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Domination Stable Graphs

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Abstract: In this paper, we study the domination polynomials of some graph and its square. We discuss nonzero real domination roots of these graphs. We also investigate whether all the domination roots of some graphs lying left half plane or not.

Key Words: Dominating set, Smarandachely k-dominating set, domination number, domination polynomial, domination root, d-number, stable.

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§1. Introduction

Let $G(V,E)$ be a simple finite graph. The order of G is the number of vertices of G. A set $S \subseteq V$ is a dominating set if every vertex $v \in V - S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of the dominating sets in G. Generally, a dominating set S is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S. Let $\mathcal{D}(G,i)$ be the family of dominating sets of G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. The polynomial

$$D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$$

is defined as domination polynomial of G. For more information on this polynomial the reader may refer to [8]. A root of $D(G,x)$ is called a domination root of G. It is easy to see that the domination polynomial is monic with no constant term. Consequently, 0 is a root of every domination polynomial (in fact, 0 is a root whose multiplicity is the domination number of the graph).

§2. d-Number

In this section we mainly focus on the number of real domination roots of some specific graphs. So we introduce a new definition as follows.

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Definition 2.1 *Let G be a graph. The number of distinct real domination roots of the graph G is called $\mathtt{d}$-number of G and is denoted by $\mathtt{d}(G)$.*

Theorem 2.1 *For any graph G, $\mathtt{d}(G) \geq 1$.*

Proof It follows from the fact that 0 is a domination root of any graph. □

Theorem 2.1 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$\mathtt{d}(G) \leq \sum_{i=1}^{m} \mathtt{d}(G_i) - m + 1.$$

Proof It follows from the fact that $D(G,x) = \prod_{i=1}^{m} D(G_i,x)$. □

Theorem 2.3 *If G and H are isomorphic, then $\mathtt{d}(G) = \mathtt{d}(H)$.*

Proof It follows from the fact that if G and H are isomorphic, then $D(G,x) = D(H,x)$. □

Theorem 2.4 *If G has exactly two distinct domination roots, then $\mathtt{d}(G) = 2$.*

Proof It follows from the fact that 0 is a domination root and complex roots occurs in conjugate pairs. □

Theorem 2.5 *Let G be a graph without pendent vertices. If G has exactly three distinct domination roots, then $\mathtt{d}(G) = 1$.*

Proof It follows from the fact that with the given conditions in theorem, $\mathbb{Z}(D(G,x)) \subseteq \{0, -2 \pm i\sqrt{2}, \frac{-3 \pm i\sqrt{3}}{2}\}$ ([8]). □

Theorem 2.6 *For all n we have the following :*

$$\mathtt{d}(K_n) = \begin{cases} 1 & ; \text{ if } n \text{ is odd,} \\ 2 & ; \text{ if } n \text{ is even.} \end{cases}$$

Proof We have known the domination polynomial of K_n is

$$D(K_n, x) = (1+x)^n - 1. \tag{1}$$

The result follows from the transformation $y = 1 + x$ in equation (1). □

Theorem 2.7 *For any graph G, $\mathtt{d}(G \circ K_1) = 2$.*

Proof Notice that $D(G \circ K_1, x) = x^n (x+2)^n$ ([8]), where n is the order of G. Therefore $\mathtt{d}(G \circ K_1) = 2$. □

Theorem 2.8 *For any graph G, $\mathtt{d}(G \circ \overline{K_2}) = 3$.*

Proof Notice that $D(G \circ \overline{K_2}, x) = x^{\frac{n}{3}}(x^2+3x+1)^{\frac{n}{3}}$ ([8]), where n is the order of G. Therefore $\mathbb{Z}(D(G,x)) = \{0, \frac{-3\pm\sqrt{5}}{2}\}$. This implies that $\mathtt{d}(G \circ \overline{K_2}) = 3$. □

Theorem 2.9 *For all n the $\mathtt{d}$-number of the star graph S_n is*

$$\mathtt{d}(S_n) = \begin{cases} 2 & ;\ \text{if } n \text{ is odd,} \\ 3 & ;\ \text{if } n \text{ is even.} \end{cases}$$

Proof We have known the domination polynomial of S_n is

$$D(S_n, x) = x(1+x)^n + x^n. \tag{2}$$

Therefore it suffices to prove that $f(x) = (1+x)^n + x^{n-1}$ has exactly one real root if n is odd and two real roots if n is even. But the number of real roots of $f(x)$ is equal to the number of real roots of $g(x) = \left(1+\frac{1}{x}\right)^n + \frac{1}{x}$. Again the number of real roots of $g(x)$ is equal to the number of real roots of $g(\frac{1}{x}) = (1+x)^n + x$. Consider $g(\frac{1}{y-1}) = y^n + y - 1$, we find the number of real roots of $h(y) = y^n + y - 1$. We have $h(0) = -1 < 0$ and $h(1) = 1 > 0$. Therefore by the intermediate value theorem, $h(y)$ has at least one real root in $(0,1)$. Also by De Gua's rule [11] for imaginary roots, there are at least $n-1$ complex roots for odd n and there are at least $n-2$ complex roots for even n. Therefore we can conclude that $h(y)$ has exactly one real root for odd n and two real roots for even n. It remains to show that all the real roots of $f(x)$ are distinct. Suppose $a \in \mathbb{R}$ is a double root of $f(x)$. Whence,

$$(1+a)^n + a^{n-1} = 0, \tag{3}$$

$$n(1+a)^{n-1} + (n-1)a^{n-2} = 0. \tag{4}$$

From equation (3) we get

$$(1+a)^{n-1} = -\frac{a^{n-1}}{1+a} \quad (\text{ since } a \neq -1). \tag{5}$$

Putting the value of $(1+a)^{n-1}$ in (4) and simplify, we obtain $a = n-1$. Which is a contradiction since $a < 0$. □

Theorem 2.10 *For all n the $\mathtt{d}$-number of $K_{2n,2n}$ is 1.*

Proof Notice that the domination polynomial of $K_{2n,2n}$ is

$$D(K_{2n,2n}, x) = \left((1+x)^{2n} - 1\right)^2 + 2x^{2n}. \tag{6}$$

Suppose for $a \in \mathbb{R}$, $\left((1+a)^{2n} - 1\right)^2 + 2a^{2n} = 0$, then $\left((1+a)^{2n} - 1\right)^2 = -2a^{2n}$. But this is true only if $a = 0$, hence $\mathtt{d}(K_{2n,2n}) = 1$. □

Theorem 2.11 *The $\mathtt{d}$-number of $K_{2n+1,2n+1}$ is greater than or equal to 3 for all n.*

Proof We have known the domination polynomial of $K_{2n+1,2n+1}$ is

$$D(K_{2n+1,2n+1}, x) = \left((1+x)^{2n+1} - 1\right)^2 + 2x^{2n+1}. \tag{7}$$

It is easy to verify that

$$D\left(K_{2n+1,2n+1}, -\frac{1}{2}\right) = 1 + \frac{1}{2^{2n-1}}\left(\frac{1}{2^{2n+3}} - 1\right) > 0$$
$$D\left(K_{2n+1,2n+1}, -1\right) = -1 < 0$$
$$D\left(K_{2n+1,2n+1}, -2\right) = 2^2(1 - 2^{2n}) < 0$$
$$D\left(K_{2n+1,2n+1}, -3\right) = (2^{2n+1}+1)^2 - 2 \times 3^{2n+1} > 0$$

Therefore by the intermediate value theorem, $K_{2n+1,2n+1}$ has at least one real domination root in $(-1, -\frac{1}{2})$ and at least one in $(-3, -2)$, hence $\mathtt{d}(K_{2n+1,2n+1}) \geq 3$. □

The Dutch-Windmill graph G_3^n is the graph obtained by selecting one vertex in each of n triangles and identifying them.

Theorem 2.12 *For $n \geq 2$ the domination polynomial of the Dutch-Windmill graph G_3^n is*

$$D(G_3^n, x) = x(1+x)^{2n} + (2x + x^2)^n.$$

Proof Let v be the center vertex of G_3^n. It is clear that $\{v\}$ is the only dominating set of cardinality 1. Therefore $\gamma(G_3^n) = 1$ and $d(G_3^n, 1) = 1$. The number of ways of selecting dominating set of cardinality which containing the center is $\binom{2n}{i-1}$. Also there are 2^n dominating sets of cardinality n which does not contain the center vertex v. Similarly there are $\binom{n}{i}2^{n-i}$ ways to select a dominating set of cardinality $n+i$ which does not contain the center vertex v. Therefore $D(G_3^n, x) = x(1+x)^{2n} + (2x+x^2)^n$. □

Theorem 2.13 *For all n the $\mathtt{d}$-number of the Dutch windmill graph G_3^{2n+1} is 1.*

Proof We have known the domination polynomial of the Dutch windmill graph G_{2n+1}^3 is

$$D(G_3^{2n+1}, x) = x(1+x)^{4n+2} + (2x + x^2)^{2n+1}.$$

Suppose there is a number $a \in \mathbb{R}$ with $a \neq 0$ such that $a(1+a)^{4n+2} + (2a + a^2)^{2n+1} = 0$. Then we have $a < 0$ and by a simple calculation we have

$$a = -\left(1 - \frac{1}{(1+a)^2}\right). \tag{7}$$

Suppose $-2 < a < 0$, then the left side of the equation (7) is negative but the right side is positive, a contradiction. Now suppose $a \leq -2$. Then the left side of the equation (7) is less than or equal to -2 but the right side is greater than -1, a contradiction. Therefore there is no nonzero real domination root for G_3^{2n+1} and hence $\mathtt{d}(G_3^{2n+1}) = 1$. □

Theorem 2.14 *The* $\mathtt{d}$*-number of* G_3^{2n} *is greater than or equal to* 3 *for all* n*.*

Proof Notice that the domination polynomial of the Dutch windmill graph G_3^{2n} is

$$D(G_3^{2n}, x) = x(1+x)^{4n} + (2x+x^2)^{2n}.$$

It is easy to verify that $D(G_3^{2n}, -1) > 0$ and $D(G_3^{2n}, -2) < 0$. Also if a is a negative real number near to 0, then $D(G_3^{2n}, a) < 0$. Therefore by the intermediate value theorem, we have G_3^{2n} has a real domination root in $(-2,-1)$ and a real domination root in $(-1,0)$ and hence $\mathtt{d}(G_3^{2n}) \geq 3$. □

The lollipop graph $L_{n,1}$ is the graph obtained by joining a complete graph K_n to a path P_1, with a bridge.

Theorem 2.15 *For* $n \geq 2$ *the domination polynomial of the lollipop graph* $L_{n,1}$ *is*

$$D(L_{n,1}, x) = x\left((1+x)^n + (1+x)^{n-1} - 1\right).$$

Proof Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of the complete graph K_n and v be the path P_1 and let v is adjacent to v_1. Clearly, $\gamma(L_{n,1}) = 1$ and $d(L_{n,1}, 1) = 1$. For $2 \leq i \leq n-1$, the only non dominating sets of i vertices of $L_{n,1}$ are the subset of $\{v_2, v_3, \cdots, v_n\}$. Therefore $d(L_{n,1}, i) = \binom{n+1}{i} - \binom{n-1}{i}$. Also $d(L_{n,1}, n) = n+1$ and $d(L_{n,1}, n+1) = 1$. □

Theorem 2.16 *For all* $n \geq 2$ *the* $\mathtt{d}$*-number of the lollipop graph* $L_{n,1}$ *is*

$$\mathtt{d}(L_{n,1}) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

Proof By Theorem 2.15 it is enough to prove that $f(y) = y^n + y^{n-1} - 1$ has only one real root if n is odd and has exactly two real roots if n is even. By De Gua's rule for imaginary roots, there are at least $n-1$ complex roots if n is odd and there are at least $n-2$ complex roots if n is even. Now, $f(0) = -1 < 0$ and $f(1) = 2 > 0$ for all n and $f(-1) = -1 < 0$ and $f(-2) = 2^{n-1} - 1 > 0$ for all even n. Therefore by the intermediate value theorem, we have the result. □

The generalized barbell graph $B_{m,n,1}$ is the simple graph obtained by connecting two complete graphs K_m and K_n by a path P_1.

Theorem 2.17 *For* $m \leq n$*, the domination polynomial of generalized barbell graph* $B_{m,n,1}$ *is*

$$D(B_{m,n,1}, x) = \left[(1+x)^m - 1\right]\left[(1+x)^n - 1\right].$$

Proof Let $V = \{v_1, v_2, \cdots, v_m\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \neq j$ every vertices V are adjacent, every vertices U are adjacent and v_m and u_n is adjacent. There is no one element dominating set and $\{v_i, u_j\}$ is a dominating set of cardinality 2 of $B_{m,n,1}$. Therefore $\gamma(B_{m,n,1}) = 2$ and $d(B_{m,n,1}, 2) = mn$. Also observe that for $2 \leq i \leq 2n$, a subset S of vertices $B_{m,n,1}$ of cardinality i is not a dominating set if either $S \subset V$ or $S \subset U$. Therefore $d(B_{m,n,1}, i) = \binom{2n}{i} - \binom{n}{i} - \binom{m}{i}$; for $2 \leq i \leq m$, $d(B_{m,n,1}, i) = \binom{2n}{i} - \binom{n}{i}$; for $m+1 \leq i \leq n$ and $d(B_{m,n,1}, i) = \binom{2n}{i}$; for $n+1 \leq i \leq 2n$. This implies that $D(B_{m,n,1}, x) = [(1+x)^m - 1]\,[(1+x)^n - 1]$. □

Theorem 2.18 *For all m, n the* d*-number of the generalized barbell graph $B_{m,n,1}$ is*

$$\mathtt{d}(B_{m,n,1}) = \begin{cases} 1 & \text{if both } m \text{ and } n \text{ are odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof The result follows from the transformation $y = 1 + x$ in the domination polynomial of $B_{m,n,1}$. □

The $n-$ barbell graph $B_{n,1}$ is the simple graph obtained by connecting two copies of complete graph K_n by a bridge.

Corollary 2.19 *The domination polynomial of the n-barbell graph $B_{n,1}$ is*

$$D(B_{n,1}, x) = \left((1+x)^n - 1\right)^2.$$

Proof It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

Corollary 2.20 *For all n, the* d*-number of the n-barbell graph $B_{n,1}$ is*

$$\mathtt{d}(B_{n,1}) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

A bi-star graph $B_{(m,n)}$ is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendant edges in u and n pendant edges in v.

Theorem 2.21 *The domination polynomial of the bi-star graph $B_{(m,n)}$ is*

$$D(B_{(m,n)}, x) = x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m.$$

Proof Let $\{u, v\}$, $U = \{u_1, u_2, \cdots, u_n\}$ and $V = \{v_1, v_2, \cdots, v_m\}$ be the vertices of $B_{m,n}$ such that u and v are adjacent, every vertices U are adjacent to u and every vertices V are adjacent to v. Clearly there is no one element dominating set. The set $\{u, v\}$ is the only dominating set of cardinality 2 of $B_{m,n}$. Therefore $\gamma(B_{m,n}) = 2$ and $d(B_{m,n}, 2) = 1$. For $3 \leq i \leq m$, the $i-$element dominating set of $B_{m,n}$ must contain $\{u, v\}$, and the $i-2$ elements can have $\binom{m+n}{i-2}$ choice. For $m+1 \leq i \leq n$, there are $\binom{m+n}{i-2}$ $i-$element dominating set of $B_{m,n}$ containing $\{u, v\}$ and $\binom{n}{i-m-1}$ $i-$element dominating set of $B_{m,n}$ containing $V\bigcup\{u\}$. For $n+1 \leq i \leq m+n-1$, there are $\binom{m+n}{i-2}$ $i-$element dominating set of $B_{m,n}$ containing $\{u, v\}$, $\binom{n}{i-m-1}$ $i-$element dominating set of $B_{m,n}$ containing $V\bigcup\{u\}$ and $\binom{m}{i-n-1}$ $i-$element dominating set of $B_{m,n}$ containing $U\bigcup\{v\}$. For $i = m+n$, there are $\binom{m+n}{i-2}$ $(m+n)-$element dominating set of $B_{m,n}$ containing $\{u, v\}$, n $(m+n)-$element dominating set of $B_{m,n}$ containing $V\bigcup\{u\}$, m $(m+n)-$element dominating set of $B_{m,n}$ containing $U\bigcup\{v\}$ and one $(m+n)-$element dominating set of $B_{m,n}$ not containing $\{u, v\}$. Also $d(B_{m,n}, m+n+1) = m+n+2$ and $d(B_{m,n}, m+n+2) = 1$. That is,

$$d(B_{m,n}, i) = \begin{cases} 1 & \text{if } i = 2, m+n+2 \\ \binom{m+n}{i-2} & \text{if } 3 \leq i \leq m \\ \binom{m+n}{i-2} + \binom{n}{i-m-1} & \text{if } m+1 \leq i \leq n \\ \binom{m+n}{i-2} + \binom{n}{i-m-1} + \binom{m}{i-n-1} & \text{if } n+1 \leq i \leq m+n-1 \\ \binom{m+n}{i-2} + n + m + 1 & \text{if } i = m+n \\ m+n+2 & \text{if } i = m+n+1 \end{cases}.$$

Hence

$$D(B_{m,n}) = x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m. \qquad \square$$

Corollary 2.22 *The domination polynomial of the bi-star graph $B_{(n,n)}$ is*

$$D(B_{(n,n)}, x) = \left(x(1+x)^n + x^n\right)^2.$$

Theorem 2.23 *For the bi-star graph $B_{(m,n)}$, $m \neq n$ we have the following :*

$$\mathtt{d}(B_{(m,n)}) = \begin{cases} 3 & \text{if both } m \text{ and } n \text{ are odd,} \\ 5 & \text{if both } m \text{ and } n \text{ are even,} \\ 4 & \text{if } m \text{ and } n \text{ have opposite parity.} \end{cases}$$

Proof By Theorem 2.21 we have,

$$\begin{aligned} D(B_{(m,n)}, x) &= x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m \\ &= x^2\left(x^{m+n-2} + (1+x)^{m+n} + x^{m-1}(1+x)^n + x^{n-1}(1+x)^m\right) \\ &= x^2\left(x^{m-1}\left((1+x)^n + x^{n-1}\right) + (1+x)^m\left((1+x)^n + x^{n-1}\right)\right) \\ &= x^2\left((1+x)^m + x^{m-1}\right)\left((1+x)^n + x^{n-1}\right). \end{aligned}$$

We have known that there is no real number satisfying both the equations $(1+x)^m+x^{m-1}=0$ and $(1+x)^n+x^{n-1}=0$ simultaneously. Therefore it suffices to prove that $(1+x)^m+x^{m-1}$ has exactly one real root for odd m and two real roots for even m. The remaining proof is similar to the proof of Theorem 2.9. □

Theorem 2.24 *For bi-star graph $B_{(n,n)}$, we have the following :*

$$\mathtt{d}(B_{(n,n)}) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

Proof The proof similar to the proof of Theorem 2.9. □

The corona $H \circ G$ of two graphs H and G is the graph formed from one copy of H and $|V(H)|$ copies of G, where the i^{th} vertex of H is adjacent to every vertex in the i^{th} copy of G.

Lemma 2.25([9]) *Let G and H be nonempty graphs of order m and n respectively. Then*

$$D(G \circ H, x) = (x(1+x)^n + D(H,x))^m .$$

Theorem 2.26 *If K_m and K_n be the complete graphs with m and n vertices respectively. Then the domination polynomial of $K_m \circ K_n$ is*

$$D(K_m \circ K_n, x) = \left((1+x)^{n+1} - 1\right)^m .$$

Theorem 2.27 *For the corona $K_m \circ K_n$, we have the following :*

$$\mathtt{d}(K_m \circ K_n) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof It follows from the transformation $y = 1 + x$ in the domination polynomial $D(K_m \circ K_n, x)$. □

Consider the graph K_m and m copies of K_n . The graph $Q(m,n)$ is obtained by identifying each vertex of K_m with a vertex of a unique K_n.

Corollary 2.28 *For $m \geq 2$, the domination polynomial of $Q(m,n)$ is*

$$D(Q(m,n), x) = ((1+x)^n - 1)^m .$$

Proof It follows from the fact that $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. □

Corollary 2.29 *For the graph $Q(m,n)$, we have the following :*

$$\mathtt{d}(Q(m,n)) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

§3. Domination Stable Graph

In this section we introduce $\mathtt{d}$-stable and $\mathtt{d}$-unstable graphs. We obtained some examples of $\mathtt{d}$-stable and $\mathtt{d}$-unstable graphs.

Definition 3.1 *Let $G = (V(G), E(G))$ be a graph. The graph G is said to be a domination stable graph or simply $\mathtt{d}$-stable graph if all the nonzero domination roots of G lie in the left open half-plane, that is, if real part of the nonzero domination roots is negative. If G is not $\mathtt{d}$-stable graph, then G is said to be a domination unstable graph or simply $\mathtt{d}$-unstable graph.*

Theorem 3.1 *If G and H are isomorphic graphs, then G is $\mathtt{d}$-stable if and only if H is $\mathtt{d}$-stable.*

Proof It follows from the fact that if G and H are isomorphic graphs then $D(G,x) = D(H,x)$. □

Corollary 3.2 *If G and H are isomorphic graphs then G is $\mathtt{d}$-unstable if and only if H is $\mathtt{d}$-unstable.*

Theorem 3.3 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}$-stable if and only if each G_i is $\mathtt{d}$-stable.*

Proof It follows from the fact that

$$D(G,x) = \prod_{i=1}^{m} D(G_i,x).$$ □

Corollary 3.4 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}$-unstable if and only if one of the G_i is $\mathtt{d}$-unstable.*

Theorem 3.5 *Let G be a connected graph of order $n > 2$ without pendent vertices. If G is $\mathtt{d}$-stable, then*

$$n < 1 + 2\,\mathtt{d}(G, n-3).$$

Proof Suppose G is $\mathtt{d}$-stable. Then by Routh-Hurwitz criteria, we have Routh-Hurwitz matrix $H_2 > 0$. This implies that

$$\mathtt{d}(G,n-1)\mathtt{d}(G,n-3) - \mathtt{d}(G,n-2) > 0.$$

Since G is connected and without pendent vertices we have

$$\mathtt{d}(G,n-1)=n \text{ and } \mathtt{d}(G,n-2)=\frac{1}{2}n(n-1).$$

This completes the proof. □

Theorem 3.6 *The complete graph K_n is* d*-stable graph for all n.*

Proof The domination polynomial of K_n is

$$D(K_n,x)=(1+x)^n-1.$$

Therefore

$$\mathbb{Z}(D(K_n,x))=\left\{\exp\left(\frac{2k\pi i}{n}\right)-1|k=0,1,\ldots,n-1\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that K_n is d-stable for all n. □

Theorem 3.7 *The complement of the complete graph K_n is* d*-stable graph for all n.*

Proof It follows from the fact that the graph $\overline{K_n}$ has no nonzero domination roots. □

We use the following definitions and results to prove some graphs which are d-unstable. These definitions and theorems are taken from [10].

Definition 3.2 *If $f_n(x)$ is a family of complex polynomials, we say that a number $z\in\mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z)=0$ for all sufficiently large n or z is a limit point of the set $\mathbb{Z}(f_n(x))$, $\mathbb{Z}(f_n(x))$ is the set of the roots of the family $f_n(x)$.*

Now, a family $f_n(x)$ of polynomials is a recursive family of polynomials if $f_n(x)$ satisfy a homogeneous linear recurrence

$$f_n(x)=\sum_{i=1}^{k}a_i(x)f_{n-i}(x), \tag{8}$$

where the $a_i(x)$ are fixed polynomials, with $a_k(x)\neq 0$. The number k is the order of the recurrence. We can form from equation (8) its associated characteristic equation

$$\lambda^k-a_1(x)\lambda^{k-1}-a_2\lambda^{k-2}-\cdots-a_k(x)=0 \tag{9}$$

whose roots $\lambda=\lambda(x)$ are algebraic functions, and there are exactly k of them counting multiplicity.

If these roots, say $\lambda_1(x),\lambda_2(x),\cdots,\lambda_k(x)$, are distinct, then the general solution to equation (8) is known to be

$$f_n(x)=\sum_{i=1}^{k}\alpha_i(x)\lambda_i(x)^n \tag{10}$$

with the usual variant if some of the $\lambda_i(x)$ are repeated. The functions

$$\alpha_1(x), \alpha_2(x), \cdots, \alpha_k(x)$$

are determined from the initial conditions, that is, the k linear equations in the α_i obtained by letting $n = 0, 1, \cdots, k-1$ in equation (10) or its variant. The details are available in [10]. Beraha, Kahane and Weiss [10] proved the following results on recursive families of polynomials and their roots.

Theorem 3.8 *If $f_n(x)$ is a recursive family of polynomials, then a complex number z is a limit of roots of $f_n(x)$ if and only if there is a sequence (z_n) in $\mathbb{C}$ such that $f_n(z_n) = 0$ for all n and $z_n \to z$ as $n \to \infty$.*

Theorem 3.9 *Under the non-degeneracy requirements that in equation (10) no $\alpha_i(x)$ is identically zero and that for no pair $i \neq j$ is it true that $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some complex number ω of unit modulus, then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either*

(1) *two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or*

(2) *for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.*

Corollary 3.10([6]) *Suppose $f_n(x)$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \ldots + \alpha_k(x)\lambda_k(x)^n, \tag{11}$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then the limits of roots of $f_n(x)$ are exactly those z satisfying (1) or (2) of Theorem 3.9.

Theorem 3.11 *The generalized barbell graph $B_{m,n,1}$ is* d*-stable for all m, n.*

Proof We have known by Theorem 2.17 that the domination polynomial of $B_{m,n,1}$ is

$$D(B_{m,n,1}, x) = [(1+x)^m - 1]\,[(1+x)^n - 1]\,.$$

Therefore

$$\begin{aligned} \mathbb{Z}(D(B_{m,n,1}, x)) &= \left\{\exp\left(\frac{2k\pi i}{m}\right) - 1 | k = 0, \cdots, m-1\right\} \\ &\bigcup \left\{\exp\left(\frac{2k\pi i}{n}\right) - 1 | k = 0, \cdots, n-1\right\}. \end{aligned}$$

Clearly, real part of all the roots are non-positive. This implies that the generalized barbell graph $B_{m,n,1}$ is d-stable for all m, n. □

The domination roots of the generalized barbell graph $B_{m,n,1}$ for $1 \leq m, n \leq 30$ are shown in Figure 1.

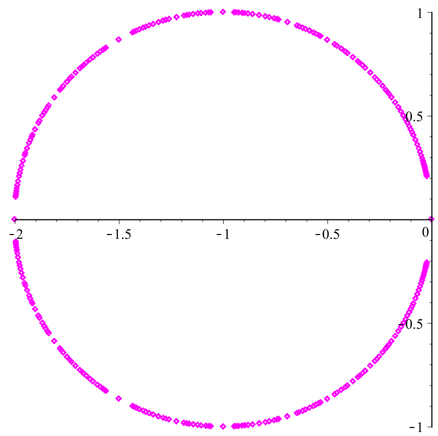

Figure 1 Domination roots of $B_{m,n,1}$ for $1 \leq m, n \leq 30$.

Corollary 3.12 *The n-barbell graph $B_{n,1}$ is* d*-stable for all n.*

Proof It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

The domination roots of the n-barbell graph $B_{n,1}$ for $1 \leq n \leq 60$ are shown in Figure 2.

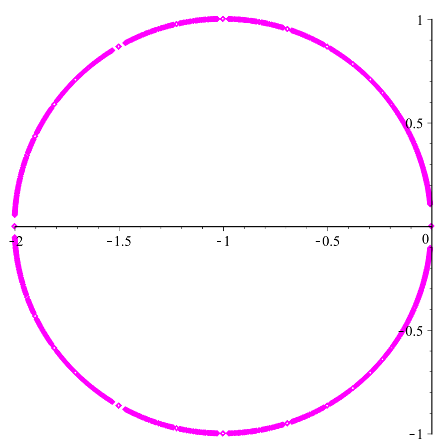

Figure 2 Domination roots of $B_{n,1}$ for $1 \leq n \leq 60$.

Theorem 3.13 *The corona $K_m \circ K_n$ is* d*-stable for all m, n.*

Proof Notice that the domination polynomial of $K_m \circ K_n$ is

$$D(K_m \circ K_n, x) = \left((1+x)^{n+1} - 1\right)^m.$$

Therefore

$$\mathbb{Z}(D(K_m \circ K_n, x)) = \left\{\exp\left(\frac{2k\pi i}{n+1}\right) - 1 | k = 0, 1, \cdots, n\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the corona $K_m \circ K_n$ is d-stable for all m, n. □

Corollary 3.14 *The graph $Q(m,n)$ is* d*-stable for all m, n.*

Proof It follows from the fact that the graph $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. □

Theorem 3.15 *Let G be a connected graph of order n and $D(G,x)$ be its domination polynomial. If $D(G,x)$ has exactly two distinct domination roots, then G is* d*-stable for all n.*

Proof It follows from the fact that the two distinct roots are real. □

Theorem 3.16 *Let G be a graph of order n, then the corona $G \circ K_1$ is* d*-stable for all n.*

Proof We have known the domination polynomial of $G \circ K_1$ is

$$D(G \circ K_1, x) = x^n (x+2)^n.$$

Therefore $\mathbb{Z}(D(G \circ K_1, x) = \{0, -2\}$, that is, $G \circ K_1$ is d-stable for all n. □

Theorem 3.17 *Let G be a graph of order n, then the corona $G \circ \overline{K_2}$ is* d*-stable for all n.*

Proof Notice that the domination polynomial of $G \circ \overline{K_2}$ is

$$D(G \circ \overline{K_2}, x) = x^{\frac{n}{3}} (x^2 + 3x + 1)^{\frac{n}{3}}.$$

Therefore $\mathbb{Z}(D(G \circ \overline{K_2}, x) = \left\{0, \frac{-3 \pm \sqrt{5}}{2}\right\}$, That is, $G \circ \overline{K_2}$ is d-stable for all n. □

Theorem 3.18 *Let G be a graph without pendent vertices and let $D(G,x)$ be its domination polynomial. If $D(G,x)$ has exactly three distinct roots, then G is* d*-stable.*

Proof Notice that

$$\mathbb{Z}(D(G,x)) \subset \left\{0, -2 \pm i\sqrt{2}, \frac{-3 \pm i\sqrt{3}}{2}\right\}.$$

This implies that G is d-stable. □

Theorem 3.19 *Any graph G with three distinct domination roots is* d*-stable.*

Proof Notice that

$$\mathbb{Z}(D(G,x)) \subset \left\{-2, 0, \frac{-3 \pm \sqrt{5}}{2}, -2 \pm i\sqrt{2}, \frac{-3 \pm i\sqrt{3}}{2}\right\}.$$

This implies that G is d-stable. □

Theorem 3.20 *The Dutch windmill graph G_3^n is not* d*-stable graph for all but finite values of n.*

Proof Using maple, we find that the Dutch windmill graph G_3^n is d-stable for $n \leq 6$. Notice that $D(G_3^n, x) = x(1+x)^{2n} + (2x+x^2)^n$. We rewrite $f_n(x) = D(G_3^n, x)$ as

$$f_n(x) = x\left((1+x)^2\right)^n + (1)(2x+x^2)^n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n,$$

where, $\alpha_1 = x,\ \lambda_1 = (1+x)^2,\ \alpha_2 = 1,\ \lambda_2 = 2x + x^2$.

Clearly, 1 and x are not identically zero and $\lambda_1 \neq \omega\lambda_2$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem .19 are satisfied. Now, for $z = a + ib \in \mathbb{C}$, $|\lambda_1(z)| = |\lambda_2(z)|$ holds if and only if $|(1+z)^2| = |2z + z^2|$. That is, $|(1+a+ib)^2| = |2(a+ib) + (a+ib)^2|$. By a simple calculation we have $(a+1)^2 + b^2 = \frac{1}{2}$. Therefore 0 and the complex numbers z such that $(1 + \mathcal{R}(z))^2 + (\mathcal{I}(z))^2 = \frac{1}{2}$ are limits of domination roots of G_3^n. This implies that the domination roots of G_3^n have unbounded positive real part. Therefore the Dutch windmill graph G_3^n is not d-stable for all but finite values of n. □

The domination roots of the Dutch windmill graph G_3^n for $1 \leq n \leq 6$ and for $1 \leq n \leq 30$ are shown in Figures 3 and 4, respectively.

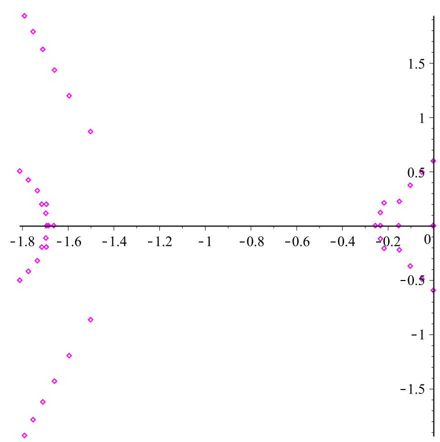

Figure 3 Domination roots of G_3^n for $1 \leq n \leq 6$.

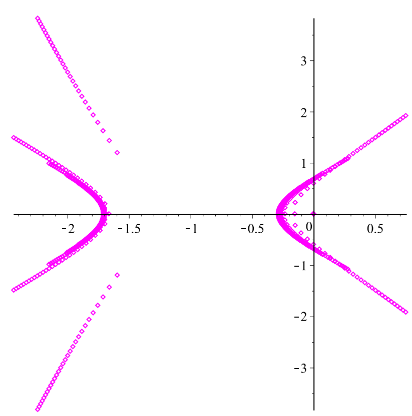

Figure 4 Domination roots of G_3^n for $1 \leq n \leq 30$.

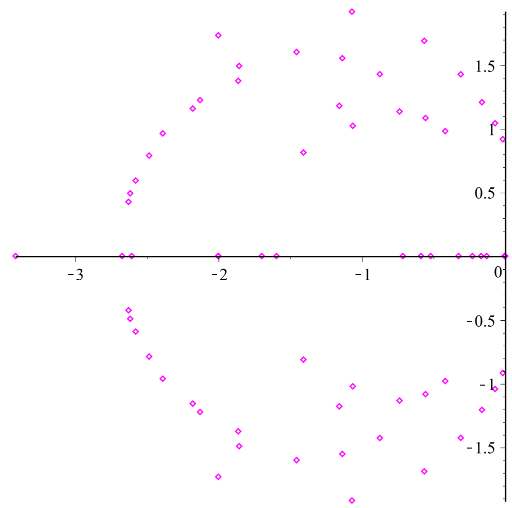

Figure 5 Domination roots of B_n for $1 \leq n \leq 9$.

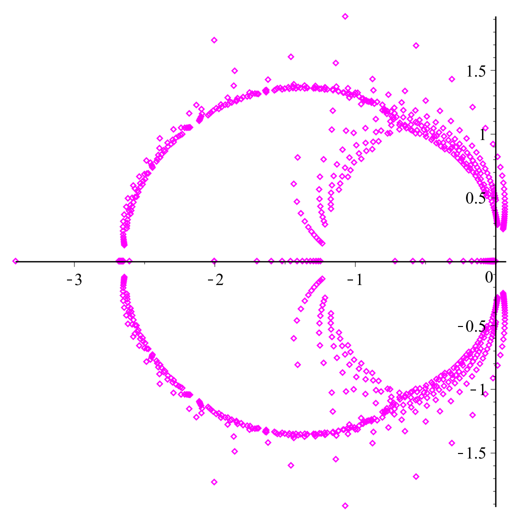

Figure 6 Domination roots of B_n for $1 \leq n \leq 30$.

The domination roots of the bipartite cocktail party graph B_n for $1 \leq n \leq 9$ and for $1 \leq n \leq 30$ are shown in Figures 5 and 6, respectively.

Remark 3.21 The domination polynomial of S_n is

$$\begin{aligned} D(S_n, x) &= x^n + x(1+x)^n \\ &= 1(x)^n + x(1+x)^n \\ &= \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n, \end{aligned}$$

where $\alpha_1 = 1$, $\lambda_1 = x$, $\alpha_2 = x$ and $\lambda_2 = 1 + x$. Clearly 1 and x are not identically zero and

$\lambda_1 \neq \omega\lambda_2$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem 3.9 are satisfied. Now, $|\lambda_1| = |\lambda_2|$ holds if and only if $|x-0| = |x-(-1)|$, that is, if and only if x is equidistant from 0 and -1. This holds if and only if real part of x is $-\frac{1}{2}$. Also α_1 is never 0 and $\alpha_2 = 0$ if and only if $x = 0$ and in this case $|\lambda_2(0)| = 1 > 0 = |\lambda_1(0)|$. By these arguments we have 0 and the complex numbers z such that $\mathcal{R}(z) = -\frac{1}{2}$ are the limits of roots of $D(S_n, x)$. Therefore we think that that there is no complex number z with positive real part is a root of $D(S_n, x)$. We conjectured that the star graph S_n is d-stable graph for all n.

The domination roots of the star graph S_n for $1 \leq n \leq 60$ are shown in Figure 7.

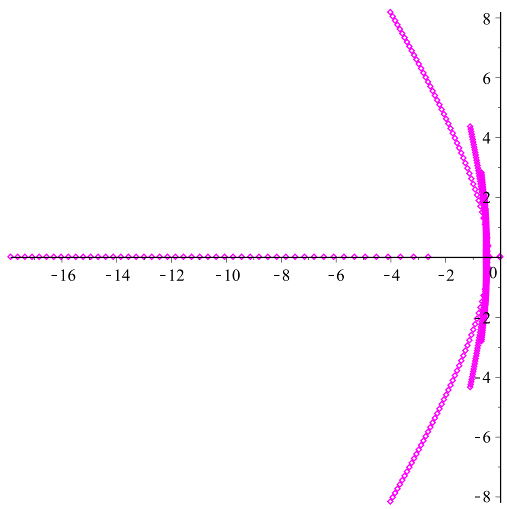

Figure 7 Domination roots of S_n for $1 \leq n \leq 60$.

Remark 3.22 The domination polynomial of $K_{m,n}$ is

$$D(K_{m,n}, x) = ((1+x)^m - 1)((1+x)^n - 1) + x^m + x^n.$$

Let m be fixed and rewrite $D(K_{m,n}, x)$ as :

$$\begin{aligned} D(K_{m,n}, x) &= ((1+x)^m - 1)(1+x)^n + ((1 + x^m - (1+x)^m))(1)^n + 1(x)^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \end{aligned}$$

where $\alpha_1 = (1+x)^m - 1$, $\lambda_1 = 1+x$, $\alpha_2 = 1 + x^m - (1+x)^m$, $\lambda_2 = 1$, $\alpha_3 = 1$ and $\lambda_3 = x$. Clearly α_1,α_2 and α_3 are not identically zero and $\lambda_i \neq \omega\lambda_j$ for $i \neq j$ and any complex number ω of modulus 1. Therefore the initial conditions of Theorem 3.9 are satisfied. Now, applying part(i) of Theorem 3.9, we consider the following four different cases:

(*i*) $|\lambda_1| = |\lambda_2| = |\lambda_3|$,
(*ii*) $|\lambda_1| = |\lambda_2| > |\lambda_3|$,
(*iii*) $|\lambda_1| = |\lambda_3| > |\lambda_2|$,
(*iv*) $|\lambda_2| = |\lambda_3| > |\lambda_1|$.

Case 1. Assume that $|1+x| = |1| = |x|$. Then $|x-(-1)| = |x-0|$ implies that x lies on the vertical line $z = -\frac{1}{2}$, $|x-(-1)| = 1$ implies that x lies on the unit circle centered at $(-1,0)$ and $1 = |x-0|$ implies that x lies on the unit circle centered at the origin. Therefore the two points of intersection, $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are limits of roots.

Case 2. Assume that $|1+x| = |1| > |x|$. Then $|x-(-1)| = 1$ implies that x lies on the unit circle centered at $(-1,0)$, $|x-(-1)| > |x-0|$ implies that x lies to the right of the vertical line $z = -\frac{1}{2}$. Therefore the complex numbers x that satisfy $|x-(-1)| = 1$ and $\mathcal{R}(x) > -\frac{1}{2}$ are limits of roots.

Case 3. Assume that $|1+x| = |x| > |1|$. Then $|x-(-1)| = |x-0|$ implies that x lies on the vertical line $x = -\frac{1}{2}$ and $|x-0| > 1$ implies that x lies outside the unit circle centered at the origin. Therefore the complex numbers x that satisfy $|x| > 1$ and $\mathcal{R}(x) > -\frac{1}{2}$ are limits of roots.

Case 4. Assume that $|1| = |x| > |1+x|$. Then $1 = |x-0|$ implies that x lies on the unit circle centered at the origin and $|x-0| > |x-(-1)|$ implies that x lies to the left of the vertical line $x = -\frac{1}{2}$. Therefore the complex numbers x that satisfy $|x| = 1$ and $\mathcal{R}(x) < -\frac{1}{2}$ are limits of roots.

Also there may be some additional isolated limits of roots, being roots of α_2 inside $|1+x| = 1$ and $|x| = 1$. The union of the curves and points above yield that for m fixed, the limits of roots of the domination polynomial of the complete bipartite graph $K_{m,n}$ consists of the part of the circle $|z| = 1$ with real part at most $-\frac{1}{2}$, the part of the circle $|z+1| = 1$ with real part at least $-\frac{1}{2}$ and the part of the line $\mathcal{R}(z) = -\frac{1}{2}$ with modulus at least 1. So we conjectured that the complete bipartite graph $K_{m,n}$ is d-stable for all m, n.

The domination roots of the complete bipartite graphs $K_{m,n}$ for $1 \le m \le 15$, $1 \le n \le 30$ and $K_{n,n}$ for $1 \le n \le 30$ are respectively shown in Figures 8 and 9.

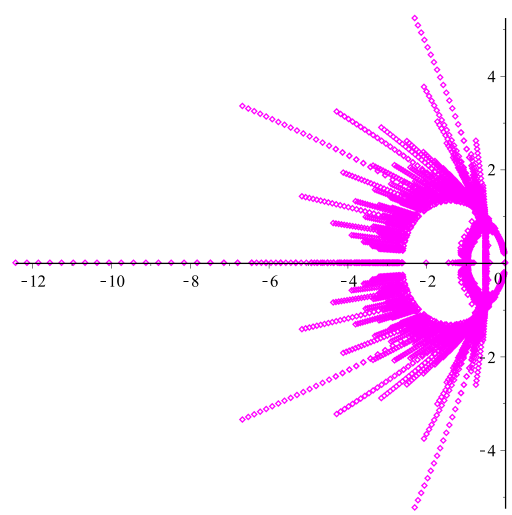

Figure 8 Domination roots of $K_{m,n}$ for $1 \le m \le 15$ and $1 \le n \le 30$.

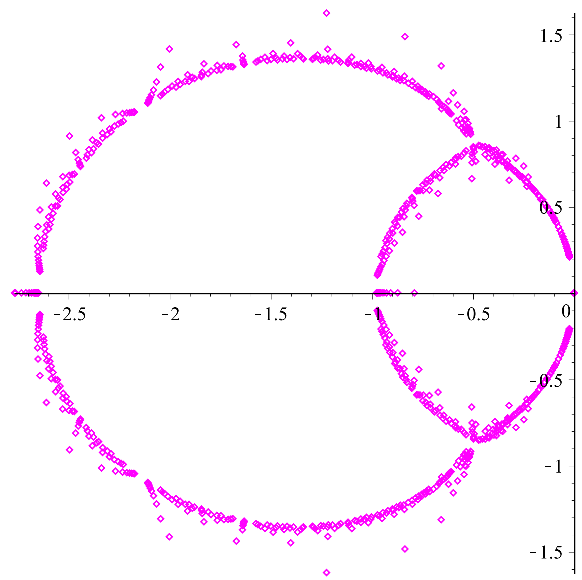

Figure 9 Domination roots of $K_{n,n}$ for $1 \leq n \leq 30$.

Remark 3.23 We have that $D(B_{(m,n)}, x) = x^2\left((1+x)^m + x^{m-1}\right)\left((1+x)^n + x^{n-1}\right)$. Let m be fixed, we rewrite $D(B_{(m,n)}, x)$ as $f_n(x)$:

$$\begin{aligned} f_n(x) &= \left(x^{m+1} + x^2(1+x)^m\right)(1+x)^n + \left(x^m + x(1+x)^m\right)x^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n, \end{aligned}$$

where

$$\alpha_1 = \left(x^{m+1} + x^2(1+x)^m\right), \ \lambda_1 = 1+x, \ \alpha_2 = \left(x^m + x(1+x)^m\right) \text{ and } \lambda_2 = x.$$

Clearly $\left(x^{m+1} + x^2(1+x)^m\right)$ and $\left(x^m + x(1+x)^m\right)$ are not identically zero and $\lambda_1 \neq \omega\lambda_2$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem 3.9 are satisfied. Now, $|\lambda_1| = |\lambda_2|$ holds if and only if $|x-(-1)| = |x-0|$, that is, if and only if x is equidistant from -1 and 0. The latter holds if and only if $\mathcal{R}(x) = -\frac{1}{2}$. Notice that $\alpha_1(0) = 0$ and $\alpha_1(0) = 1 + 0 = 1$ has modulus strictly greater than $\lambda_2(0) = 0$.

Note that there may be some additional limits of roots, being roots of α_1 and α_2. But from the Remark 3.21, we can conclude that α_1 and α_2 have no roots in the right-half plane. By these arguments we have 0 and the complex numbers z that satisfy $\mathcal{R}(z) = -\frac{1}{2}$ are the limits of roots of $D(B_{(m,n)}, x)$. So we conjectured that the bi-star graph $B_{(m,n)}$ is $\mathtt{d}$-stable for all m, n.

The domination roots of the bi-star graph $B_{(n,n)}$ for $1 \leq n \leq 50$ are shown in Figure 10.

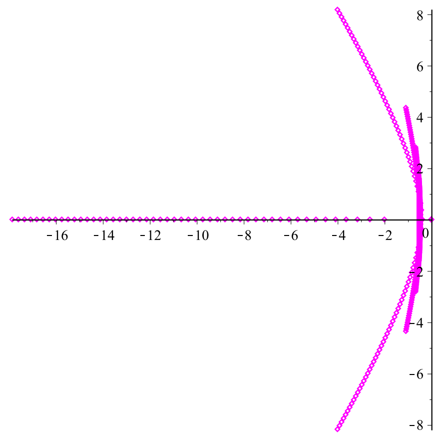

Figure 10 Domination roots of bi-star graph $B_{(n,n)}$ for $1 \leq n \leq 50$.

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