem 3.3 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be complex curve of constant breadth with pseudo arc-length in $\mathbb{C}^4$. There is no constant breadth of complex curve which lies fully in the $\boldsymbol{e}_1\boldsymbol{e}_4$ subspace, and $\boldsymbol{x} = \boldsymbol{x}(s)$ is isotropic cubic.*

Proof Let $\mathbf{x} = \mathbf{x}(s)$ be the pseudo arc-length parameter of constant breadth of complex curve in $\mathbb{C}^4$. From equation (3.5), we get $m_1 = 0$, $m_4 = c$ and $k = 0$. So $\mathbf{x} = \mathbf{x}(s)$ lies fully in the $\mathbf{e}_1\mathbf{e}_4$ subspace. From Definition 2.1, $\mathbf{x} = \mathbf{x}(s)$ is an isotropic cubic. □

Theorem 3.4 *A pseudo arc-length isotropic $\boldsymbol{x} = \boldsymbol{x}(s)$ in $\mathbb{C}^4$ is of constant breadth if and only if it satisfies the following third order differential equation.*

Proof From equation $(3.5)_1$, we get

$$m_2 = \frac{ck'' + c\xi k - f(s) + \frac{dm_1}{ds}}{-ik}.$$

Substituting into $(3.5)_2$, this expression m_3 is obtained

$$m_3 = \frac{\frac{d}{ds}\left[\frac{ck''+c\xi k-f(s)+\frac{dm_1}{ds}}{-ik}\right] - m_1 i}{ik}.$$

Taking the derivative of this expression, we obtain

$$\frac{dm_3}{ds} = \frac{d}{ds}\left[\frac{\frac{d}{ds}\left(\frac{ck''+c\xi k-f(s)+\frac{dm_1}{ds}}{-ik}\right)-m_1}{k}\right].$$

Substituting into equation $(3.5)_3$, this expression, we have a differential equation of third order with complex variable coefficients as follows:

$$\frac{d}{ds}\left[-\frac{1}{ik}\frac{d}{ds}\left(\frac{ck''+c\xi k-f(s)+\frac{dm_1}{ds}}{-ik}\right)\right]+\frac{d}{ds}\left(\frac{m_1}{k}\right) \\ -\frac{1}{k}\left(ck''+c\xi k-f\left(s\right)+\frac{dm_1}{ds}\right)+ck=0. \tag{3.6}$$

The differential equation of third order with variable coefficients in equation (3.6) is characterized for the constant breadth of isotropic curve $\mathbf{x}=\mathbf{x}(s)$.

Now, we characterize the distance between opposite points of the curves of constant breadth in $\mathbb{C}^4$. Remember the equation (3.1)

$$\mathbf{X}^*(s)=\mathbf{X}(s)+m_1(s)\mathbf{e}_1+m_2(s)\mathbf{e}_2+m_3(s)\mathbf{e}_3 \\ +m_4(s)\mathbf{e}_4,(0\leq s\leq 1).$$

If the distance between opposite points of (C) and (C^*) is constant, then we can write that

$$\|x^*-x\|=m_1^2+2m_1m_3-m_4^2=l^2=constant. \tag{3.7}$$

Hence, we write

$$m_2\frac{dm_2}{ds}+m_3\frac{dm_1}{ds}+m_1\frac{dm_3}{ds}-m_4\frac{dm_4}{ds}=0 \tag{3.8}$$

from equations (3.5) since $m_4=c$ is constant. Rearranging the equation (3.8), we obtain

$$m_2\frac{dm_2}{ds}+m_3\frac{dm_1}{ds}+m_1\frac{dm_3}{ds}=0. \tag{3.9}$$

Considering equations (3.5), we have

$$m_3\left[\mu\left(s\right)-k^2i-\frac{m_2'ck}{m_3}\right]=0. \tag{3.10}$$

We write $m_3=0$ or $\mu\left(s\right)-k^2i-\frac{m_2'ck}{m_3}=0$, obviously, $m_3\neq 0$. Then it can be expressed in the following cases:

Case 1. Let us suppose $m_3=c_1\neq 0$ constant. From equations $(3.5)_2$ and $(3.5)_3$ we easily have $m_2=cki, m_1=-c_1k$. Then the isotropic position vector of φ^* can be written as follows:

$$\varphi^*=\varphi+c_1k\mathbf{e}_1+cki\mathbf{e}_2+c_1\mathbf{e}_3+c\mathbf{e}_4.$$

Case 2. Let us suppose that m_3 is constant and φ is isotropic helix. Thus, the equation (3.6) takes the form

$$\frac{d^2g(s)}{ds^2} - kh(s) + ck^3 = 0, \tag{3.11}$$

where $h(s) = c\xi k - f(s)$. The solution of the equation (3.11) is

$$h(s) = L_1 e^{\sqrt{k}x} + L_2 e^{-\sqrt{k}x} + \frac{1}{2} - \frac{1}{\sqrt{2}}, \tag{3.12}$$

where L_1 and L_2 are real numbers.

Case 3. Let us suppose

$$\mu(s) - k^2 i - \frac{m_2' ck}{m_3} = 0. \tag{3.13}$$

In this case, (C^*) is transformed by the constant vector $\eta = m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3 + m_4\mathbf{e}_4$ of (C). Now, let us investigate the solution to Case 3.

Suppose that μ is an isotropic cubic. Then, we get from equation (3.13) $\mu(s) = 0$ and from equation (3.5) we get m_1 =constant, $m_2 = 0, m_3 = -\frac{c}{k}$. □

§4. Involute and Evolute of Isotropic Curves in $\mathbb{C}^4$

Theorem 4.1 *Let φ and δ be complex curves and φ be an evolute of δ. The Cartan apparatus of $\varphi\{\boldsymbol{e}_{1\varphi}, \boldsymbol{e}_{2\varphi}, \boldsymbol{e}_{3\varphi}, \boldsymbol{e}_{4\varphi}, k_\varphi\}$ can be formed according to the Cartan apparatus of $\delta\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, k\}$.*

Proof Let φ and δ be complex curves and φ be an evolute of δ. According to the property of involute-evolute curve couples, we have

$$\varphi = \delta + \lambda\mathbf{e}_1. \tag{4.1}$$

Differentiating both sides of the equation (4.1) with respect to s, we obtain

$$\frac{d\varphi}{ds_\varphi}.\frac{ds_\varphi}{ds} = \mathbf{e}_1 + \frac{d\lambda}{ds}\mathbf{e}_1 + \lambda(-i\mathbf{e}_2). \tag{4.2}$$

Rearranging equation (4.2), we have

$$\frac{d\varphi}{ds_\varphi}\frac{ds_\varphi}{ds} = (1 + \frac{d\lambda}{ds})\mathbf{e}_1 - \lambda i\mathbf{e}_2. \tag{4.3}$$

Similarly, based on the definition of involute and evolute curves, we can say $\mathbf{e}_{1\varphi} \perp \mathbf{e}_1$. Obviously, we get

$$1 + \frac{d\lambda}{ds} = 0. \tag{4.4}$$

We get $\lambda = c - s$, where c is constant. Rearranging the equation (4.1), we get

$$\varphi = \delta + (c - s)\mathbf{e}_1. \tag{4.5}$$

By differentiating the equation (4.5), we have the following equation

$$\varphi' = \mathbf{e}_{1\varphi}.\frac{ds_\varphi}{ds} = (c-s)(-i\mathbf{e}_2). \tag{4.6}$$

Taking the norm of both sides, we get

$$\mathbf{e}_{1\varphi} = -\mathbf{e}_2 \tag{4.7}$$

and

$$\frac{ds_\varphi}{ds} = (c-s)i. \tag{4.8}$$

Differentiating the equation (4.6) two times with respect to s, we get

$$\varphi'' = -(c-s)k\mathbf{e}_1 + i\mathbf{e}_2 + (c-s)\mathbf{e}_3 \tag{4.9}$$

and

$$\varphi''' = [-2k + (c-s)k']\,\mathbf{e}_1 + [-2i(c-s)k]\,\mathbf{e}_2 - 2\mathbf{e}_3. \tag{4.10}$$

Thus, we have the following expressions for $\mathbf{e}_{2\varphi}, \mathbf{e}_{3\varphi}$ and k_φ.

$$\begin{aligned} \mathbf{e}_{2\varphi} &= (c-s)ki\mathbf{e}_1 - \mathbf{e}_2 + (c-s)i\mathbf{e}_3 \\ \mathbf{e}_{3\varphi} &= [-2k + (c-s)k']\,\mathbf{e}_1 + i(c-s)(\frac{\beta}{2} - 2k)\mathbf{e}_2 - 2\mathbf{e}_3 \end{aligned} \tag{4.11}$$

and

$$k_\varphi = -2\,[-2k + (c-s)k'] + [-2(c-s)k]^2\,. \tag{4.12}$$

Using the exterior product $\sigma(\mathbf{e}_{1\varphi} \wedge \mathbf{e}_{2\varphi} \wedge \mathbf{e}_{3\varphi})$, we get

$$\mathbf{e}_{4\varphi} = \sigma\,[2(c-s)ik(1 + (c-s)k)\mathbf{e}_4]\,, \tag{4.13}$$

where $\sigma = \pm 1$. □

Since from equation (4.7), it follows that $\mathbf{e}_{1\varphi}$ is not an isotropic vector, we can state the following.

Remark 4.1 Let φ be an evolute of a complex curve in $\mathbb{C}^4$. The curve φ cannot be an isotropic curve.

Theorem 4.2 *Let φ and δ be complex curve and φ be an evolute of δ in $\mathbb{C}^4$. The evolute φ cannot be an isotropic helix in $\mathbb{C}^4$.*

Proof Considering the definition of isotropic helix, we write

$$\mathbf{e}_{1\varphi}.\mathbf{v} = \text{constant}, \tag{4.14}$$

where $\mathbf{v}$ is a constant isotropic vector. From equation (4.7), we easily have

$$-\mathbf{e}_2.\mathbf{v} = \text{constant}, \tag{4.15}$$

Differentiating both sides of equation (4.15), we get

$$-(ik\mathbf{e}_1 + i\mathbf{e}_3).\mathbf{v} = 0. \tag{4.16}$$

Therefore $\mathbf{v} \perp \mathbf{e}_1$ and $\mathbf{v} \perp \mathbf{e}_3$. Let us decompose $\mathbf{v}$ as

$$\mathbf{v} = t_1\mathbf{e}_2 + t_2\mathbf{e}_4. \tag{4.17}$$

Differentiating equation (4.17) consecutively and using Cartan equations, we have $t_1 = 0$ and $t_2 = 0$. According to the result, we write

$$\mathbf{v} = 0. \tag{4.18}$$

Equations (4.14) and (4.18) yield a contradiction. Therefore, evolute φ cannot be an isotropic helix in space $\mathbb{C}^4$. □

§5. Bertrand Couple Curves of Isotropic Curves in $\mathbb{C}^4$

Theorem 5.1 *Let α^* and α be Bertrand curves in complex space $\mathbb{C}^4$. The Cartan apparatus of $\alpha^*\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \boldsymbol{e}_3^*, \boldsymbol{e}_4^*, k^*\}$ can be formed by the Cartan apparatus of $\alpha\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, k\}$.*

Proof Suppose that $\{\alpha(s), \alpha^*(s^*)\}$ is an isotropic Bertrand pair of curves. Then $\alpha^*(s^*)$ can be expressed by

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)\mathbf{e}_2, \tag{5.1}$$

where $\lambda(s)$ is the non zero analytic function and s^* is the pseudo arc-length parameter of $\alpha^*(s^*)$. Differentiating both sides of the equation (5.1) with respect to s, we get

$$\alpha^* = \frac{d\alpha^*}{ds^*}\frac{ds^*}{ds} = \mathbf{e}_1^*\frac{ds^*}{ds} = (1 + \lambda ki)\,\mathbf{e}_1 + \frac{d\lambda}{ds}\mathbf{e}_2 + \lambda i\mathbf{e}_3. \tag{5.2}$$

The definition of Bertrand curves yields $\mathbf{e}_1^* \perp \mathbf{e}_2$. Multiplying both sides of equation (5.2) with $\mathbf{e}_2$ we have

$$\frac{d\lambda}{ds} = 0 \tag{5.3}$$

which implies that λ is constant. Using this in the equation (5.2) and taking the norm of the both sides, we get

$$\frac{ds^*}{ds} = \sqrt{2(1 + \lambda ki)\lambda i}$$

and the tangent vector $\mathbf{e}_1^*$ is equal to

$$\mathbf{e}_1^* = \frac{(1+\lambda ki)}{\sqrt{2\,(1+\lambda ki)\,\lambda i}}\mathbf{e}_1 + \frac{\lambda i}{\sqrt{2\,(1+\lambda ki)\,\lambda i}}\mathbf{e}_3. \tag{5.4}$$

Taking the derivative of the equation (5.2) two times with respect to s, we get

$$\alpha^{*''} = (1+\lambda k' i)\mathbf{e}_1 + (-1-\lambda ki+\lambda k)\mathbf{e}_2 \tag{5.5}$$

and

$$\alpha^{*'''} = (1+\lambda k'' i - ki + \lambda k^2 + \lambda k^2 i)\mathbf{e}_1 + (-1-\lambda k' i + \lambda k')\mathbf{e}_2 + (-i-\lambda k+\lambda ki)\mathbf{e}_3. \tag{5.6}$$

Using the equation (5.6), we get the vectors $\mathbf{e}_2^*$, $\mathbf{e}_3^*$ and pseudo curvature k^*, as follows:

$$\mathbf{e}_2^* = \frac{1}{i}[(1+\lambda k' i)\mathbf{e}_1 + (-1-\lambda ki+\lambda k)\mathbf{e}_2],$$

$$\begin{aligned}\mathbf{e}_3^* = \frac{1}{2}\{&[-1-\lambda k' i + 2(1+\lambda k'' i - ki + \lambda k^2 i)(-i-\lambda k+\lambda ki)]\mathbf{e}_1 \\ &+(-1-\lambda k' i+\lambda k')\mathbf{e}_2 + [-1-\lambda ki+\lambda k]\mathbf{e}_3\}\end{aligned}$$

and

$$k^* = \frac{1}{2}\{-1-\lambda k' i + \lambda k' + 2(1+\lambda k'' i - ki + \lambda k^2 + \lambda k^2 i)(-i-\lambda k+\lambda ki)\}.$$

So, the pseudo curvature $k^*(s)$ is a non zero constant. □

Remark 5.1 Obviously, $\mathbf{e}_1^*$ isn't an isotropic vector from equation (5.4). So, the Bertrand curve α^* cannot be an isotropic curve.

Remark 5.2 Let α^* and α be Bertrand curves in $\mathbb{C}^4$. If one of the Bertrand curves lies fully in $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ subspace, then the Bertrand mate also lies fully in the same subspace of $\mathbb{C}^4$.

Theorem 5.2 *Let* $\mathbf{x} = \mathbf{x(s)}$ *be an isotropic curve in* $\mathbb{C}^4$*. Then,* $\mathbf{x(s)}$ *is a pseudo isotropic helix if and only if the following statements are equivalent:*

(a) $\det(\mathbf{x}''(s), \mathbf{x}'''(s), \mathbf{x}^{(iv)}(s)) = 0$;

(b) $\det(\mathbf{e}_1'(s), \mathbf{e}_1''(s), \mathbf{e}_1'''(s)) = 0$

c) $\det(\mathbf{e}_3'(s), \mathbf{e}_3''(s), \mathbf{e}_3'''(s)) = 0$.

Proof Taking the 2^{nd}, 3^{rd} and 4^{th} derivatives of equation (2.1), we obtain

$$\mathbf{x}' = \mathbf{e}_1, \quad \mathbf{x}'' = -i\mathbf{e}_2, \quad \mathbf{x}''' = k\mathbf{e}_1 + \mathbf{e}_3, \quad \mathbf{x}^{(iv)} = k'\mathbf{e}_1 - 2ik\mathbf{e}_2. \tag{5.7}$$

We calculate that

$$\det(\mathbf{x}''(s),\mathbf{x}'''(s),\mathbf{x}^{(iv)}(s)) = \begin{vmatrix} 0 & -i & 0 \\ k & 0 & 1 \\ k' & -2ik & 0 \end{vmatrix} = -ik'.$$

Since $\mathbf{x} = \mathbf{x}(s)$ is an isotropic helix, then $\det(\mathbf{x}''(s),\mathbf{x}'''(s),\mathbf{x}^{(iv)}(s)) = 0$. Conversely, let the statement "**a)**" be true. Then $\det(\mathbf{x}''(s),\mathbf{x}'''(s),\mathbf{x}^{(iv)}(s)) = -ik' = 0$. Thus, k is constant and $\mathbf{x}(s)$ is an isotropic helix. This completes the proof "**a)**". Similarly, denoting $\mathbf{x}' = \mathbf{e}_1, \mathbf{x}'' = \mathbf{e}_1', \mathbf{x}''' = \mathbf{e}_1''$ and $\mathbf{x}^{(iv)} = \mathbf{e}_1'''$, we easily see that "**a)**" and "**b)**" are equivalent. Also, because of the fact that the equations

$$\begin{aligned} \mathbf{e}_3' &= -ik\mathbf{e}_2 \\ \mathbf{e}_3'' &= k^2\mathbf{e}_1 - ik'\mathbf{e}_2 + k\mathbf{e}_3 \\ \mathbf{e}_3''' &= 3kk'\mathbf{e}_1 - (2k^2 + ik'')\mathbf{e}_2 + 2k'\mathbf{e}_3, \end{aligned}$$

are hold, we can calculate that

$$\det(\mathbf{e}_3'(s),\mathbf{e}_3''(s),\mathbf{e}_3'''(s)) = -k^3k' = 0.$$

Since $\mathbf{x} = \mathbf{x}(s)$ is an isotropic helix, then $\det(\mathbf{e}_3'(s),\mathbf{e}_3''(s),\mathbf{e}_3'''(s)) = 0$. Conversely, let us say that in the determinant,

$$\det(\mathbf{e}_3'(s),\mathbf{e}_3''(s),\mathbf{e}_3'''(s)) = \begin{vmatrix} 0 & ik & 0 \\ k^2 & -ik' & k \\ 3kk' & -\left(2k^2 + ik''\right) & 2k' \end{vmatrix} = -k^3k' = 0, \tag{5.8}$$

we get $\frac{dk}{ds} = 0$ then k is a constant. As an immediate consequence of Definition 2.2, $\mathbf{x} = \mathbf{x}(s)$ is an isotropic helix. □

§6. Isotropic rectifying curves in $\mathbb{C}^4$

In this section, we firstly characterize the rectifying curves in $\mathbb{C}^4$ in terms of their pseudo curvature. In analogy with Euclidean four dimensional case, we define the rectifying curves in complex space $\mathbb{C}^4$ as a curve whose position vector always lies in the orthogonal complement $\mathbf{e}_2^{\perp}$ of its principal normal vector field $\mathbf{e}_2$. Hence, $\mathbf{e}_2^{\perp}$ is a three dimensional subspace of $\mathbb{C}^4$, spanned by vector filed $\mathbf{e}_1, \mathbf{e}_3$ and $\mathbf{e}_4$. Therefore the position vector with respect to some chosen origin of a rectifying curve α in $\mathbb{C}^4$, satisfies the equation

$$\alpha(s) = \lambda(s)\mathbf{e}_1(s) + \mu\mathbf{e}_3(s) + \delta(s)\mathbf{e}_4(s) \tag{6.1}$$

for differentiable functions $\lambda(s), \mu(s)$ and $\delta(s)$ with pseudo arc-length parameter s. Firstly, let us characterize the rectifying curve α in $\mathbb{C}^4$ in terms of its pseudo curvature. Let $\alpha = \alpha(s)$ be a unit speed complex rectifying curve in $\mathbb{C}^4$, with non zero pseudo curvature $k(s)$. By definition, the position vector of complex curve α satisfies equation (6.1) for some differentiable functions $\lambda(s), \mu(s)$ and $\delta(s)$. Differentiating the equation (6.1) and using Cartan derivative formulas (2.2), we get

$$[\lambda' - 1 - \delta\xi(k'' + \xi k)]\mathbf{e}_1 + [\lambda i - \mu i k]\mathbf{e}_2 + [\lambda' - \delta\xi k]\mathbf{e}_3 + [\delta' + \delta\frac{\xi'}{\xi}]\mathbf{e}_4 = 0.$$

It follows that

$$\begin{aligned} \lambda'(s) - \delta(s)\xi(s)(k''(s) + \xi(s)k(s)) &= 1 \\ \lambda(s)i - \mu(s)k(s)i &= 0 \\ \lambda'(s) - \delta(s)\xi(s)k(s) &= 0 \\ \delta'(s) + \delta(s)\frac{\xi'(s)}{\xi(s)} &= 0 \end{aligned} \tag{6.2}$$

and thus

$$\begin{aligned} \lambda(s) &= c\int_0^s k(s)ds \\ \mu(s) &= \frac{c}{k(s)}\int_0^s k(s)ds \\ \delta(s) &= \frac{c}{\xi(s)}. \end{aligned} \tag{6.3}$$

Conversely, assuming that the pseudo curvature $k(s)$ of an arbitrary unit speed complex curve α in $\mathbb{C}^4$, satisfied the following equation

$$\alpha(s) = \left(c\int_0^s k(s)ds\right)\mathbf{e}_1(s) + \left(\frac{c}{k(s)}\int_0^s k(s)ds\right)\mathbf{e}_3(s) + \left(\frac{c}{\xi(s)}\right)\mathbf{e}_4(s)$$

Remark 6.1 (i) α cannot be an isotropic cubic, since $\frac{c}{k(s)} \neq 0$;

(ii) If α is a helix, then $\alpha(s) = s\left[(ck)\mathbf{e}_1 + (c)\mathbf{e}_3 + \left(\frac{c}{\xi(s)}\right)\mathbf{e}_4\right]$.

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