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Generalized abc-Block Edge Transformation Graph $Q^{abc}(G)$ When $abc = +0-$

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Abstract: The generalized abc-block edge transformation graph $Q^{+0-}(G)$ is a graph whose vertex set is the union of the edges and blocks of G, in which two vertices are adjacent whenever corresponding edges of G are adjacent or one corresponds to an edge and other to a block of G are not incident with each other. In this paper, we study the girth, covering invariants and the domination number of $Q^{+0-}(G)$. We present necessary and sufficient conditions for $Q^{+0-}(G)$ to be planar, outerplanar, minimally nonouterplanar and maximal outerplanar. Further, we establish a necessary and sufficient condition for the generalized abc-block edge transformation graph $Q^{+0-}(G)$ have crossing number one.

Key Words: Line graph, abc-block edge transformation, generalized abc-block edge transformation graph, Smarandachely block-edge H-graph.

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§1. Introduction

Throughout the paper, we only consider simple graphs without isolated vertices. Definitions not given here may be found in [5]. A *cut vertex* of a connected graph is the one whose removal increases the number of components. A *nonseparable* graph is connected, nontrivial and has no cut vertices. A *block* of a graph is a maximal nonseparable subgraph. Let $G = (V, E)$ be a graph with block set $U(G)$=$\{B_i;\ B_i$ is a block of $G\}$. If a block $B \in U(G)$ with the edge set $\{e_1, e_2, \cdots, e_m; m \geq 1\}$, then we say that the edge e_i and block B are incident with each other, where $1 \leq i \leq m$. The *girth* of a graph G, denoted by $g(G)$, is the length of the shortest cycle if any in G. Let $\lceil x\rceil(\lfloor x\rfloor)$ denote the least (greatest) integer greater (less) than or equal to x.

A vertex and an edge are said to *cover* each other if they are incident. A set of vertices in a graph G is a *vertex covering set*, which covers all the edges of G. The *vertex covering number* $\alpha_0(G)$ of G is the minimum number of vertices in a vertex covering set of G. A set of edges in a graph G is an *edge covering set*, which covers all vertices of G. The *edge covering number*

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$\alpha_1(G)$ of G is the minimum number of edges in an edge covering set of G. A set of vertices in a graph G is *independent* if no two of them are adjacent. The maximum number of vertices in such a set is called the *vertex independence number* of G and is denoted by $\beta_0(G)$. The set of edges in a graph G is *independent* if no two of them are adjacent. The maximum number of edges in such a set is called the *edge independence number* of G and is denoted by $\beta_1(G)$.

The *line graph* $L(G)$ of a graph G is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common [5]. The *plick graph* $P(G)$ of a graph G is the graph whose set of vertices is the union of the set of edges and blocks of G and in which two vertices are adjacent if and only if the corresponding edges of G are adjacent or one is corresponds to an edge and other is corresponds to a block are incident [8]. In [2], we generalized the concept of plick graph and were termed as generalized abc-block edge transformation graphs $Q^{abc}(G)$ of a graph G and obtained 64 kinds of graphs. In this paper, we consider one among those 64 graph which is defined as follows:

Definition 1.1 *The generalized abc-block edge transformation graph $Q^{+0-}(G)$ is a graph whose vertex set is the union of the edges and blocks of G, in which two vertices are adjacent whenever corresponding edges of G are adjacent or one corresponds to an edge and other to a block of G are not incident with each other.*

Generally, a *Smarandachely block-edge H-graph* is such a graph with vertex set $E(G)\bigcup B(G)$ and two vertices $e_1, e_2 \in E(G)\bigcup B(G)$ are adjacent if $e_1, e_2 \in E(H)$ are adjacent, or at least one of e_1, e_2 not in $E(H)$ and they are non-adjacent, or one in $E(H)$ and other in $B(G)$ which are not incident, where H is a subgraph of G with property $\mathscr{P}$. Clearly, a Smarandachely block-edge $E(G)\bigcup B(G)$-graph is nothing else but a generalized abc-block edge transformation graph.

In this paper, we study the girth, covering invariants and the domination number of $Q^{+0-}(G)$. We present necessary and sufficient conditions for $Q^{+0-}(G)$ to be planar, outerplanar, minimally nonouterplanar and maximal outerplanar. Further, we establish a necessary and sufficient condition for the generalized abc-block edge transformation graph $Q^{+0-}(G)$ have crossing number one. Some other graph valued functions were studied in [3, 4, 7, 8, 9, 11, 12]. In Figure 1, a graph G and its generalized abc-block edge transformation graph $Q^{+0-}(G)$ are shown.

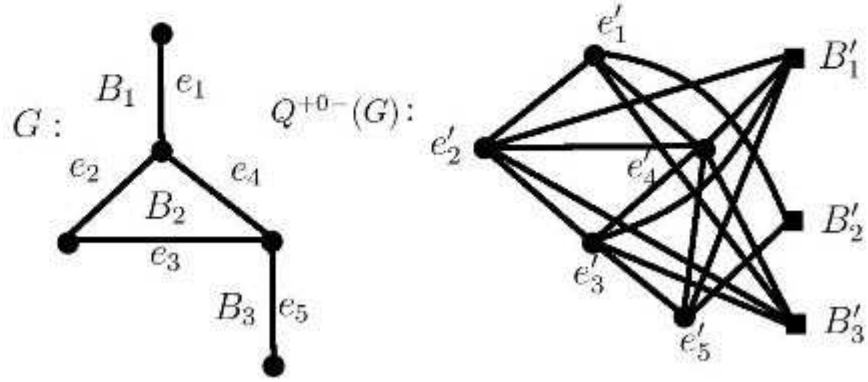

Figure 1. Graph G and its $Q^{+0-}(G)$.

In $Q^{+0-}(G)$, the vertices correspond to edges of G denoted by circles and vertices correspond to blocks of G denoted by squares. The vertex $e_i^{'}$ $(B_i^{'})$ of $Q^{+0-}(G)$ corresponding to edge e_i (block B_i) of G and is refereed as edge (block)-vertex.

The following theorems will be useful in the proof of our results.

Theorem 1.1([8]) *If G is a nontrivial connected (p, q) graph whose vertices have degree d_i and if b_i the number of blocks to which vertex v_i belongs in G, then $P(G)$ has $q - p + 1 + \sum\limits_{i=1}^{p} b_i$ vertices and $\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2$ edges.*

Theorem 1.2([5]) *For any nontrivial connected graph G with p vertices,*

$$\alpha_0(G) + \beta_0(G) = p = \alpha_1(G) + \beta_1(G).$$

Theorem 1.3([6]) *If $L(G)$ is the line graph of a nontrivial connected graph G with q edges, then*

$$\alpha_1(L(G)) = \lceil \frac{q}{2} \rceil.$$

§2. Basic Results on $Q^{+0-}(G)$

We start with preliminary remarks.

Remark 2.1 $L(G)$ is an induced subgraph of $Q^{+0-}(G)$.

Remark 2.2 If G is a block, then $Q^{+0-}(G) = L(G) \cup K_1$.

Remark 2.3 Let G be a graph with edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$ and r blocks. Then $d_{Q^{+0-}(G)}e_i' = d_G e_i + r - 1$.

Remark 2.4 Let G be a (p, q)-graph with block set $U(G) = \{B_1, B_2, \cdots, B_r\}$ such that $\mid E(B_i) \mid = n_i$. Then $d_{Q^{+0-}(G)}B_i' = q - n_i$.

Theorem 2.1 *Let G be a $(p, q)-$connected graph whose vertices have degree d_i with $r \geq 1$ blocks and b_i $(1 \leq i \leq p)$ the number of blocks to which vertex v_i belongs in G. Then*

(1) *The order of $Q^{+0-}(G)$ $= q - p + 1 + \sum\limits_{i=1}^{p} b_i$;*

(2) *The size of $Q^{+0-}(G)$ $= q(r - 2) + \frac{1}{2}\sum\limits_{i=1}^{p} d_i^2$.*

Proof It is shown in [5] that for a connected graph G with p vertices and b_i number of blocks to which vertex v_i $(1 \leq i \leq p)$ belongs in G. Then the number of blocks of G is given by $b(G) = 1 + \sum\limits_{i=1}^{p}(b_i - 1)$. The order of $Q^{+0-}(G)$ is the sum of the number of edges of G and number of blocks of G. Hence the order of $Q^{+0-}(G)$

$$= \quad q+1+\sum_{i=1}^{p}(b_i-1) = q-p+1+\sum_{i=1}^{p} b_i.$$

The total number of edges formed by joining each of the r block-vertices to all the q edge-vertices is rq. The number of edges in line graph $L(G)$ is $-q+\frac{1}{2}\sum_{i=1}^{p} d_i^2$. Thus, the size of

$$Q^{+0-}(G) = rq-q-q+\frac{1}{2}\sum_{i=1}^{p} d_i^2 = q(r-2)+\frac{1}{2}\sum_{i=1}^{p} d_i^2. \qquad \square$$

An immediate consequence of the above theorem is the following corollary.

Corollary 2.2 *Let G be a (p,q) graph whose vertices have degree d_i with r blocks and m components. If b_i $(1 \le i \le p)$ is the number of blocks to which vertex v_i belongs in G, then*

(1) *The order of* $Q^{+0-}(G) = q-p+m+\sum_{i=1}^{p} b_i$;

(2) *The size of* $Q^{+0-}(G) = q(r-2)+\frac{1}{2}\sum_{i=1}^{p} d_i^2$.

Theorem 2.3 *Let G be a graph. The graphs $Q^{+0-}(G)$ and $P(G)$ are isomorphic if and only if G has two blocks.*

Proof Let G be a (p,q) graph with $r \ge 1$ blocks. Suppose $Q^{+0-}(G) = P(G)$. Then $|E(Q^{+0-}(G))| = |E(P(G))|$. By Theorems 1.1 and 2.1, we have

$$\begin{aligned} q(r-2)+\frac{1}{2}\sum_{i=1}^{p} d_i^2 &= \frac{1}{2}\sum_{i=1}^{p} d_i^2 \\ q(r-2) &= 0. \end{aligned}$$

Since G has at least one edge and hence equality holds only when $r=2$. Therefore G has two blocks.

Conversely, suppose G has two blocks B_1 and B_2. Then by definitions of $Q^{+0-}(G)$ and $P(G)$, $L(G)$ is induced subgraph of $Q^{+0-}(G)$ and $P(G)$. In $Q^{+0-}(G)$, block-vertex B_1' is adjacent all the edge-vertices corresponding to edges of B_2 and block-vertex B_2' is adjacent to all the edge-vertices corresponding to edges of B_1. In $P(G)$, block-vertex B_1' is adjacent all the edge-vertices corresponding to edges of B_1 and block-vertex B_2' is adjacent to all the edge-vertices corresponding to edges of B_2. This implies that there exist a one-to-one correspondence between vertices of $Q^{+0-}(G)$ and $P(G)$ which preserves adjacency. Therefore the graphs $Q^{+0-}(G)$ and $P(G)$ are isomorphic. $\square$

The following theorem gives the girth of $Q^{+0-}(G)$.

Theorem 2.4 *For a graph $G \neq 2K_2, K_2, P_3$,*

$$g(Q^{+0-}(G)) = \begin{cases} 3 \ if \ G \ contains \ K_{1,3} \ or \ K_3 \ or \ G = P_n; \ n \geq 4 \ or \ G \ is \ union \ of \ at \ least \\ two \ cycles \ or \ paths \ or \ G \ is \ union \ of \ paths \ and \ cycles, \\ 4 \ if \ G = mK_2, m \geq 4, \\ 6 \ if \ G = 3K_2, \\ n \ if \ G = C_n, n \geq 4. \end{cases}$$

Proof If G contains a triangle or $K_{1,3}$, then the line graph $L(G)$ of G contains triangle. By Remark 2.1, it follows that girth of $Q^{+0-}(G)$ is 3. Assume that G is triangle free and $K_{1,3}$ free. Then we have the following cases:

Case 1. Assume G has every vertex of degree is 2. We have two subcases:

Subcase 1.1 If G is connected, then clearly $G = C_n$; $n \geq 4$, we have $Q^{+0-}(G) = C_n \bigcup K_1$, $n \geq 4$. Therefore girth of $Q^{+0-}(G)$ is n.

Subcase 1.2 If G is disconnected, then G is union of at least two cycles and $Q^{+0-}(G)$ contains at least two wheels. Therefore girth of $Q^{+0-}(G)$ is 3.

Case 2. Assume that $G \neq 2K_2, K_2$ has every vertex of degree is one. It is easy to see that

$$g(Q^{+0-}(G)) = \begin{cases} 6 \text{ if } G = 3K_2, \\ 4 \text{ if } G = mK_2; \ m \geq 4. \end{cases}$$

Case 3. Assume that $G \neq P_3$ has vertices of degree one or two. Then G is either union of paths P_n or union of paths and cycles. Therefore girth of $Q^{+0-}(G)$ is 3. □

§3. Covering Invariants of $Q^{+0-}(G)$

Theorem 3.1 *For a connected (p, q)-graph G with r blocks, if $Q^{+0-}(G)$ is connected, then $\alpha_0(Q^{+0-}(G)) = q$ and $\beta_0(Q^{+0-}(G)) = r$.*

Proof Let G be a connected (p, q)-graph. By Remark 2.1, $L(G)$ is an induced subgraph of $Q^{+0-}(G)$. Therefore by definition of $Q^{+0-}(G)$, the edge-vertices covers all the edges of $L(G)$. Since $Q^{+0-}(G)$ is connected, it follows that for each block-vertex B' of $Q^{+0-}(G)$, there exists a edge-vertex e' such that e' and B are adjacent in $Q^{+0-}(G)$. Therefore the vertex set of $L(G)$ covers all the edges of $Q^{+0-}(G)$ and this is minimum covering. Hence $\alpha_0(Q^{+0-}(G)) = q$. Since $Q^{+0-}(G)$ is connected. By Theorem 1.2, we have $\alpha_0(Q^{+0-}(G)) + \beta_0(Q^{+0-}(G)) = q + r$. Thus $\beta_0(Q^{+0-}(G)) = r$. □

Theorem 3.2 *Let G be a connected (p, q)-graph with r blocks. If $Q^{+0-}(G)$ is connected, then*

$$\alpha_1(Q^{+0-}(G)) = \begin{cases} r & if \ G \ is \ a \ tree, \\ r + \lceil \frac{q-r}{2} \rceil & otherwise. \end{cases}$$

and

$$\beta_1(Q^{+0-}(G))= \begin{cases} q & if\ G\ is\ a\ tree, \\ q-\lceil \frac{q-r}{2} \rceil & otherwise. \end{cases}$$

Proof Let T be the set of minimum edges covering all block-vertices of $Q^{+0-}(G)$. i.e., $|T| = r$. Let S be the set of minimum edge cover of $L(G)$. By Theorem 1.3, $|S| = \lceil \frac{q}{2} \rceil$. We consider the following two cases:

Case 1. If G is a tree, then $q = r$. By the definition, T covers all block-vertices and edge-vertices of $Q^{+0-}(G)$. Thus $\alpha_1(Q^{+0-}(G)) = r$.

Case 2. If G is not a tree, then $q > r$. By the definition, T covers all block-vertices and only r edge-vertices of $Q^{+0-}(G)$. Therefore there exists a set of edge-vertices F, say of $Q^{+0-}(G)$ such that no element of T is incident with any element of F in $Q^{+0-}(G)$. i.e., $|F| = q - r$. Since each element of S covers two elements of $L(G)$ and $F \subset V(Q^{+0-}(G))$, it follows that we need $\left\lceil \frac{|F|}{2} \right\rceil$ elements from S to cover all elements of F. Thus $\alpha_1(Q^{+0-}(G))= r + \lceil \frac{q-r}{2} \rceil$.

Therefore, $\alpha_1(Q^{+0-}(G))= \begin{cases} r & \text{if } G \text{ is a tree,} \\ r+\lceil \frac{q-r}{2} \rceil & \text{otherwise.} \end{cases}$

Since $Q^{+0-}(G)$ is connected. By Theorem 1.2, we have $\alpha_1(Q^{+0-}(G)) + \beta_1(Q^{+0-}(G)) = q + r$. Thus

$$\beta_1(Q^{+0-}(G))= \begin{cases} q & \text{if } G \text{ is a tree,} \\ q-\lceil \frac{q-r}{2} \rceil & \text{otherwise.} \end{cases}$$ □

§4. Domination Number of $Q^{+0-}(G)$

A set D of vertices in a graph $G = (V, E)$ is called a *dominating set* of G if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D is called *minimal dominating set* if no proper subset of D is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G ([10]).

The following result is immediate from Remark 2.2.

Theorem 4.1 *If G is a block, then $\gamma(Q^{+0-}(G)) = \gamma(L(G)) + 1$.*

Theorem 4.2 *If G has two blocks, then $\gamma(Q^{+0-}(G)) = 2$.*

Proof Suppose G has two blocks B_1 and B_2. Then B_1' dominates all the edge-vertices in $Q^{+0-}(G)$ corresponding to edges of B_2 and B_2' dominates all the edge-vertices in $Q^{+0-}(G)$ corresponding to edges of B_1. Therefore $\gamma(Q^{+0-}(G))= |\{B_1', B_2'\}| = 2$ where $\{B_1', B_2'\}$ is a minimal dominating set in $Q^{+0-}(G)$. □

Theorem 4.3 *For any graph G with at least three blocks,*

$$\gamma(Q^{+0-}(G))=\begin{cases} 2 & if\ G\ contain\ an\ edge\ is\ adjacent\ to\ every\ other\ edge\ of\ its\ block, \\ 3 & otherwise. \end{cases}$$

Proof Let G be a graph having at least three blocks. We consider following two cases:

Case 1. If G contain an edge e is adjacent to every other edge of its block B, then block-vertex B' dominates the edge-vertices corresponding to the edges not in B. And edge-vertex e' dominates the block-vertices except B' and dominates the edge-vertices corresponding to edges of B. Therefore $\gamma(Q^{+0-}(G)) = |\{e', B'\}| = 2$ where $\{e', B'\}$ is a minimal dominating set in $Q^{+0-}(G)$.

Case 2. If G contain no edge is adjacent to every other edge of its block, then there exist two block-vertices B', B'_1 and one edge-vertex e', where e is in B in G, such that B' dominates the edge-vertices corresponding to the edges not in B and edge-vertex e' dominates all the block-vertices except B' and block vertex B'_1 dominates the edge-vertices which are not dominated from e' and B'. Therefore $\gamma(Q^{+0-}(G))= |\{e', B', B'_1\}| = 3$ where $\{e', B', B'_1\}$ is a minimal dominating set in $Q^{+0-}(G)$. □

§5. Planarity of Graphs $Q^{+0-}(G)$

A graph is *planar* if it can be drawn on the plane in such a way that no two of its edges intersect. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the exterior region. In [1], Kulli introduced the concept of a minimally nonouterplanar graph. The *inner vertex number* $i(G)$ of a planar graph G is the minimum possible number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. Obviously G is outerplanar if and only if $i(G) = 0$. A graph G is *minimally nonouterplanar* if $i(G) = 1$. An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity. The *crossing number* $Cr(G)$ of a graph G is the minimum number of pairwise intersections of its edges when G is drawn in the plane. Obviously, $Cr(G) = 0$ if and only if G is planar. A *cactus* is a connected graph in which every block is an edge or a cycle. If G and H are graphs with the property that the identification of any vertex of G with an arbitrary vertex of H results in a unique graph, then we write $G \cdot H$ for this graph.

The condition for the planar, outerplanar, minimally nonouterplanar, maximal outerplanar and crossing number of line graph of G and generalized abc-block edge transformation graph $Q^{+0-}(G)$ are same when G is a block. So that in this section we assume graph G under consideration is not a block in what follows.

Lemma 5.1 *If G is not a tree having more than two blocks, then $Q^{+0-}(G)$ is nonplanar.*

Proof Let G be not a tree having more than two blocks, i.e., G has a block B contains a cycle C. Then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(2K_2 \cup K_3)$, and $Cr(Q^{+0-}(2K_2 \cup K_3)) = 1$. Therefore $Q^{+0-}(G)$ is nonplanar. □

Theorem 5.2 *Let G be a connected graph with more than one block. Then generalized abc-*

block edge transformation graph $Q^{+0-}(G)$ is planar if and only if G satisfies one of the following conditions:

(1) *G is a cactus having two blocks;*

(2) *G is a tree of order ≤ 5.*

Proof Suppose $Q^{+0-}(G)$ is planar. Assume a connected graph G has atleast 5 blocks. We consider the following cases:

Case 1. If G is not a tree, then by Lemma 5.1, $Q^{+0-}(G)$ is nonplanar, a contradiction.

Case 2. If G is a tree, i.e., every block of G is K_2, then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(5K_2)$ and $Cr(Q^{+0-}(5K_2)) = 4$. Therefore $Q^{+0-}(G)$ is nonplanar, a contradiction.

In either case we arrive at a contradiction. Hence G contains at most four blocks. We discuss two possibilities on number of blocks:

Subcase 2.1 If G is not a cactus having two blocks, i.e., some block B of G contains a subgraph homeomorphic to $C_n + e$, then edge-vertices corresponding to edges of $C_n + e$ and block-vertex corresponding to block other than B forms a subgraph with at least one crossing in $Q^{+0-}(G)$. Therefore $Q^{+0-}(G)$ is nonplanar, a contradiction. This proves (1).

Subcase 2.2 If G is not a tree having 3 or 4 blocks, then by Lemma 5.1, $Q^{+0-}(G)$ is nonplanar, a contradiction. This proves (2).

Conversely, suppose G satisfies (1) or (2). Then $G = C_n \cdot K_2$ or $C_n \cdot C_m$ or P_4 or $K_{1,3}$ or $K_{1,3} \cdot K_2$ or P_3 or P_5. Therefore it is easy to check that $Q^{+0-}(G)$ is planar. □

Theorem 5.3 *Let G be a connected graph with more than one block. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ is outerplanar if and only if G is a tree of order ≤ 4.*

Proof Suppose $Q^{+0-}(G)$ is outerplanar. Then $Q^{+0-}(G)$ is planar. By Theorem 5.2, we have, G is a cactus having two blocks or G is a tree of order ≤ 5. Assume G is a tree of order 5. Then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(4K_2)$ and $i(Q^{+0-}(4K_2)) = 4$. Therefore $Q^{+0-}(G)$ is nonouterplanar, a contradiction. Assume $G = C_m \cdot C_m$ or $C_n \cdot K_2$. Then $Q^{+0-}(G)$ is nonouterplanar, a contradiction. In either case we arrive at a contradiction. Hence G is a tree of order ≤ 4.

Assume G is not a tree of order ≤ 4, i.e., G has a block B contains a cycle C. Then edge-vertices corresponding to edges of C and a block-vertex corresponding to block other than B forms a subgraph wheel in $Q^{+0-}(G)$. Therefore $Q^{+0-}(G)$ is nonouterplanar, a contradiction. Hence G is a tree of order ≤ 4.

Conversely, suppose G is a tree of order ≤ 4. Then $G = P_3$ or P_4 or $K_{1,3}$. Therefore $Q^{+0-}(G)$ is outerplanar. □

Theorem 5.4 *Let G be a connected graph with more than one block. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ is minimally nonouterplanar if and only if $G = C_n \cdot K_2$.*

Proof Suppose $Q^{+0-}(G)$ is minimally nonouterplanar. Then $Q^{+0-}(G)$ is planar. By Theorem 5.2, we have, G is either cactus having two blocks or tree of order ≤ 5. If G is a tree of order ≤ 4, then by Theorem 5.3, $Q^{+0-}(G)$ is outerplanar, a contradiction. If G is a tree of order 5, then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(4K_2)$, and $i(Q^{+0-}(4K_2)) = 4$. Therefore $Q^{+0-}(G)$ is not minimally outerplanar, a contradiction.

Suppose $G \neq C_n \cdot K_2$ is cactus having two blocks. Then $G = P_3$ or $C_n \cdot C_m$. Therefore $Q^{+0-}(G)$ is not minimally nonouterplanar, a contradiction. Thus G is $C_n \cdot K_2$.

Conversely, suppose $G = C_n \cdot K_2$. Then $Q^{+0-}(G)$ is minimally nonouterplanar. □

Theorem 5.5 *Let G be a connected graph with more than one block. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ is maximal outerplanar if and only if $G = K_{1,3}$.*

Proof Suppose $Q^{+0-}(G)$ is maximal outerplanar. Then $Q^{+0-}(G)$ is outerplanar. By Theorem 5.3, we have, G is a tree of order ≤ 4. Assume $G \neq K_{1,3}$ is a tree of order ≤ 4. Then $G = P_3$ or P_4. Therefore $Q^{+0-}(G)$ is not maximal outerplanar, a contradiction. Hence $G = K_{1,3}$.

Conversely, suppose $G = K_{1,3}$. Then $Q^{+0-}(G)$ is maximal outerplanar. □

§6. Graphs $Q^{+0-}(G)$ and Crossing Number One

Lemma 6.1 *Let G be a connected graph having two blocks. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ has crossing number one if and only if G is either $C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$.*

Proof Suppose $Q^{+0-}(G)$ has crossing number one. Assume $G \neq C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$. Then we have the following cases:

Case 1. If G is a cactus, then by Theorem 5.2, $Q^{+0-}(G)$ is planar, a contradiction.

Case 2. If G is not a cactus, then G is homeomorphic to $K_2 \cdot (C_t + 2e)$ or $K_2 \cdot (\overline{K_2 \cup K_3})$ or $(C_t + e) \cdot (C_s + e)$ or $C_t \cdot (C_s + e)$ with $\triangle(G) = 5$. Therefore $Cr(K_2 \cdot (C_t + 2e)) \geq 2$, $Cr(K_2 \cdot (\overline{K_2 \cup K_3})) \geq 2$, $Cr((C_t + e) \cdot (C_s + e)) \geq 2$, $Cr(C_t \cdot (C_s + e)) = 2$. Hence $Cr(Q^{+0-}(G)) \geq 2$, a contradiction.

Conversely, suppose G is either $C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$. Then $Cr(Q^{+0-}(G)) = 1$. □

Theorem 6.2 *Let G be a connected graph with more than one block. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ has crossing number one if and only if $G = C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$ or $C_n \cdot P_3$.*

Proof Suppose $Q^{+0-}(G)$ has crossing number one. Assume $G \neq C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$ or $C_n \cdot P_3$. We consider the following cases:

Case 1. If G is a tree, then we consider following subcases:

Subcase 1.1 If G is a tree of order ≤ 5, then by Theorem 5.2, $Q^{+0-}(G)$ is planar, a contradiction.

Subcase 1.2 If G is a tree of order at least 6, then $Q^{+0-}(G)$ has a subgraph homoemorphic to $Q^{+0-}(5K_2)$ and $Cr(Q^{+0-}(5K_2)) = 4$. Therefore $Cr(Q^{+0-}(G)) \geq 4$, a contradiction.

Case 2. If G is not a tree, then G contains at least one cycle. We consider the following subcases:

Subcase 2.1 If G has more than 3 blocks, then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(3K_2 \cup K_3)$ and $Cr(Q^{+0-}(3K_2 \cup K_3)) = 5$. Therefore $Cr(Q^{+0-}(G)) \geq 5$, a contradiction.

Subcase 2.2 If G has three blocks, then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}((C_4+e) \cdot P_3)$ or G_1 where $G_1 = K_3^+ - e$, e is pendant edge, and $Cr(Q^{+0-}((C_4+e) \cdot P_3)) \geq 4$, $Cr(Q^{+0-}(G_1)) = 2$. Therefore $Cr(Q^{+0-}(G)) \geq 2$, a contradiction.

Subcase 2.3 If G has two blocks, then by Lemma 6.1, crossing number of $Q^{+0-}(G)$ is not equal to one, a contradiction.

Conversely, suppose G=$C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$ or $C_n \cdot P_3$. Then $Q^{+0-}(G)$ has crossing number one. □

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