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Energy, Wiener index and Line Graph of Prime Graph of a Ring

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Abstract: Let $\mathbb{Z}_n$ be the commutative ring of residue classes modulo n, $PG(\mathbb{Z}_n)$ be the prime graph of a ring over a ring $\mathbb{Z}_n$. In this paper we study Energy and Wiener index of $PG(\mathbb{Z}_n)$ and give some results of line graph of prime graph of a ring over a ring $\mathbb{Z}_n$, denote it by $L(PG(\mathbb{Z}_n))$.

Key Words: Prime graph of a ring $PG(R)$, line graph, energy, Wiener index.

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§1. Introduction

Prime graph of a ring first introduced by Satyanarayana et al. [3]. Prime graph of a ring is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. The concept of energy and Wiener index of zero divisor graph was introduced by Mohammad Reza and Reza Jahani in [4]. Motivated from the article in [4] in Section 2 of this paper we discuss energy of prime graph of a ring and give general MATLAB code for our calculation. In section 3, We calculate Wiener index of $PG(\mathbb{Z}_n)$, for $n = p$, $n = p^2$ and $n = p^3$. In last section of paper, we introduce Line Graph of Prime Graph of a Ring denoted by $L(PG(\mathbb{Z}_n))$ and discuss Planerity, Girth and degree of all vertices in $L(PG(\mathbb{Z}_n))$. Also, we find center, eccentricity, point covering number, independence number, Energy, Wiener index and Chromatic number of $L(PG(\mathbb{Z}_n))$, where $n = p$, p prime. Here, we also discuss complement of line graph of prime graph of a ring over a ring $\mathbb{Z}_n$, denote it by $L(PG(\mathbb{Z}_n))^c$. We study Girth of $L(PG(\mathbb{Z}_n))^c$ and also find Eulerianity and degree of all vertices in $L(PG(\mathbb{Z}_n))^c$, where $n = p$, p prime.

For more preliminary definitions and Notations the reader is referred to [5]-[8].

§2. Energy of Prime Graph of a Ring

In this section we give some examples and calculate the Energy of prime graph of a ring.

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Definition 2.1 *The energy of the prime graph of a ring $PG(\mathbb{Z}_n)$ is defined as the sum of the absolute values of all the eigen values of its adjacency matrix $M(PG[R])$. i.e. if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are n eigen values of $M(PG[R])$, then the energy of $PG(\mathbb{Z}_n)$ is -*

$$E(PG[R]) = \sum_{i=1}^{n} |\lambda_i| .$$

Example 2.2 For $p = 2$, the adjacency matrix of $PG(\mathbb{Z}_2)$ is

$$M(PG[\mathbb{Z}_2]) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^2 - 1$.The eigen values are $\lambda_1 = 1, \lambda_2 = -1$. Therefore, $E(PG[\mathbb{Z}_2]) = 2$.

Example 2.3 For $p = 3$, the adjancency matrix of $PG(\mathbb{Z}_3)$ is

$$M(PG[\mathbb{Z}_3]) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^3 - 2\lambda$. The eigen values are $\lambda_1 = -1.4142, \lambda_2 = 1.4142, \lambda_3 = 0$. Therefore, $E(PG[\mathbb{Z}_3]) = 2.8284$.

Example 2.4 For $p = 4$, the adjancency matrix of $PG(\mathbb{Z}_4)$ is

$$M(PG[\mathbb{Z}_4]) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^4 - 3\lambda^2$. The eigen values are $\lambda_1 = 1.7321, \lambda_2 = -1.7321, \lambda_3 = 0, \lambda_4 = 0$. Therefore, $E(PG[\mathbb{Z}_4]) = 3.4641$.

Example 2.5 For $p = 5$, the adjancency matrix of $PG(\mathbb{Z}_5)$ is

$$M(PG[\mathbb{Z}_5]) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^5 - 4\lambda^3$. The eigen values are $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 =$

$0, \lambda_4 = 0,\ \lambda_5 = 0$. Therefore, $E(PG[\mathbb{Z}_5]) = 4$.

From the above Discussion we conclude the following theorem.

Theorem 2.6 *If p is a prime number then energy of $PG(\mathbb{Z}_p)$ is $2\sqrt{p-1}$.*

General MATLAB code to find Energy of a Graph

syms λ To create Symbolic Variables;
$A = [\cdots;\cdots;\cdots;\cdots]$ To create a matrix that has multiple rows, separate the rows with semicolons;
$charpoly(A, \lambda)$ Returns the characteristic polynomial of A in terms of variable λ;
$p = [\quad]$ To input the coefficients of characteristic polynomial;
$r = roots(p)$ Gives the eigen Values of matrix A;
$s = sum(abs(r))$ Gives the energy of a graph.

The values of $E(PG[\mathbb{Z}_n])$ for $n = 2, 3, 4, 5, 6, 9$ and 10 are given in table below.

Sr.No.	n	Characteristic Polynomial	Energy
1	2	$\lambda^2 - 1$	2
2	3	$\lambda^3 - 2\lambda$	2.8284
3	4	$\lambda^4 - 3\lambda^2$	3.4641
4	5	$\lambda^5 - 4\lambda^3$	4
5	6	$\lambda^6 - 7\lambda^4 - 4\lambda^3 + 4\lambda^2$	6.6858
6	9	$\lambda^9 - 9\lambda^7 - 2\lambda^6 + 6\lambda^5$	7.4641
7	10	$\lambda^{10} - 13\lambda^8 - 8\lambda^7 + 16\lambda^6$	9.2058

§3. Wiener Index of Prime Graph of a Ring

In this section, We calculate Wiener index of $PG(\mathbb{Z}_n)$, for $n = p$, $n = p^2$ and $n = p^3$.

Definition 3.1 *Let $PG(R)$ be a Prime Graph of a Ring with vertex set V. We denote the length of the shortest path between every pair of vertices $x, y \in V$ with $d(x, y)$. Then the Wiener index of $PG(R)$ is the sum of the distances between all pair of vertices of $PG(R)$, i.e.*

$$W(PG[R]) = \sum_{x,y \in V} d(x, y).$$

The following results can be easily verified.

Theorem 3.2 $W(PG[\mathbb{Z}_p]) = (p-1)^2$ *if p is a prime.*

Theorem 3.3 $W(PG[\mathbb{Z}_{p^2}]) = \frac{p\cdot(p-1)}{2} \cdot \left[2p^2 - 2p + 1\right]$ *if p is a prime.*

Theorem 3.4 $W(PG[\mathbb{Z}_{p^3}]) = \frac{p\cdot(p-1)}{2} \left[2p^4 + 2p^3 - 2p - 3\right]$ *if p is a prime.*

§4. Line Graph of Prime Graph of a Ring

In this section we define line graph of prime graph of a ring, presented some examples and give some results.

Definition 4.1 *The line graph $L(PG(\mathbb{Z}_n))$ of the graph $PG(\mathbb{Z}_n)$ is defined to the graph whose set of vertices constitutes of the edges of $PG(\mathbb{Z}_n)$, where two vertices are adjacent if the corresponding edges have a common vertex in $PG(\mathbb{Z}_n)$.*

Consider $\mathbb{Z}_n$, the ring of integers modulo n.

Example 4.2 $L(PG(\mathbb{Z}_2))$ is a single vertex graph, there is no edge in $L(PG(\mathbb{Z}_2))$.

Example 4.3 In $L(PG(\mathbb{Z}_3))$, there is an edge between the vertices [0,1] to [0,2], as shown in figure below.

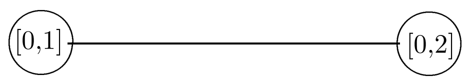

Figure 1

Example 4.4 In $L(PG(\mathbb{Z}_4))$, there is an edge between the vertices [0,1] to [0,2], [0,2] to [0,3] and [0,3] to [0,1] as shown in figure below.

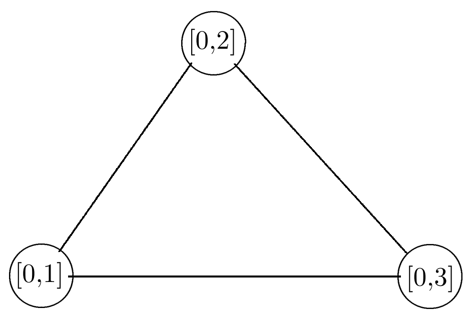

Figure 2

i.e. $L(PG(\mathbb{Z}_4))$ is a complete graph k_3.

Example 4.5 In $L(PG(\mathbb{Z}_5))$, there is an edge between the vertices [0,1] to [0,2], [0,2] to [0,3], [0,3] to [0,4], [0,4] to [0,1], [0,1] to [0,3] and [0,2] to [0,4] as shown in figure below.

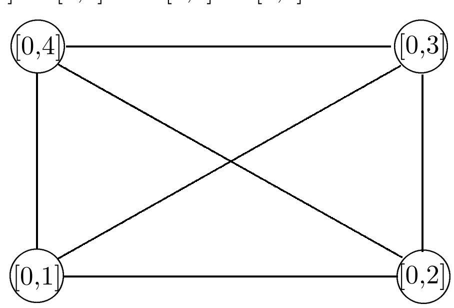

Figure 3

i.e. $L(PG(\mathbb{Z}_5))$ is a complete graph k_4.

Example 4.6 Let us construct $L(PG(\mathbb{Z}_6))$.

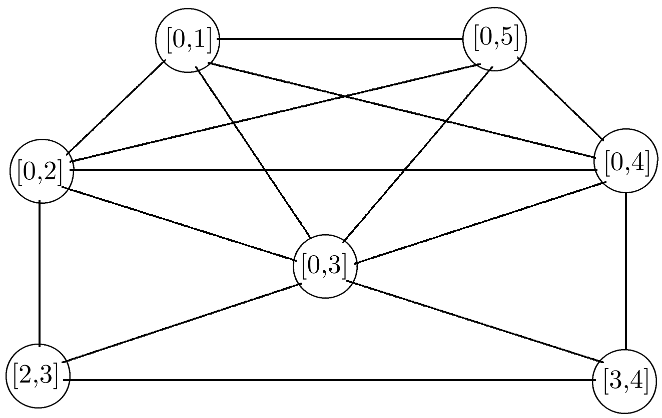

Figure 4

i.e. $L(PG(\mathbb{Z}_6))$ contains a complete subgraph k_5.

Example 4.7 Let us construct $L(PG(\mathbb{Z}_7))$.

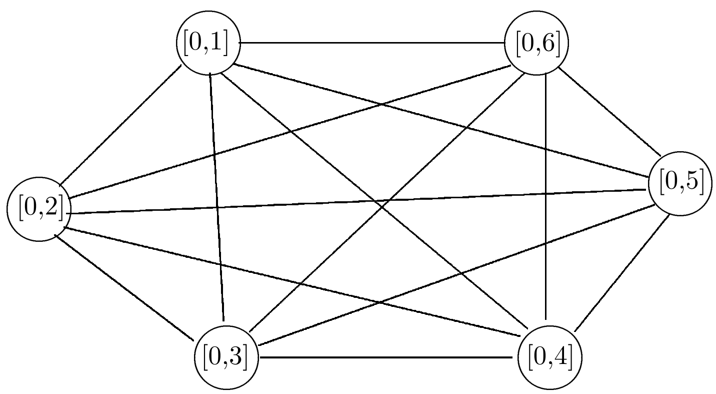

Figure 5

i.e. $L(PG(\mathbb{Z}_7))$ is a complete graph k_6.

Observations 4.8 Every $L(PG(\mathbb{Z}_n))$ contains a complete subgraph on $n-1$ vertices.

Observations 4.9 If $\mathbb{Z}_n$ is a prime ring then $L(PG(\mathbb{Z}_n))$ is a regular graph.

Observations 4.10 If $n = p$, a prime number then $PG(\mathbb{Z}_n)$ is a star graph. So, its line graph $L(PG(\mathbb{Z}_n))$ is a complete graph and hence its eccentricity $e(v) = 1, \forall v \in V(L(PG(\mathbb{Z}_n)))$. Therefore, centre is $L(PG(\mathbb{Z}_n))$.

Theorem 4.11 *The graph $L(PG(\mathbb{Z}_n))$ is Hamiltonian if and only if $n = p$, a prime number and $n \geq 4$.*

Proof When $n = 2$, $L(PG(\mathbb{Z}_n))$ is a single vertex graph, hence there is no cycle. For $n = 3$, $L(PG(\mathbb{Z}_n))$ is a single edge graph, hence there is no cycle exist. For $n = 4$, $L(PG(\mathbb{Z}_n))$ is a triangle graph and there exist a cycle which containing every vertex. So, $L(PG(\mathbb{Z}_4))$ is a

Hamiltonian graph. Now, for $n = p$, a prime number then $L(PG(\mathbb{Z}_n))$ is Hamiltonian graph because there exist a cycle containing every vertex. Hence, the graph $L(PG(\mathbb{Z}_n))$ is Hamiltonian if and only if $n = p$, a prime number and $n \geq 4$. □

Theorem 4.12 *Let $L(PG(\mathbb{Z}_n))$ be a line graph of prime graph of a ring, where $n = p$ and p is an odd prime number then point covering number and independence number of $L(PG(\mathbb{Z}_n))$ both are one.*

Proof When $n = p$, $PG(\mathbb{Z}_n)$ is a star graph. So, there is a common vertex which is adjacent to all other vertices and that vertex is called center of the graph. When we draw the line graph of $PG(\mathbb{Z}_n)$, for $n = p$, and let $a_1 = 0$ be the common vertex of $PG(\mathbb{Z}_n)$ which is the end point of every edge of $PG(\mathbb{Z}_n)$. Then a_1 appears in every vertex of the line graph. $[a_1, v_i] \in V(L(PG(\mathbb{Z}_n)))$, where $i = 1, 2, 3, \cdots, (p-1)$ forms a complete line graph of $PG(\mathbb{Z}_n)$ and here, $[a_1, v_1]$ is adjacent with all other vertices of line graph. In other words, we can say that single vertex cover all other vertices of line graph of $PG(\mathbb{Z}_n)$. Thus, the point cover is one and from that vertex an independence number is also one. □

The following results can be immediately verified.

Theorem 4.13 *The general formula for degree of vertex in $L(PG(\mathbb{Z}_n))$ is:*

$$\begin{aligned} deg[u,v] &= gcd(u,n) + gcd(v,n) - 2, && \text{if } u^2 \neq 0 \text{ and } v^2 \neq 0 \\ &= gcd(u,n) + gcd(v,n) - 3, && \text{if either } u^2 = 0\ ,\ v^2 = 0 \\ &= gcd(u,n) + gcd(v,n) - 4, && \text{if } u^2 = 0 \text{ and } v^2 = 0 \end{aligned}$$

Theorem 4.14 *$L(PG(\mathbb{Z}_n))$ is planer if and only if $n = 2, 3, 4, 5$ and is non-planer for $n \geq 6$.*

Theorem 4.15 *The girth $gr(L(PG(\mathbb{Z}_n))) = 3$ if and only if $n \geq 4$. If $n = 2, 3$ then $gr(L(PG(\mathbb{Z}_n))) = \infty$.*

Theorem 4.16 *The chromatic number $\chi(L(PG(\mathbb{Z}_p))) = p - 1$ for $p = 2, 3, 5, \cdots$.*

Theorem 4.17 *The chromatic number $\chi(L(PG(\mathbb{Z}_{p^n}))) = p^n - 1$, p prime.*

Theorem 4.18 *The energy $E(L(PG(\mathbb{Z}_p))) = 2p - 4$, for $p = 3, 5, \cdots$ and $n = 4$.*

Theorem 4.19 *The Wiener index $W(L(PG(\mathbb{Z}_p))) = \frac{p(p-1)}{2}$, for $p = 3, 5, \cdots$ and $n = 4$.*

Theorem 4.20 *The graph $L(PG(\mathbb{Z}_n))^c$ is Eulerian if and only if $n = p$, a prime number and $n \geq 4$.*

Proof When $n = 2$, there is no graph, as there is no edge between the vertices 0 and 1 in $(PG(\mathbb{Z}_n))^c$. For $n = 3$, $L(PG(\mathbb{Z}_n))^c$is a single vertex graph. For $n = 4$, $L(PG(\mathbb{Z}_n))^c$ is triangle graph and every vertex is of even degree. Now, For $n = p$, a prime number, every vertex of $L(PG(\mathbb{Z}_n))^c$ have even degree. Hence, the graph $L(PG(\mathbb{Z}_n))^c$ is Eulerian if and only if $n = p$, a prime number and $n \geq 4$. □

Theorem 4.21 *The general formula for degree of vertex in* $L(PG(\mathbb{Z}_n))^c$, *where* $n = p$ *a prime number and* $n \geq 5$ *is:*

$$deg[u, v] = n + \phi(n) - 5$$

Theorem 4.22 *The girth* $gr(L(PG(\mathbb{Z}_n))^c) = 3$ *if and only if* $n \geq 4$. *If* $n = 2, 3$ *then* $gr(L(PG(\mathbb{Z}_n))^c) = \infty$.

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