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C-Geometric Mean Labeling of Some Ladder Graphs

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Abstract: A function f is called a C-geometric mean labeling of a graph $G(V,E)$ if $f : V(G) \to \{1,2,3,\cdots,q+1\}$ is injective and the induced function $f^* : E(G) \to \{2,3,4,\cdots,q+1\}$ defined by $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$ for all $uv \in E(G)$ is bijective. A graph that admits a C-geometric mean labeling is called a C-geometric mean graph. In this paper, we have discussed the C-geometric meanness of some ladder graphs.

Key Words: Labeling, C-geometric mean labeling, C-Geometric mean graph, Smarandache k-mean graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [5]. For a detailed survey on graph labeling we refer to [4].

Path on n vertices is denoted by P_n. $G \odot S_m$ is the graph obtained from G by attaching m pendant vertices at each vertex of G. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices respectively. Then the cartesian product $G_1 \times G_2$ has p_1p_2 vertices which are $\{(u,v) : u \in G_1, v \in G_2\}$. The edges are obtained as follows: (u_1,v_1) and (u_2,v_2) are adjacent in $G_1 \times G_2$ if either $u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent in G_1 and $v_1 = v_2$. The ladder graph L_n is a graph obtained from the cartesian product of P_2 and P_n. The triangular ladder $TL_n, n \geq 2$ is a graph obtained by completing the ladder L_n by the edges u_iv_{i+1} for $1 \leq i \leq n-1$, where L_n is the graph $P_2 \times P_n$. The slanting ladder SL_n is a graph that consists of two copies of P_n having vertex set $\{u_i : 1 \leq i \leq n\} \bigcup \{v_i : 1 \leq i \leq n\}$ and edge set is formed by adjoining u_{i+1} and v_i for all $1 \leq i \leq n-1$ ([2]).

Let P_n be a path on n vertices denoted by $u_{1,1}, u_{1,2}, u_{1,3}, \cdots, u_{1,n}$ and with $n-1$ edges denoted by $e_1, e_2, \cdots, e_{n-1}$ where e_i is the edge joining the vertices $u_{1,i}$ and $u_{1,i+1}$. On each edge e_i, erect a ladder with $n-(i-1)$ steps including the edge e_i, for $i = 1,2,3,\ldots,n-1$. The graph thus obtained is called a one sided step graph and it is denoted by ST_n. Let P_{2n} be

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a path on $2n$ vertices $u_{1,1}, u_{1,2}, u_{1,3}, \cdots, u_{1,2n}$ and with $2n-1$ edges $e_1, e_2, \ldots, e_{2n-1}$ where e_i is the edge joining the vertices $u_{1,i}$ and $u_{1,i+1}$. On each edge e_i, we erect a ladder with '$i+1$' steps including the edge e_i, for $i = 1, 2, 3, \ldots, n$ and on each e_i erect a ladder with $2n+1-i$ steps including e_i, for $i = n+1, n+2, \cdots, 2n-1$. The graph thus obtained is called a double sided step graph and it is denoted by $2ST_{2n}$.

The study of graceful graphs and graceful labeling methods was first introduced by Rosa [7]. The concept of mean labeling was first introduced by S. Somasundaram and R. Ponraj [8] and it was developed in [6] and [9]. In [11], R. Vasuki et al. discussed the mean labeling of cyclic snake and armed crown. In [1, 3], some graph labelings of step graphs were analyzed.

In a study of traffic, the labeling problems in graph theory can be used by considering the crowd at every junctions as the weights of a vertex and expected average traffic in each street as the weight of the corresponding edge. If we assume the expected traffic at each street as the arithmetic mean of the weight of the end vertices, that eases mean labeling of the graph. When we consider a geometric mean instead of arithmetic mean in a large population of a city, the rate of growth of traffic in each street will be more accurated. Motivated by this, we establish the geometric mean labeling on graphs.

Motivated by the works of so many authors in the area of graph labeling, we introduced a new type of labeling called C-geometric mean labeling. A function f is called a C-geometric mean labeling of a graph G if $f : V(G) \rightarrow \{1, 2, 3, \cdots, q+1\}$ is injective and the induced function $f^* : E(G) \rightarrow \{2, 3, 4, \cdots, q+1\}$ defined as

$$f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil \quad \text{for all } uv \in E(G)$$

is bijective. A graph that admits a C-geometric mean labeling is called a C-geometric mean graph.

In [10], S. Somasundaram et al. defined the geometric mean labeling as follows.

A graph $G = (V, E)$ with p vertices and q edges is said to be a geometric mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \cdots, q+1$ in such way that when each edge $e = uv$ is labeled with $f(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ or $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ then the edge labels are distinct.

In the above definition, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function. Generally, a graph $G = (V, E)$ with p vertices and q edges is said to be a *Smarandache k-mean graph* for an integer $k \geq 2$ if it is labeled vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \cdots, q+1$ in such way that edge $e = uv$ is labeled with $f(uv) = \left\lfloor \sqrt[k]{f(u)^k f(v)^k} \right\rfloor$ or $\left\lceil \sqrt[k]{f(u)^k f(v)^k} \right\rceil$ then the edge labels are distinct. Clearly, a Smarandache 2-mean graph is nothing else but a geometric mean labeling graph.

In [10], S. Somasundaram et al. have given the geometric mean labeling of the graph $C_5 \cup C_7$ as shown in Figure 1.

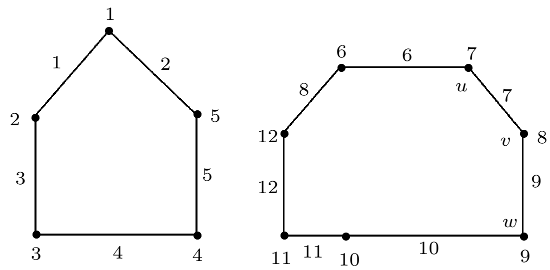

Figure 1 A geometric mean labeling of $C_5 \cup C_7$.

From the above figure, for the edge uv, they have used flooring function $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ and for the edge vw, they have used ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for fulfilling their requirement. To avoid the confusion of assigning the edge labels in their definition, we just consider the ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for our discussion. Based on our definition, the C-geometric mean labeling of the same graph $C_5 \cup C_7$ is given in Figure 2.

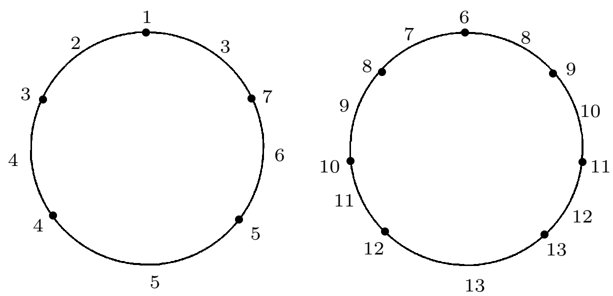

Figure 2 A C-geometric mean labeling of $C_5 \cup C_7$

In this paper, we have discussed the C-geometric mean labeling of the ladder graphs L_n for $n \geq 2$, $L_n \odot S_m$ for $n \geq 2$ and $m \leq 2$, TL_n for $n \geq 2$, $TL_n \odot S_m$ for $n \geq 2$ and $m \leq 2$, SL_n for $n \geq 2$, $SL_n \odot S_m$ for $n \geq 2$ and $m \leq 2$, step graph ST_n and double sided step graph $2ST_{2n}$.

§2. Main Results

Theorem 2.1 *The graph L_n is a C-geometric mean graph for $n \geq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of $L_n = P_n \times P_2$. Then the ladder graph L_n having $2n$ vertices and $3n - 2$ edges.

Define $f : V(L_n) \to \{1, 2, 3, \ldots, 3n - 1\}$ as follows:

$$f(u_1) = 1,$$
$$f(u_i) = 3i - 1, \text{ for } 2 \leq i \leq n \text{ ,}$$
$$f(v_1) = 3 \text{ and}$$
$$f(v_i) = 3i - 2, \text{ for } 2 \leq i \leq n.$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned} f^*(u_1u_2) &= 3, \\ f^*(u_iu_{i+1}) &= 3i+1, \text{ for } 2 \leq i \leq n-1, \\ f^*(v_1v_2) &= 4, \\ f^*(v_iv_{i+1}) &= 3i, \text{ for } 2 \leq i \leq n-1 \text{ and} \\ f^*(u_iv_i) &= 3i-1, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Hence, f is a C-geometric mean labeling of the ladder $P_n \times P_2$. Thus the ladder $P_n \times P_2$ is a C-geometric mean graph for $n \geq 2$. □

Theorem 2.2 *The graph $L_n \odot S_m$ is a C-geometric mean graph for $n \geq 2$ and $m \leq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of $L_n = P_n \times P_2$. Let $w_1^{(i)}, w_2^{(i)}, \cdots, w_m^{(i)}$ and $x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)}$ be the pendent vertices attached at each vertex u_i and v_i of the ladder L_n, for $1 \leq i \leq n$.

Case 1. $m = 1$.

Define $f : V(L_n \odot S_1) \to \{1, 2, 3, \cdots, 5n-1\}$ as follows:

$$\begin{aligned} f(u_1) &= 3, \\ f(u_i) &= 5i-3, \text{ for } 2 \leq i \leq n\ , \\ f(v_1) &= 4, \\ f(v_i) &= 5i-2, \text{ for } 2 \leq i \leq n\ , \\ f(w_1^{(i)}) &= 5i-4, \text{ for } 1 \leq i \leq n\ , \\ f(x_1^{(1)}) &= 2 \text{ and} \\ f(x_1^{(i)}) &= 5i-1, \text{ for } 2 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned} f^*(u_iu_{i+1}) &= 5i, \text{ for } 1 \leq i \leq n-1\ , \\ f^*(v_iv_{i+1}) &= 5i+1, \text{ for } 1 \leq i \leq n-1\ , \\ f^*(u_1v_1) &= 4, \\ f^*(u_iv_i) &= 5i-2, \text{ for } 2 \leq i \leq n\ , \\ f^*(u_iw_1^{(i)}) &= 5i-3, \text{ for } 1 \leq i \leq n\ , \\ f^*(v_1x_1^{(1)}) &= 3 \text{ and} \\ f^*(v_ix_1^{(i)}) &= 5i-1, \text{ for } 2 \leq i \leq n. \end{aligned}$$

Case 2. $m = 2$.

Define $f : V(L_n \odot S_2) \rightarrow \{1, 2, 3, \cdots, 7n-1\}$ as follows:

$$f(u_i) = \begin{cases} 3 & i = 1 \\ 7i-2 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-5 & 2 \leq i \leq n \text{ and } i \text{ is odd ,} \end{cases}$$

$$f(v_i) = \begin{cases} 5 & i = 1 \\ 7i-4 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-1 & 2 \leq i \leq n \text{ and } i \text{ is odd ,} \end{cases}$$

$$f(w_1^{(i)}) = \begin{cases} 1 & i = 1 \\ 7i-3 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-6 & 2 \leq i \leq n \text{ and } i \text{ is odd ,} \end{cases}$$

$$f(x_1^{(i)}) = \begin{cases} 3i+1 & 1 \leq i \leq 2 \\ 7i-6 & 3 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-3 & 3 \leq i \leq n \text{ and } i \text{ is odd} \end{cases}$$

$$\text{and } f(x_2^{(i)}) = \begin{cases} 8 & i = 1 \\ 7i-5 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-2 & 2 \leq i \leq n \text{ and } i \text{ is odd .} \end{cases}$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 6 & i = 1 \\ 7i & 2 \leq i \leq n-1, \end{cases}$$

$$f^*(v_i v_{i+1}) = 7i+1, \text{ for } 1 \leq i \leq n-1 \text{ ,}$$

$$f^*(u_i v_i) = 7i-3, \text{ for } 1 \leq i \leq n \text{ ,}$$

$$f^*(u_i w_1^{(i)}) = \begin{cases} 2 & i = 1 \\ 7i-2 & 2 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-5 & 2 \leq i \leq n \text{ and } i \text{ is odd ,} \end{cases}$$

$$f^*(u_i w_2^{(i)}) = \begin{cases} 7i-1 & 1 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-4 & 1 \leq i \leq n \text{ and } i \text{ is odd ,} \end{cases}$$

$$f^*(v_i x_1^{(i)}) = \begin{cases} 7i-5 & 1 \leq i \leq n \text{ and } i \text{ is even} \\ 7i-2 & 1 \leq i \leq n \text{ and } i \text{ is odd} \end{cases}$$

$$\text{and } f^*(v_i x_2^{(i)}) = \begin{cases} 7 & i = 1 \\ 7i-4 & 2 \le i \le n \text{ and } i \text{ is even} \\ 7i-1 & 2 \le i \le n \text{ and } i \text{ is odd} . \end{cases}$$

Hence, f is a C-geometric mean labeling of the graph $L_n \odot S_m$. Thus the graph $L_n \odot S_m$ is a C-geometric mean graph for $n \ge 2$ and $m \le 2$. □

Theorem 2.3 *The graph TL_n is a C-Geometric mean graph for $n \ge 2$.*

Proof Let the vertex set of TL_n be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and the edge set of TL_n be $\{u_i u_{i+1}; 1 \le i \le n-1\} \cup \{v_i v_{i+1}; 1 \le i \le n-1\} \cup \{u_i v_i; 1 \le i \le n\} \cup \{v_i u_{i+1}; 1 \le i \le n-1\}$. Then TL_n has $2n$ vertices and $4n-3$ edges.

Define $f : V(TL_n) \to \{1, 2, 3, \cdots, 4n-2\}$ as follows:

$$f(u_i) = 4i-3, \text{ for } 1 \le i \le n,$$
$$f(v_i) = 4i-1, \text{ for } 1 \le i \le n-1 \text{ and}$$
$$f(v_n) = 4n-2.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = 4i-1, \text{ for } 1 \le i \le n-1,$$
$$f^*(u_i v_i) = 4i-2, \text{ for } 1 \le i \le n,$$
$$f^*(v_i v_{i+1}) = 4i+1, \text{ for } 1 \le i \le n-1 \text{ and}$$
$$f^*(v_i u_{i+1}) = 4i, \text{ for } 1 \le i \le n-1.$$

Hence f is a C-geometric mean labeling of TL_n. Thus the triangular ladder TL_n is a C-geometric mean graph for $n \ge 2$. □

Theorem 2.4 *The graph $TL_n \odot S_m$ is a C-geometric mean graph for $n \ge 2$ and $m \le 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of TL_n. Let $w_1^{(i)}, w_2^{(i)}, \cdots, w_m^{(i)}$ *and* $x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)}$ be the pendent vertices attached at each vertex u_i and v_i of the ladder L_n, for $1 \le i \le n$.

Case 1. $m = 1$.

Define $f : V(TL_n \odot S_1) \to \{1, 2, 3, \cdots, 6n-2\}$ as follows:

$$f(u_1) = 3,$$
$$f(u_i) = 6i-4, \text{ for } 2 \le i \le n \text{ ,}$$

$$f(v_i) = 6i - 2, \text{ for } 1 \le i \le n \ ,$$
$$f(w_1^{(i)}) = 6i - 5, \text{ for } 1 \le i \le n \ ,$$
$$f(x_1^{(1)}) = 2 \text{ and}$$
$$f(x_1^{(i)}) = 6i - 3, \text{ for } 2 \le i \le n.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = 6i - 1, \text{ for } 1 \le i \le n-1 \ ,$$
$$f^*(v_iv_{i+1}) = 6i + 1, \text{ for } 1 \le i \le n-1 \ ,$$
$$f^*(v_iu_{i+1}) = 6i, \text{ for } 1 \le i \le n-1 \ ,$$
$$f^*(u_1v_1) = 4,$$
$$f^*(u_iv_i) = 6i - 3, \text{ for } 2 \le i \le n \ ,$$
$$f^*(u_iw_1^{(i)}) = 6i - 4, \text{ for } 1 \le i \le n \ ,$$
$$f^*(v_1x_1^{(1)}) = 3 \text{ and}$$
$$f^*(v_ix_1^{(i)}) = 6i - 2, \text{ for } 2 \le i \le n.$$

Case 2. $m = 2$.

Define $f : V(TL_n \odot S_2) \to \{1, 2, 3, \cdots, 8n-2\}$ as follows:

$$f(u_1) = 3,$$
$$f(u_i) = 8i - 3, \text{ for } 2 \le i \le n \ ,$$
$$f(v_1) = 5,$$
$$f(v_i) = 8i - 5, \text{ for } 2 \le i \le n \ ,$$
$$f(w_1^{(1)}) = 1,$$
$$f(w_1^{(i)}) = 8i - 4, \text{ for } 2 \le i \le n \ ,$$
$$f(w_2^{(1)}) = 2,$$
$$f(w_2^{(i)}) = 8i - 2, \text{ for } 2 \le i \le n \ ,$$
$$f(x_1^{(1)}) = 4,$$
$$f(x_1^{(i)}) = 8i - 7, \text{ for } 2 \le i \le n \ ,$$
$$f(x_2^{(1)}) = 6 \text{ and}$$
$$f(x_2^{(i)}) = 8i - 6, \text{ for } 2 \le i \le n.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_1u_2) = 7,$$
$$f^*(u_iu_{i+1}) = 8i + 1, \text{ for } 2 \le i \le n-1 \ ,$$

$$f^*(v_1v_2) = 8,$$
$$f^*(v_iv_{i+1}) = 8i-1, \text{ for } 2 \le i \le n-1 ,$$
$$f^*(u_iv_i) = 8i-4, \text{ for } 1 \le i \le n ,$$
$$f^*(v_1u_2) = 9,$$
$$f^*(v_iu_{i+1}) = 8i, \text{ for } 2 \le i \le n-1 ,$$
$$f^*(u_1w_1^{(1)}) = 2,$$
$$f^*(u_iw_1^{(i)}) = 8i-3, \text{ for } 2 \le i \le n ,$$
$$f^*(u_1w_2^{(1)}) = 3,$$
$$f^*(u_iw_2^{(i)}) = 8i-2, \text{ for } 2 \le i \le n ,$$
$$f^*(v_1x_1^{(1)}) = 5,$$
$$f^*(v_ix_1^{(i)}) = 8i-6, \text{ for } 2 \le i \le n ,$$
$$f^*(v_1x_2^{(1)}) = 6 \text{ and}$$
$$f^*(v_ix_2^{(i)}) = 8i-5, \text{ for } 2 \le i \le n.$$

Hence, f is a C-geometric mean labeling of the graph $TL_n \odot S_m$. Thus the graph $TL_n \odot S_m$ is a C-geometric mean graph for $n \ge 2$ and $m \le 2$. □

Theorem 2.5 *The graph SL_n is a C-geometric mean graph for $n \ge 2$.*

Proof Let the vertex set of SL_n be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and the edge set of SL_n be $\{u_iu_{i+1}; 1 \le i \le n-1\} \cup \{v_iv_{i+1}; 1 \le i \le n-1\} \cup \{v_iu_{i+1}; 1 \le i \le n-1\}$. Then SL_n has $2n$ vertices and $3n-3$ edges.

Define $f : V(SL_n) \to \{1, 2, 3, \cdots, 3n-2\}$ as follows:

$$f(u_1) = 1,$$
$$f(u_i) = 3i-4, \text{ for } 2 \le i \le n,$$
$$f(v_i) = 3i, \text{ for } 1 \le i \le n-1 \text{ and}$$
$$f(v_n) = 3n-2.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_1u_2) = 2,$$
$$f^*(u_iu_{i+1}) = 3i-2, \text{ for } 2 \le i \le n-1,$$
$$f^*(v_iv_{i+1}) = 3i+2, \text{ for } 1 \le i \le n-2,$$
$$f^*(v_{n-1}v_n) = 3n-2 \text{ and}$$
$$f^*(v_iu_{i+1}) = 3i, \text{ for } 1 \le i \le n-1.$$

Hence f is a C-geometric mean labeling of SL_n. Thus the slanting ladder SL_n is a C-geometric mean graph for $n \ge 2$. □

Theorem 2.6 *The graph $SL_n \odot S_m$ is a C-geometric mean graph for $n \geq 2$ and $m \leq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of SL_n. Let $w_1^{(i)}, w_2^{(i)}, \cdots, w_m^{(i)}$ *and* $x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)}$ be the pendent vertices attached at each vertex u_i and v_i of the ladder L_n, for $1 \leq i \leq n$.

Case 1. $m = 1$ and $n \geq 3$.

Define $f : V(SL_n \odot S_1) \to \{1, 2, 3, \cdots, 5n-2\}$ as follows:

$$\begin{aligned}
f(u_1) &= 2,\\
f(u_i) &= 5i-6, \text{ for } 2 \leq i \leq n \ ,\\
f(v_1) &= 6,\\
f(v_i) &= 5i, \text{ for } 2 \leq i \leq n-1 \ ,\\
f(v_n) &= 5n-2,\\
f(w_1^{(1)}) &= 1,\\
f(w_1^{(i)}) &= 5i-7, \text{ for } 2 \leq i \leq n \ ,\\
f(x_1^{(1)}) &= 7,\\
f(x_1^{(i)}) &= 5i+1, \text{ for } 2 \leq i \leq n-1 \text{ and}\\
f(x_1^{(n)}) &= 5n-3.
\end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned}
f^*(u_iu_{i+1}) &= \begin{cases} 3i & 1 \leq i \leq 2 \\ 5i-3 & 3 \leq i \leq n-1, \end{cases}\\
f^*(v_iv_{i+1}) &= 5i+3, \text{ for } 1 \leq i \leq n-2 \ ,\\
f^*(v_{n-1}v_n) &= 5n-3,\\
f^*(v_iu_{i+1}) &= 5i, \text{ for } 1 \leq i \leq n-1 \ ,\\
f^*(u_1w_1^{(1)}) &= 2,\\
f^*(u_iw_1^{(i)}) &= 5i-6, \text{ for } 2 \leq i \leq n \ ,\\
f^*(v_1x_1^{(1)}) &= 7,\\
f^*(v_ix_1^{(i)}) &= 5i+1, \text{ for } 2 \leq i \leq n-1 \text{ and}\\
f^*(v_nx_1^{(n)}) &= 5n-2.
\end{aligned}$$

Case 2. $m = 2$ and $n \geq 3$.

Define $f : V(SL_n \odot S_2) \to \{1, 2, 3, \cdots, 7n-2\}$ as follows:

$$f(u_i) = \begin{cases} 2i+1 & 1 \leq i \leq 2 \\ 7i-6 & 3 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 7i-9 & 3 \leq i \leq n-1 \text{ and } i \text{ is odd ,} \end{cases}$$

$$f(u_n) = \begin{cases} 7n-10 & n \text{ is even} \\ 7n-9 & n \text{ is odd ,} \end{cases}$$

$$f(v_i) = \begin{cases} 9 & i = 1 \\ 7i+2 & 2 \leq i \leq n-3 \text{ and } i \text{ is even} \\ 7i-1 & 2 \leq i \leq n-3 \text{ and } i \text{ is odd ,} \end{cases}$$

$$f(v_{n-2}) = \begin{cases} 7n-13 & n \text{ is even} \\ 7n-15 & n \text{ is odd ,} \end{cases}$$

$$f(v_{n-1}) = 7n-5,$$

$$f(v_n) = 7n-3,$$

$$f(w_1^{(i)}) = \begin{cases} 1 & i = 1 \\ 6i-8 & 2 \leq i \leq 3 \\ 7i-7 & 4 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 7i-10 & 4 \leq i \leq n-1 \text{ and } i \text{ is odd ,} \end{cases}$$

$$f(w_1^{(n)} = \begin{cases} 7n-11 & n \text{ is even} \\ 7n-10 & n \text{ is odd ,} \end{cases}$$

$$f(w_2^{(i)}) = \begin{cases} 4i-2 & 1 \leq i \leq 2 \\ 7i-5 & 3 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 7i-8 & 3 \leq i \leq n-1 \text{ and } i \text{ is odd ,} \end{cases}$$

$$f(w_2^{(n)} = \begin{cases} 7n-7 & n \text{ is even} \\ 7n-8 & n \text{ is odd ,} \end{cases}$$

$$f(x_1^{(i)}) = \begin{cases} 8 & i = 1 \\ 7i & 2 \leq i \leq n-3 \text{ and } i \text{ is even} \\ 7i-3 & 2 \leq i \leq n-3 \text{ and } i \text{ is odd ,} \end{cases}$$

$$f(x_1^{(n-2)}) = \begin{cases} 7n-12 & n \text{ is even} \\ 7n-17 & n \text{ is odd ,} \end{cases}$$

$$f(x_1^{(n-1)}) = \begin{cases} 7n-8 & n \text{ is even} \\ 7n-7 & n \text{ is odd ,} \end{cases}$$

$$f(x_1^{(n)}) = 7n-4,$$

$$f(x_2^{(i)}) = \begin{cases} 11 & i=1 \\ 7i+1 & 2\leq i\leq n-3 \text{ and } i \text{ is even} \\ 7i-2 & 2\leq i\leq n-3 \text{ and } i \text{ is odd} \end{cases},$$

$$f(x_2^{(n-2)}) = \begin{cases} 7n-9 & n \text{ is even} \\ 7n-16 & n \text{ is odd} \end{cases},$$

$$f(x_2^{(n-1)}) = 7n-6$$

$$\text{and } f(x_2^{(n)}) = 7n-2.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 4i & 1\leq i\leq 2 \\ 7i-4 & 3\leq i\leq n-2 \end{cases},$$

$$f^*(u_{n-1}u_n) = \begin{cases} 7n-13 & n \text{ is even} \\ 7n-11 & n \text{ is odd} \end{cases},$$

$$f^*(v_iv_{i+1}) = \begin{cases} 12 & i=1 \\ 7i+4 & 2\leq i\leq n-3 \end{cases},$$

$$f^*(v_{n-2}v_{n-1}) = \begin{cases} 7n-9 & n \text{ is even} \\ 7n-10 & n \text{ is odd} \end{cases},$$

$$f^*(v_{n-1}v_n) = 7n-4,$$

$$f^*(v_iu_{i+1}) = 7i, \text{ for } 1\leq i\leq n-1,$$

$$f^*(u_iw_1^{(i)}) = \begin{cases} 2 & i=1 \\ 6i-7 & 2\leq i\leq 3 \\ 7i-6 & 4\leq i\leq n-1 \text{ and } i \text{ is even} \\ 7i-9 & 4\leq i\leq n-1 \text{ and } i \text{ is odd} \end{cases},$$

$$f^*(u_nw_1^{(n)}) = \begin{cases} 7n-10 & n \text{ is even} \\ 7n-9 & n \text{ is odd} \end{cases},$$

$$f^*(u_iw_2^{(i)}) = \begin{cases} 3i & 1\leq i\leq 2 \\ 7i-5 & 3\leq i\leq n-1 \text{ and } i \text{ is even} \\ 7i-8 & 3\leq i\leq n-1 \text{ and } i \text{ is odd} \end{cases},$$

$$f^*(u_nw_2^{(n)}) = 7n-8,$$

$$f^*(v_ix_1^{(i)}) = \begin{cases} 9 & i=1 \\ 7i+1 & 2\leq i\leq n-3 \text{ and } i \text{ is even} \\ 7i-2 & 2\leq i\leq n-3 \text{ and } i \text{ is odd} \end{cases},$$

$$f^*(v_{n-2}x_1^{(n-2)}) = \begin{cases} 7n-12 & n \text{ is even} \\ 7n-16 & n \text{ is odd}, \end{cases}$$

$$f^*(v_{n-1}x_1^{(n-1)}) = 7n-6,$$

$$f^*(v_nx_1^{(n)}) = 7n-3,$$

$$f^*(v_ix_2^{(i)}) = \begin{cases} 10 & i=1 \\ 7i+2 & 2 \leq i \leq n-3 \text{ and } i \text{ is even} \\ 7i-1 & 2 \leq i \leq n-3 \text{ and } i \text{ is odd}, \end{cases}$$

$$f^*(v_{n-2}x_2^{(n-2)}) = \begin{cases} 7n-11 & n \text{ is even} \\ 7n-15 & n \text{ is odd}, \end{cases}$$

$$f^*(v_{n-1}x_2^{(n-1)}) = 7n-5$$

and $f^*(v_nx_2^{(n)}) = 7n-2.$

Case 3. $m = 1, 2$ and $n = 2$.

The C-geometric mean labeling of $SL_2 \odot S_1$ and $SL_2 \odot S_2$ is given in Figure 3.

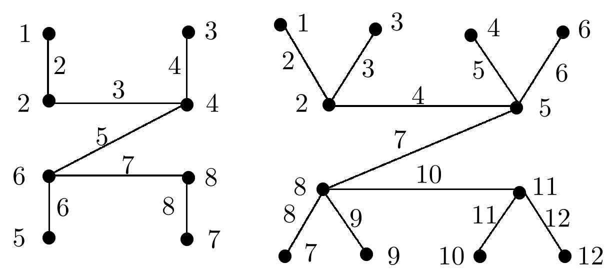

Figure 3 The C-geometric mean labeling of $SL_2 \odot S_1$ and $SL_2 \odot S_2$.

Hence, f is a C-geometric mean labeling of the graph $SL_n \odot S_m$. Thus the graph $SL_n \odot S_m$ is a C-geometric mean graph for $n \geq 2$ and $m \leq 2$. □

Theorem 2.7 *The graph ST_n is a C-geometric mean graph for $n \geq 2$.*

Proof Let $u_{1,1}, u_{1,2}, u_{1,3}, \cdots, u_{1,n}, u_{2,1}, u_{2,2}, u_{2,3}, \cdots, u_{2,n}, u_{3,1}, u_{3,2}, u_{3,3}, \cdots, u_{3,n-1}, u_{4,1}, u_{4,2}, u_{4,3}, \cdots, u_{4,n-2}, \cdots, u_{n,1}, u_{n,2}$ be the vertices of the step graph ST_n.

In $u_{i,j}$, i denotes the row (counted from bottom to top) and j denotes the column (counted from left to right) in which the vertex occurs.

Define $f : V(ST_n) \rightarrow \{1, 2, 3, \cdots, n^2+n-1\}$ as follows: For $1 \leq i \leq n-1$,

$$f(u_{i,j}) = \begin{cases} (n+1-i)^2 + 2(j-1) & 1 \leq j \leq \left\lfloor \frac{n+2-i}{2} \right\rfloor \\ (n+1-i)(n+3-i) - 2j + 1 & \left\lfloor \frac{n+2-i}{2} \right\rfloor + 1 \leq j \leq n+1-i, \end{cases}$$

$$f(u_{i,n+2-i}) = (n+1-i)(n+3-i), \text{ for } 2 \leq i \leq n-1\ ,$$
$$f(u_{n,1}) = 3 \text{ and}$$
$$f(u_{n,2}) = 1.$$

Then the induced edge labeling is obtained as follows:

For $1 \leq i \leq n-2$,

$$f^*(u_{i,j}u_{i,j+1}) = \begin{cases} (n+1-i)^2 + 2j - 1 & 1 \leq j \leq \left\lfloor \frac{n+2-i}{2} \right\rfloor - 1 \\ (n+1-i)^2 + 2j - 1 & j = \left\lfloor \frac{n+2-i}{2} \right\rfloor \text{ and } i \text{ is odd} \\ (n+1-i)(n+3-i) - 2j & j = \left\lfloor \frac{n+2-i}{2} \right\rfloor \text{ and } i \text{ is even} \\ (n+1-i)(n+3-i) - 2j & \left\lfloor \frac{n+2-i}{2} \right\rfloor + 1 \leq j \leq n-i, \end{cases}$$

$$f^*(u_{n-1,1}u_{n-1,2}) = 5,$$
$$f^*(u_{n,1}u_{n,2}) = 2,$$
$$f^*(u_{i,n+1-i}u_{i+1,n+2-i}) = (n+1-i)(n+2-i), \text{ for } 2 \leq i \leq n-1,$$
$$f^*(u_{i,1}u_{i+1,1}) = (n+1-i)(n-i), \text{ for } 1 \leq i \leq n-2,$$
$$f^*(u_{n-1,1}u_{n,1}) = 4,$$

For $1 \leq i \leq n-3$,

$$f^*(u_{i,j}u_{i+1,j}) = \begin{cases} (n+1-i)(n-i) + 2j - 1 & 2 \leq j \leq \left\lfloor \frac{n+2-i}{2} \right\rfloor - 1 \\ (n+1-i)(n-i) + 2j - 1 & j = \left\lfloor \frac{n+2-i}{2} \right\rfloor \text{ and } i \text{ is odd} \\ (n+1-i)(n+2-i) - 2j & j = \left\lfloor \frac{n+2-i}{2} \right\rfloor \text{ and } i \text{ is even} \\ (n+1-i)(n+2-i) - 2j & \left\lfloor \frac{n+2-i}{2} \right\rfloor + 1 \leq j \leq n-i, \end{cases}$$

$$f^*(u_{n-2,2}u_{n-1,2}) = 8,$$
$$f^*(u_{n-1,2}u_{n,2}) = 3,$$
$$f^*(u_{i,n+1-i}u_{i+1,n+1-i}) = (n+1-i)^2, \text{ for } 1 \leq i \leq n-2$$
$$\text{and } f^*(u_{n-1,2}u_{n,2}) = 3.$$

Hence, f is a C-geometric mean labeling of ST_n. Thus the step graph ST_n is a C-geometric mean graph, for $n \geq 2$. □

Theorem 2.8 *The graph $2ST_{2n}$ is a C-geometric mean graph, for $n \geq 2$.*

Proof Let $u_{1,1}, u_{1,2}, u_{1,3}, \cdots, u_{1,n}, u_{2,1}, u_{2,2}, u_{2,3}, \cdots, u_{2,2n}, u_{3,1}, u_{3,2}, u_{3,3}, \cdots, u_{3,2n-2}, u_{4,1}, u_{4,2}, u_{4,3}, \cdots, u_{4,2n-4}, \cdots, u_{n+1,1}, u_{n+1,2}$ be the vertices of the double sided step graph $2ST_{2n}$.

In $u_{i,j}$, i denotes the row (counted from bottom to top) and j denotes the column (counted from left to right) in which the vertex occurs.

Define $f : V(2ST_{2n}) \rightarrow \{1, 2, 3, \cdots, 2n^2 + 3n\}$ as follows:

$$f(u_{1,j}) = \begin{cases} 2n^2 + n + 1 + 2(j-1) & 1 \leq j \leq n \\ 2n^2 + 3n - 2(j-n-1) & n+1 \leq j \leq 2n, \end{cases}$$

for $2 \leq i \leq n$ and $2 \leq j \leq n+2-i$,

$$f(u_{i,j}) = 2(n+1-i)^2 + (n+2-i) + 2(j-2),$$

for $2 \leq i \leq n$ and $n+3-i \leq j \leq 2n+3-2i$,

$$\begin{aligned} f(u_{i,j}) &= 2(n+1-i)^2 + 3(n+1-i) - 2(i+j-n-3), \\ f(u_{2,1}) &= 2n^2 + n - 2, \\ f(u_{1,1}) &= 3, \\ f(u_{1,2}) &= 1, \\ f(u_{i,1}) &= 2(n+2-i)^2 + n - i, \text{ for } 3 \leq i \leq n \text{ and} \\ f(u_{i,2n+4-2i}) &= 2(n+2-i)^2 + n + 1 - i, \text{ for } 2 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_{1,j}u_{1,j+1}) = \begin{cases} 2n^2 + n + 2 + 2(j-1) & 1 \leq j \leq n \\ 2n^2 + 3n + 1 - 2(j-n) & n+1 \leq j \leq 2n-1, \end{cases}$$

for $2 \leq i \leq n-1$ and $2 \leq j \leq n+2-i$,

$$f^*(u_{i,j}u_{i,j+1}) = 2(n+1-i)^2 + (n+3-i) + 2(j-2),$$

for $2 \leq i \leq n-1$ and $n+3-i \leq j \leq 2n+2-2i$,

$$\begin{aligned} f^*(u_{i,j}u_{i,j+1}) &= 2(n+1-i)^2 + 3(n+1-i) + 1 - 2(i+j-n-2), \\ f^*(u_{i,2n+3-2i}u_{i+1,2n+2-2i}) &= 2(n+1-i)^2 + (n+2-i), \text{ for } 2 \leq i \leq n-1, \\ f^*(u_{n,3}u_{n+1,2}) &= 3, \\ f^*(u_{n,2}u_{n,3}) &= 5, \\ f^*(u_{n+1,1}u_{n+1,2}) &= 2, \\ f^*(u_{1,1}u_{2,1}) &= 2n^2 + n, \\ f^*(u_{1,2n}u_{2,2n}) &= 2n^2 + n + 1, \\ f^*(u_{i,2}u_{i+1,1}) &= 2(n+1-i)^2 + n + 1 - i, \text{ for } 2 \leq i \leq n-1, \\ f^*(u_{n,2}u_{n+1,1}) &= 4, \end{aligned}$$

$$f^*(u_{1,j}u_{2,j}) = \begin{cases} 2n^2 - n + 2 + 2(j-2) & 2 \leq j \leq n \\ 2n^2 + n - 1 - 2(j-n-1) & n+1 \leq j \leq 2n-1, \end{cases}$$

for $2 \leq i \leq n-1$ and $3 \leq j \leq n+2-i$,

$$f^*(u_{i,j}u_{i+1,j-1}) = 2(n+1-i)^2 - (n+1-i) + 2(j-2),$$

for $2 \leq i \leq n-1$ and $n+3-i \leq j \leq 2n+2-2i$,

$$f^*(u_{i,j}u_{i+1,j-1}) = 2(n+1-i)^2 + (n+4-i) - 2(i+j-n-1),$$
$$f^*(u_{i,1}u_{i,2}) = 2(n+1-i)^2 + 3(n+1-i) + 1, \text{ for } 2 \leq i \leq n \text{ and}$$
$$f^*(u_{i,2n+3-2i}u_{i,2n+4-2i}) = 2(n+1-i)^2 + 3(n+1-i) + 2, \text{ for } 2 \leq i \leq n.$$

Hence, f is a C-geometric mean labeling of $2ST_{2n}$. Thus the double sided step graph $2ST_{2n}$ is a C-geometric mean graph, for $n \geq 2$. □

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