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The Pebbling Number of Jahangir Graph $J_{2,m}$

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Abstract: The t-pebbling number, $f_t(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, t pebbles can be moved to any specified target vertex by a sequence of pebbling moves, each move taking two pebbles off a vertex and placing one on an adjacent vertex. When $t = 1$, we call it as the pebbling number of G, and we denote it by $f(G)$. In this paper, we are going to give an alternate proof for the pebbling number of the graph $J_{2,m}$ $(m \geq 3)$.

Key Words: Graph pebbling, pebbling move, Jahangir graph.

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§1. Introduction

An n-dimensional cube Q_n, or n-cube for short, consists of 2^n vertices labelled by $(0, 1)$-tuples of length n. Two vertices are adjacent if their labels are different in exactly one entry. Saks and Lagarias (see [1]) propose the following question: suppose 2^n pebbles are arbitrarily placed on the vertices of an n-cube. Does there exist a method that allows us to make a sequence of moves, each move taking two pebbles off one vertex and placing one pebble on an adjacent vertex, in such a way that we can end up with a pebble on any desired vertex? This question is answered in the affirmative in [1].

We begin by introducing relevant terminology and background on the subject. Here, the term graph refers to a simple graph without loops or multiple edges. A configuration C of pebbles on a graph $G = (V, E)$ can be thought of as a function $C : V(G) \to N \cup \{0\}$. The value $C(v)$ equals the number of pebbles placed at vertex v, and the size of the configuration is the number $|C| = \sum_{v \in V(G)} C(v)$ of pebbles placed in total on G. Suppose C is a configuration of pebbles on a graph G. A pebbling move (step) consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex v, the target vertex, if we can apply pebbling moves repeatedly (if necessary) so that in the resulting configuration the vertex v has at least one pebble.

Definition 1.1([2]) *The t-pebbling number of a vertex v in a graph G, $f_t(v, G)$, is the smallest positive integer n such that however n pebbles are placed on the vertices of the graph, t pebbles can be moved to v in finite number of pebbling moves, each move taking two pebbles off one*

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vertex and placing one on an adjacent vertex. The t-pebbling number of G, $f_t(G)$, is defined to be the maximum of the pebbling numbers of its vertices.

Thus the t-pebbling number of a graph G, $f_t(G)$, is the least n such that, for any configuration of n pebbles to the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the pebbling number of G.

Fact 1.2 ([12], [13]) For any vertex v of a graph G, $f(v, G) \geq n$ where $n = |V(G)|$.

Fact 1.3 ([12]) The pebbling number of a graph G satisfies

$$f(G) \geq max\{2^{diam(G)}, |V(G)|\}.$$

Saks and Lagarias question then reduces to asking whether $f(Q_n) \leq n$, where Q_n is the n-cube. Chung [1] answered this question in the affirmative, by proving a stronger result.

Theorem 1.4 ([1]) *In an n-cube with a specified vertex v, the following are true:*

(1) *If 2^n pebbles are assigned to vertices of the n-cube, one pebble can be moved to v;*

(2) *Let q be the number of vertices that are assigned an odd number of pebbles. If there are all together more than $2^{n+1} - q$ pebbles, then two pebbles can be moved to v.*

With regard to t-pebbling number of graphs, we find the following theorems.

Theorem 1.5 ([9]) *Let K_n be the complete graph on n vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.*

Theorem 1.6 ([3]) *Let $K_1 = \{v\}$. Let $C_{n-1} = (u_1, u_2, \cdots, u_{n-1})$ be a cycle of length $n - 1$. Then the t-pebbling number of the wheel graph W_n is $f_t(W_n) = 4t + n - 4$ for $n \geq 5$.*

Theorem 1.7 ([5]) *For $G = K^*_{s_1,s_2,\cdots,s_r}$,*

$$f_t(G) = \begin{cases} 2t + n - 2, & \text{if } 2t \leq n - s_1 \\ 4t + s_1 - 2, & \text{if } 2t \geq n - s_1 \end{cases}.$$

Theorem 1.8 ([9]) *Let $K_{1,n}$ be an n-star where $n > 1$. Then $f_t(K_{1,n}) = 4t + n - 2$.*

Theorem 1.9 ([9]) *Let C_n denote a simple cycle with n vertices, where $n \geq 3$. Then $f_t(C_{2k}) = t2^k$ and $f_t(C_{2k+1}) = \frac{2^{k+1}-(-1)^{k+2}}{3} + (t-1)2^k$.*

Theorem 1.10 ([9]) *Let P_n be a path on n vertices. Then $f_t(P_n) = t(2^{n-1})$.*

Theorem 1.11 ([9]) *Let Q_n be the n-cube. Then $f_t(Q_n) = t(2^n)$.*

Now, we state the known pebbling results of the Jahangir graph $J_{2,m}$ and then we give an alternate proof for those results in Section 2.

Definition 1.12 ([11]) *Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm+1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.*

A labeling for $J_{2,m}$ for $m \geq 3$ is defined as follows:

Let v_{2m+1} be the label of the center vertex and $v_1, v_2, \cdots, v_{2m}$ be the label of the vertices that are incident clockwise on cycle C_{2m} so that $deg(v_1) = 3$.

The pebbling number of Jahangir graph $J_{2,m}$ $(m \geq 3)$ is determined as follows:

Theorem 1.13 ([6]) *For the Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$.*

Theorem 1.14 ([6]) *For the Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$.*

Theorem 1.15 ([6]) *For the Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.*

Theorem 1.16 ([6]) *For the Jahangir graph $J_{2,6}$, $f(J_{2,6}) = 21$.*

Theorem 1.17 ([6]) *For the Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.*

Theorem 1.18 ([7]) *For the Jahangir graph $J_{2,m}$ where $m \geq 8$, $f(J_{2,m}) = 2m + 10$.*

The t-pebbling number of Jahangir graph $J_{2,m}$ $(m \geq 3)$is as follows:

Theorem 1.19 ([8]) *For the Jahangir graph $J_{2,3}$, $f_t(J_{2,3}) = 8t$.*

Theorem 1.20 ([8]) *For the Jahangir graph $J_{2,4}$, $f_t(J_{2,4}) = 16t$.*

Theorem 1.21 ([8]) *For the Jahangir graph $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.*

Theorem 1.22 ([8]) *For the Jahangir graph $J_{2,m}$, $f_t(J_{2,m}) = 16(t-1) + f(J_{2,m})$ where $m \geq 6$.*

Notation 1.23 Let $p(v)$ denote the number of pebbles on the vertex v and $p(A)$ denote the number of pebbles on the vertices of the set $A \subseteq V(G)$. We define the sets $S_1 = \{v_1, v_3, \cdots, v_{2m-1}\}$ and $S_2 = \{v_2, v_4, \cdots, v_{2m}\}$ from the labelling of $J_{2,m}$.

Remark 1.24 Consider a graph G with n vertices and $f(G)$ pebbles on it and we choose a target vertex v from G. If $p(v) = 1$ or $p(u) \geq 2$ where $uv \in E(G)$, then we can move one pebble to v easily. So, we always assume that $p(v) = 0$ and $p(u) \leq 1$ for all $uv \in E(G)$ when v is the target vertex.

§2. Alternate Proof for the Pebbling Number of $J_{2,m}$

Theorem 2.1 *For the Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$.*

Proof Put seven pebbles at v_4. Clearly we cannot move a pebble to v_1, since $d(v_4, v_1) = 3$. Thus $f(J_{2,3}) \geq 8$.

We have three cases to prove $f(J_{2,3}) \leq 8$.

Case 1. Let v_7 be the target vertex.

Clearly, $p(v_7) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq 5$, there exists a vertex, say v_2, such that $p(v_2) \geq 2$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_7 easily. Also, we can move one pebble to v_7, if $p(v_2) \geq 4$. Assume that $p(v_1) = 0$, $p(v_3) = 0$ and $p(v_2) = 2$ or 3. Thus either $p(v_4) \geq 2$ or $p(v_6) \geq 2$ and hence we can move one pebble to v_7 through v_3 or v_1.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$, $p(v_2) \leq 1$, $p(v_6) \leq 1$ and $p(v_7) \leq 1$, by Remark **??**. If $p(v_3) \geq 4$ or $p(v_5) \geq 4$ or $p(v_3) \geq 2$ and $p(v_5) \geq 2$ then we can move one pebble to v_1 through v_7. Without loss of generality, let $p(v_3) \geq 2$ and so $p(v_5) \leq 1$. If $p(v_2) = 1$ or $p(v_7) = 1$ then also we can move one pebble to v_1. So, we assume $p(v_2) = p(v_7) = 0$. Clearly, $p(v_4) \geq 3$. If $p(v_3) = 3$ then we move one pebble to v_3 from v_4 and hence we are done. Let $p(v_3) = 2$ and thus we move two pebbles to v_3 from v_4 and hence we are done. Assume $p(v_3) \leq 1$. In a similar way, we may assume that $p(v_5) \leq 1$ and hence $p(v_4) \geq 3$. Let $p(v_2) = 1$. If $p(v_3) = 1$ then clearly we can move one pebble to v_1. If $p(v_3) = 0$ then $p(v_4) \geq 4$ and hence we can move one pebble to v_2 and so one pebble is moved to v_1. Assume $p(v_2) = 0$. In a similar way, we may assume that $p(v_6) = 0$ and hence $p(v_4) \geq 5$. If $p(v_7) = 1$ then we are done easily. Let $p(v_7) = 0$. If $p(v_3) = 1$ or $p(v_5) = 1$ then we move three pebbles to v_3 or v_5, respectively. Thus we can move one pebble to v_1. Assume $p(v_3) = p(v_5) = 0$. Then $p(v_4) = 8$ and hence we can move one pebble to v_1 easily.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$, by Remark 1.24. Let $p(v_4) \geq 2$. If $p(v_4) \geq 4$ then clearly we are done.

Assume $p(v_4) = 2$ or 3 then clearly $p(v_3) = 0$ and $p(v_7) \leq 1$ (otherwise, we can move one pebble to v_2). Since, $p(v_5) + p(v_6) \geq 3$, first we let $p(v_6) \geq 2$. Clearly we are done if $p(v_1) = 0$ and $p(v_6) \geq 4$. Assume $p(v_1) = 0$ and $p(v_6) = 2$ or 3. If $p(v_7) = 1$ then we move one pebble to v_7 from v_4 since $p(v_4) \geq 2$ and $p(v_5) = 1$ and thus we move one pebble to v_1. Then we move one more pebble to v_1 from v_6 and hence one pebble can be moved to v_2. Assume $p(v_7) = 0$ and so $p(v_5) \geq 2$. If $p(v_4) = 3$ or $p(v_6) = 3$ then clearly we can move one pebble to v_2 by moving one pebble to v_3 or v_6. Thus we assume $p(v_4) = 2$ and $p(v_6) = 2$ and so $p(v_5) = 4$ and hence we are done. Assume $p(v_6) \leq 1$ and so $p(v_4) = 2$. Clearly, we are done if $p(v_5) \geq 4$. Assume $p(v_5) = 3$ and hence we move one pebble to v_2 since $p(v_7) = p(v_1) = 1$.

Assume $p(v_4) \leq 1$. In a similar way, we may assume that $p(v_6) \leq 1$ and so $p(v_7) \leq 1$. Let $p(v_1) = 1$. Clearly we are done if $p(v_7) = 1$ or $p(v_6) = 1$. Assume $p(v_6) = p(v_7) = 0$ and so $p(v_5) \geq 4$. Thus we move one pebble to v_1 and hence we are done. Assume $p(v_1) = 0$. In a similar way, we assume that $p(v_3) = 0$. We have $p(v_5) \geq 5$. Let $p(v_5) = 5$. Clearly, $p(v_6) = p(v_7) = 1$ and hence we can move one pebble to v_2 through v_1. Let $p(v_5) \geq 6$. If $p(v_4) = 1$ or $p(v_6) = 1$ or $p(v_7) = 1$ then we move three pebbles to v_4 or v_6 or v_7 and hence we are done. Assume $p(v_4) = p(v_6) = p(v_7) = 0$ and so $p(v_5) = 8$. Thus we can move one pebble to v_2 easily. □

Theorem 2.2 *For the Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$.*

Proof Put fifteen pebbles at v_8. Clearly we cannot move a pebble to v_4, since $d(v_8, v_4) = 4$. Thus $f(J_{2,4}) \geq 16$.

We have three cases to prove $f(J_{2,4}) \leq 16$.

Case 1. Let v_9 be the target vertex.

Clearly, $p(v_9) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark **??**. Since, $p(S_2) \geq 12$, there exists a vertex, say v_2, such that $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_9 easily. Assume $p(v_1) = 0$ and $p(v_3) = 0$. So, we can move one pebble to v_9 easily, since $p(v_2) \geq 4$.

Case 2: Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$, $p(v_2) \leq 1$, $p(v_8) \leq 1$ and $p(v_9) \leq 1$, by Remark **??**. If $p(v_3) \geq 4$ or $p(v_5) \geq 4$ or $p(v_7) \geq 4$ then we can move one pebble to v_1 through v_9. Assume $p(v_i) \leq 3$ for all $i \in \{3, 5, 7\}$. Let $p(v_3) \geq 2$ and if $p(v_9) = 1$ or $p(v_2) = 1$ or $p(v_5) \geq 2$ or $p(v_7) \geq 2$ then we can move one pebble to v_1 through v_9 easily. Assume $p(v_2) = 0$, $p(v_9) = 0$, $p(v_5) \leq 1$ and $p(v_7) \leq 1$. Clearly, either $p(v_4) \geq 4$ or $p(v_6) \geq 4$ and hence we can move one pebble to v_1 through v_9.

Assume $p(v_3) \leq 1$. In a similar way, we may assume that $p(v_5) \leq 1$ and $p(v_7) \leq 1$ and hence either $p(v_4) \geq 5$ or $p(v_6) \geq 5$. Without loss of generality, let $p(v_4) \geq 5$. If $p(v_2) = 1$ or $p(v_9) = 1$ then we move one pebble to v_2 or v_9 from v_4 and hence we can move one pebble to v_1. Assume $p(v_2) = 0$ and $p(v_9) = 0$ then clearly $p(v_4) \geq 6$. If $p(v_3) = 1$ or $p(v_5) = 1$ or $p(v_6) \geq 2$ then we can move one pebble to v_1 easily by moving three pebbles to v_3 or v_5 from v_4. Let $p(v_3) = 0$, $p(v_5) = 0$ and $p(v_6) \leq 1$ and hence $p(v_4) \geq 13$. Thus we can move one pebble to v_1 easily.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$, by Remark 1.24. Let $p(v_4) \geq 2$. If $p(v_4) \geq 4$ then clearly we are done.

Assume $p(v_4) = 2$ or 3 then clearly $p(v_3) = 0$ and $p(v_9) \leq 1$ (otherwise, we can move one pebble to v_2). If $p(v_5) \geq 4$ or $p(v_7) \geq 4$ or $p(v_5) \geq 2$ and $p(v_7) \geq 2$ then we can move one pebble to v_3 and then we move one pebble to v_3 from v_4 and hence one pebble can be moved to v_2 from v_3. Asssume $p(v_5) \leq 3$ and $p(v_7) \leq 4$ such that we cannot move one pebble to v_9. So, $p(v_5) + p(v_7) \leq 4$. Clearly, $p(v_8) + p(v_1) \leq 3$ and hence $p(v_6) \geq 6$. If $p(v_5) = 1$ or $p(v_7) = 1$ then we move three pebbles to v_5 or v_7 and then we can move two pebbles to v_3 from v_5 and v_4 and hence we are done. Assume $p(v_5) = 0$ and $p(v_7) = 0$. So, $p(v_6) \geq 8$. We move two pebbles to v_4 from v_6 and hence we can move one pebble to v_2 from v_4 easily.

Assume $p(v_4) \leq 1$. In a similar way, we may assume that $p(v_8) \leq 1$ and so $p(v_9) \leq 1$. Clearly, $p(v_5) + p(v_6) + p(v_7) \geq 11$ and so we can move two pebbles to v_9. If $p(v_1) = 1$ or $p(v_3) = 1$ then we move one more pebble to v_1 or v_3 from v_9 and hence we are done. Assume $p(v_1) = 0$ and $p(v_3) = 0$ then we have $p(v_5) + p(v_6) + p(v_7) \geq 13$. Let $p(v_5) \geq 4$. Clearly, we are done if $p(v_6) + p(v_7) \geq 8$. Assume $p(v_6) + p(v_7) \leq 7$ and so $p(v_5) \geq 6$. If $p(v_4) = 1$ or $p(v_9) = 1$ then we move three pebbles to v_4 or v_9 from v_6 and hence we are done. Let

$p(v_4) = 0$ and $p(v_9) = 0$ and hence we can move one pebble to v_2 since $p(v_5) \geq 8$. Assume $p(v_5) = 2$ or 3 and so $p(v_6) + p(v_7) \geq 8$. Thus we can move one pebble to v_1 or v_3 from the vertices v_6 and v_7. Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$. Otherwise, we can move one pebble to v_1 or v_3 if $p(v_4) = 1$ or $p(v_9) = 1$. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) = 0$ and $p(v_9) = 0$ and so $p(v_6) + p(v_7) \geq 12$. Thus we can move four pebbles to v_9 from the vertices v_5, v_6 and v_7. Assume $p(v_5) \leq 1$. In a similar way, we assume that $p(v_7) \leq 1$. Thus we have $p(v_6) \geq 9$. If $p(v_1) = 1$ or $p(v_3) = 1$ then clearly we are done. Let $p(v_1) = 0$ and $p(v_3) = 0$ and so $p(v_6) \geq 11$. Let $p(v_5) = 1$. We move five pebbles to v_5 from v_6. Clearly, we are done if $p(v_4) = 1$ or $p(v_9) = 1$. Assume $p(v_4) = p(v_9) = 0$ and so $p(v_6) \geq 13$. If $p(v_7) = 1$ then we move one pebble to v_7 and then we move three pebbles to v_5 from v_6 and hence we are done since v_9 receives four pebbles from v_5 and v_7. Let $p(v_7) = 0$ and so $p(v_6) \geq 14$. We move seven pebbles to v_5 from v_6 and hence we are done easily. Assume $p(v_5) = 0$. In a similar way, we may assume that $p(v_7) = 0$. Thus, $p(v_6) \geq 13$. If $p(v_4) = 1$ or $p(v_8) = 1$ or $p(v_9) = 1$ then we move three pebbles to v_4 or v_8 or v_9 and hence we are done. Assume $p(v_4) = p(v_8) = p(v_9) = 0$ and so $p(v_6) = 16$. Thus we can move one pebble to v_2 easily. □

Theorem 2.3 *For the Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.*

Proof Put fifteen pebbles at v_6 and one pebble each at v_8 and v_{10}. Clearly we cannot move a pebble to v_2. Thus $f(J_{2,5}) \geq 18$.

To prove that $f(J_{2,5}) \leq 18$, we have the following cases:

Case 1. Let v_{11} be the target vertex.

Clearly, $p(v_{11}) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq 13$, there exists a vertex, say v_2, such that $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_{11} easily. If $p(v_{10}) \geq 2$ or $p(v_4) \geq 2$ then also we can move one pebble to v_{11}. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$ and $p(v_{10}) \leq 1$. Thus, we can move one pebble to v_{11} easily, since $p(v_2) \geq 4$.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $i \in \{2, 10, 11\}$ by Remark 1.24. Let $p(v_3) \geq 2$. If $p(v_3) \geq 4$ or a vertex of $S_1 - \{v_1, v_3\}$ contains two or more pebbles then we can move one pebble to v_1 easily through v_{11}. So, assume $p(v_3) = 2$ or 3 and no vertex of $S_1 - \{v_1, v_3\}$ contain more than one pebble. Clearly, $p(v_6) + p(v_8) \geq 7$ and hence we can move one pebble to v_{11} from v_6 or v_8 and hence we are done, since $p(v_3) \geq 2$. Assume $p(v_3) \leq 1$. In a similar way, we assume that $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3\}$. Clearly, $p(v_4) + p(v_6) + p(v_8) \geq 11$. Let $p(v_4) \geq 4$. If $p(v_6) \geq 4$ or $p(v_8) \geq 4$ or $p(v_6) \geq 2$ and $p(v_8) \geq 2$ then we can move one pebble to v_{11}. Since $p(v_4) \geq 4$, we can move another one pebble to v_{11} from v_4 and hence one pebble can be moved to v_1. Assume $p(v_6) \leq 3$ and $p(v_8) \leq 3$ such that we cannot move two pebbles to v_7. Thus $p(v_6) + p(v_8) \leq 4$ and so $p(v_4) \geq 8$ and hence we can move one pebble to v_1 from v_4. Assume $p(v_4) \leq 3$. Similarly, $p(v_8) \leq 3$. We have $p(v_6) \geq 6$. Clearly, we are done if $p(v_5) = 1$ or $p(v_7) = 1$. Otherwise, $p(v_6) \geq 8$ and hence we can move one pebble to v_1 easily.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$ by Remark 1.24. Let $p(v_5) \geq 4$. If $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$ or $p(v_{11}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{10}) \leq 1$ and $p(v_{11}) \leq 1$. Also we assume that $p(v_7)+p(v_9) \leq 4$ such that we cannot move two pebbles to v_{11}. Let $p(v_7) \geq 2$ and so $p(v_9) \leq 1$. If $p(v_{11}) = 1$ or $p(v_5) \geq 6$ then clearly, we are done. Assume $p(v_{11}) = 0$ and $p(v_5) = 4$ or 5. Thus $p(v_6)+p(v_8) \geq 7$ and we can move one pebble to v_{11} from v_6 or v_8 and hence we are done. Assume $p(v_7) \leq 1$. In a similar way, we may assume that $p(v_9) \leq 1$. Let $p(v_5) = 6$ or 7 and so $p(v_6) + p(v_8) \geq 6$. Thus we can move one pebble to v_{11} from v_6 and v_8. Assume $p(v_5) = 4$ or 5 and so $p(v_6) + p(v_8) \geq 8$. If $p(v_7) = 1$ or $p(v_{11}) = 1$ then we can move one pebble to v_2 easily through v_{11}. Let $p(v_7) = p(v_{11}) = 0$ and so $p(v_6)+p(v_8) \geq 10$. Clearly, we can move two pebbles to v_{11} from v_6 and v_8 and hence we are done since $p(v_5) \geq 4$. Assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_7) \leq 3$ and $p(v_9) \leq 3$.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_{11}) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$. Assume $p(v_1) = p(v_3) = p(v_{11}) = 0$ and $p(v_4) \leq 1$, $p(v_{10}) \leq 1$. Clearly, $p(v_6) + p(v_8) \geq 7$ and hence we can move one pebble to v_{11} from v_6 or v_8. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1 - \{v_1, v_3\}$.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$ or $p(v_{11}) \geq 2$. Let $p(v_{11}) = 1$ and so we can move three pebbles to v_{11} from the two vertices of $S_1 - \{v_1, v_3\}$ and v_6 or v_8. Assume $p(v_{11}) = 0$ and so $p(v_6) + p(v_8) \geq 9$. Thus we can move two pebbles to v_{11} from the vertices v_6 and v_8 and then we move two more pebbles to v_{11} from the two vertices of $S_1 - \{v_1, v_3\}$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$ or $p(v_{11}) \geq 2$. Let $p(v_{11}) = 1$ and so $p(v_6) + p(v_8) \geq 10$. Thus we can move three pebbles to v_{11} from the vertex of $S_1 - \{v_1, v_3\}$ and the vertices v_6 and v_8. Assume $p(v_{11}) = 0$ and let v_5 is the vertex of $S_1 - \{v_1, v_3\}$ contains more than one pebble on it. So $p(v_6) + p(v_8) \geq 12$. If $p(v_7) = 1$ then we can move three pebbles to v_{11} from v_6 and v_8 and hence we are done since $p(v_5) \geq 2$. Assume $p(v_7) = 0$ and so we can move three pebbles to v_{11} from v_6 and v_8 and hence we are done. In a similar way, we can move one pebble to v_2 if $p(v_9) \geq 2$ and $p(v_7) \geq 2$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$ or $p(v_{11}) \geq 2$. Thus we have $p(v_6) + p(v_8) \geq 12$. Let $p(v_{11}) = 1$. Clearly we can move three pebbles to v_{11} if $p(v_7) = 1$. Assume $p(v_7) = 0$ and so we can move three pebbles to v_{11} since $p(v_6) + p(v_8) \geq 13$ and hence we are done. Assume $p(v_{11}) = 0$. Without loss of generality, we let $p(v_6) \geq 7$. If $p(v_4) = 1$ or $p(v_5) = 1$ or $p(v_7) = 1$ then we can move two pebbles to v_3 and hence we are done. Assume $p(v_4) = p(v_5) = p(v_7) = 0$. Let $p(v_8) \geq 2$. If $p(v_9) = 1$ then we move one pebble to v_{11} and then we move another three pebbles to v_{11} from v_6 and v_8 since $p(v_6) + p(v_8) - 2 \geq 14$ and hence we are done. Assume $p(v_9) = 0$ and so $p(v_6) + p(v_8) \geq 17$. Clearly we can move one pebble to v_2 from v_6 and v_8. □

Theorem 2.4 *For the Jahangir graph $J_{2,6}$, $f(J_{2,6}) = 21$.*

Proof Put fifteen pebbles at v_6, three pebbles at v_{10} and one pebble each at v_8 and v_{12}. Then, we cannot move a pebble v_2. Thus, $f(J_{2,6}) \geq 21$.

To prove that $f(J_{2,6}) \leq 21$, we have the following cases:

Case 1. Let v_{13} be the target vertex.

Clearly, $p(v_{13}) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq 15$, there exists a vertex, say v_2, such that $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_{13} easily. If $p(v_{12}) \geq 2$ or $p(v_4) \geq 2$ then also we can move one pebble to v_{13}. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$ and $p(v_{10}) \leq 1$. Thus, we can move one pebble to v_{13} easily, since $p(v_2) \geq 4$.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $i \in \{2, 12, 13\}$ by Remark 1.24. Let $p(v_3) \geq 2$. If $p(v_2) = 1$ or $p(v_{13}) = 1$ or a vertex of $S_1 - \{v_1, v_3\}$ has more than one pebble then we can move one pebble to v_1 easily. Otherwise, there exists a vertex, say v_6, of $S_2 - \{v_2, v_{12}\}$, contains more than three pebbles and hence we are done. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1\}$. Clearly, $S_2 - \{v_2, v_{12}\} \geq 13$, and so we can move two pebbles to v_{13} and hence we are done.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$ by Remark 1.24. Let $p(v_5) \geq 4$. If $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$ or $p(v_{13}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{12}) \leq 1$ and $p(v_{13}) \leq 1$. Also we assume that $p(v_7) + p(v_9) + p(v_{11}) \leq 5$ such that we cannot move two pebbles to v_{13}. Let $p(v_7) \geq 2$ and so $p(v_9) \leq 1$ and $p(v_{11}) \leq 1$. If $p(v_{13}) = 1$ or $p(v_5) \geq 6$ then clearly, we are done. Assume $p(v_{13}) = 0$ and $p(v_5) = 4$ or 5. Thus $p(v_6) + p(v_8) + p(v_{10}) \geq 9$ and we can move one pebble to v_{13} from v_6, v_8 and v_{10} and hence we are done. Assume $p(v_7) \leq 1$. In a similar way, we may assume that $p(v_9) \leq 1$ and $p(v_{11}) \leq 1$. Let $p(v_5) = 6$ or 7 and so $p(v_6) + p(v_8) + p(v_{10}) \geq 8$. Thus we can move one pebble to v_{13} from v_6, v_8 and v_{10}. Assume $p(v_5) = 4$ or 5 and so $p(v_6) + p(v_8) + p(v_{10}) \geq 10$. If $p(v_7) = 1$ or $p(v_9) = 1$ or $p(v_{13}) = 1$ then we can move one pebble to v_2 easily through v_{13}. Let $p(v_7) = p(v_9) = p(v_{13}) = 0$ and so $p(v_6) + p(v_8) + p(v_{10}) \geq 13$. Clearly, we can move two pebbles to v_{13} from v_6, v_8 and v_{10} and hence we are done since $p(v_5) \geq 4$. Assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_{11}) \leq 3$, $p(v_7) \leq 3$, and $p(v_9) \leq 3$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move four pebbles to v_{13} and hence one pebble can be moved to v_2 from v_{13}.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_{13}) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$. Assume $p(v_1) = p(v_3) = p(v_{13}) = 0$ and $p(v_4) \leq 1$, $p(v_{12}) \leq 1$. Clearly, $p(v_6)+p(v_8)+p(v_{10}) \geq 9$ and hence we can move one pebble to v_{13} from v_6, v_8 and v_{10}. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1 - \{v_1, v_3\}$.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$ or $p(v_{13}) \geq 2$. Let $p(v_{13}) = 1$ and so we can move three pebbles to v_{13} from the two vertices of $S_1 - \{v_1, v_3\}$ and v_6, v_8 and v_{10}. Assume $p(v_{13}) = 0$ and so $p(v_6) + p(v_8) + p(v_{10}) \geq 11$. Thus we can move two pebbles to v_{13} from the vertices v_6, v_8 and v_{10} and then we move two more pebbles to v_{13} from the two vertices of $S_1 - \{v_1, v_3\}$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$ or $p(v_{13}) \geq 2$. Let $p(v_{13}) = 1$ and so $p(v_6) + p(v_8) + p(v_{10}) \geq 12$. Thus we can move three pebbles to v_{13} from the vertex of $S_1 - \{v_1, v_3\}$ and the vertices v_6, v_8 and v_{10}. Assume $p(v_{13}) = 0$ and let v_5 is the vertex of $S_1 - \{v_1, v_3\}$ contains more than one pebble on it. So $p(v_6) + p(v_8) + p(v_{10}) \geq 13$. If $p(v_7) = 1$ or $p(v_9) = 1$ then we can move three pebbles to v_{13} from v_6, v_8 and v_{10} and hence we are done since $p(v_5) \geq 2$. Assume $p(v_7) = p(v_9) = 0$ and so we can move three pebbles to v_{13} from v_6, v_8 and v_{10} and hence we are done. In a similar way, we can move one pebble to v_2 if $p(v_{11}) \geq 2$, $p(v_7) \geq 2$ and $p(v_9) \geq 2$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$ or $p(v_{13}) \geq 2$. Thus we have $p(v_6) + p(v_8) + p(v_{10}) \geq 14$. Let $p(v_{13}) = 1$. Clearly we can move three pebbles to v_{13} if $p(v_7) = 1$ or $p(v_9) = 1$. Assume $p(v_7) = p(v_9) = 0$ and so we can move three pebbles to v_{13} since $p(v_6) + p(v_8) + p(v_{10}) \geq 15$ and hence we are done. Assume $p(v_{13}) = 0$. Without loss of generality, we let $p(v_6) \geq 5$. If $p(v_4) = 1$ or $p(v_5) = 1$ or $p(v_7) = 1$ then we can move two pebbles to v_3 and hence we are done. Assume $p(v_4) = p(v_5) = p(v_7) = 0$. Let $p(v_8) \geq 2$. If $p(v_9) = 1$ then we move one pebble to v_{13} and then we move another three pebbles to v_{13} from v_6, v_8 and v_{10} since $p(v_6) + p(v_8) + p(v_{10}) - 2 \geq 16$ and hence we are done. Assume $p(v_9) = 0$ and so $p(v_6) + p(v_8) + p(v_{10}) \geq 20$. Clearly we can move one pebble to v_2 from v_6, v_8 and v_{10}.□

Theorem 2.5 *For the Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.*

Proof Put fifteen pebbles at v_6, three pebbles at v_{10} and one pebble each at v_8, v_{14}, v_{12}, and v_{13}. Then, we cannot move a pebble to v_2. Thus, $f(J_{2,7}) \geq 23$.

To prove that $f(J_{2,7}) \leq 23$, we have the following cases:

Case 1. Let v_{15} be the target vertex.

Clearly, $p(v_{15}) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq 16$, there exists a vertex, say v_2, such that $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_{15} easily. If $p(v_{14}) \geq 2$ or $p(v_4) \geq 2$ then also we can move one pebble to v_{15}. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$ and $p(v_{14}) \leq 1$. Thus, we can move one pebble to v_{15} easily, since $p(v_2) \geq 4$.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $i \in \{2, 14, 15\}$ by Remark 1.24. Let $p(v_3) \geq 2$. If

$p(v_2) = 1$ or $p(v_{15}) = 1$ or a vertex of $S_1 - \{v_1, v_3\}$ has more than one pebble then we can move one pebble to v_1 easily. Otherwise, there exists a vertex, say v_6, of $S_2 - \{v_2, v_{14}\}$, contains more than two pebbles and hence we are done if $p(v_5) = 1$ or $p(v_7) = 1$. Let $p(v_5) = p(v_7) = 0$ and so $p(v_6) \geq 4$ and hence we are done. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1\}$. Clearly, $p(S_2 - \{v_2, v_{14}\}) \geq 14$, and so we can move two pebbles to v_{15} and hence we are done.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$ by Remark 1.24. Let $p(v_5) \geq 4$. If $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$ or $p(v_{15}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{14}) \leq 1$ and $p(v_{15}) \leq 1$.Also we assume that $p(v_7) + p(v_9) + p(v_{11}) + p(v_{13}) \leq 6$ such that we cannot move two pebbles to v_{15}. Let $p(v_7) \geq 2$ and so $p(v_9) \leq 1$, $p(v_{11}) \leq 1$ and $p(v_{13}) \leq 1$. If $p(v_{15}) = 1$ or $p(v_5) \geq 6$ then clearly, we are done. Assume $p(v_{15}) = 0$ and $p(v_5) = 4$ or 5. Thus $p(v_6) + p(v_8) + p(v_{10}) + p(v_{12}) \geq 10$ and we can move one pebble to v_{15} from v_6, v_8, v_{10} and v_{12} and hence we are done. Assume $p(v_7) \leq 1$. In a similar way, we may assume that $p(v_9) \leq 1$, $p(v_{11}) \leq 1$ and $p(v_{13}) \leq 1$. Let $p(v_5) = 6$ or 7 and so $p(v_6) + p(v_8) + p(v_{10}) + p(v_{12}) \geq 10$. Thus we can move one pebble to v_{15} from v_6, v_8, v_{10} and v_{12}. Assume $p(v_5) = 4$ or 5 and so $p(v_6) + p(v_8) + p(v_{10}) + p(v_{12}) \geq 12$. If $p(v_7) = 1$ or $p(v_9) = 1$ or $p(v_{11}) = 1$ or $p(v_{15}) = 1$ then we can move one pebble to v_2 easily through v_{15}. Let $p(v_7) = p(v_9) = p(v_{11}) = p(v_{15}) = 0$ and so $p(v_6) + p(v_8) + p(v_{10}) + p(v_{12}) \geq 15$. Clearly, we can move two pebbles to v_{15} from v_6, v_8, v_{10} and v_{12} and hence we are done since $p(v_5) \geq 4$. Assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_{13}) \leq 3$, $p(v_7) \leq 3$, $p(v_9) \leq 3$ and $p(v_{11}) \leq 3$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move four pebbles to v_{15} and hence one pebble can be moved to v_2 from v_{15}.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_{15}) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$. Assume $p(v_1) = p(v_3) = p(v_{15}) = 0$ and $p(v_4) \leq 1$, $p(v_{14}) \leq 1$. Clearly, $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 10$ and hence we can move one pebble to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1 - \{v_1, v_3\}$.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$ or $p(v_{15}) \geq 2$. Let $p(v_{15}) = 1$ and so we can move three pebbles to v_{15} from the two vertices of $S_1 - \{v_1, v_3\}$ and the vertices of $S_2 - \{v_2, v_4, v_{14}\}$. Assume $p(v_{15}) = 0$ and so $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 12$. Thus we can move two pebbles to v_{15} from the vertices $p(S_2 - \{v_2, v_4, v_{14}\})$ and then we move two more pebbles to v_{15} from the two vertices of $S_1 - \{v_1, v_3\}$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$ or $p(v_{15}) \geq 2$. Let $p(v_{15}) = 1$ and so $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 13$. Thus we can move three pebbles to v_{15} from the vertex of $S_1 - \{v_1, v_3\}$ and the vertices $S_2 - \{v_2, v_4, v_{14}\}$. Assume $p(v_{15}) = 0$ and let v_5 is the vertex of $S_1 - \{v_1, v_3\}$ contains more than one pebble on it. So $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 14$. If $p(v_7) = 1$ or $p(v_9) = 1$ or $p(v_{11}) = 1$ then we can move three pebbles to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$ and hence we are done since $p(v_5) \geq 2$. Assume $p(v_7) = p(v_9) = p(v_{11}) = 0$

and so we can move three pebbles to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$ and hence we are done. In a similar way, we can move one pebble to v_2 if $p(v_i) \geq 2$, where $v_i \in S_1 - \{v_1, v_3, v_5\}$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$ or $p(v_{15}) \geq 2$. Thus we have $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 15$. Let $p(v_{15}) = 1$. Clearly we can move three pebbles to v_{15} if $p(v_7) = 1$ or $p(v_9) = 1$ or $p(v_{11}) = 1$. Assume $p(v_7) = p(v_9) = p(v_{11}) = 0$ and so we can move three pebbles to v_{15} since $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 18$ and hence we are done. Assume $p(v_{15}) = 0$. Without loss of generality, we let $p(v_6) \geq 5$. If $p(v_4) = 1$ or $p(v_5) = 1$ or $p(v_7) = 1$ then we can move two pebbles to v_3 and hence we are done. Assume $p(v_4) = p(v_5) = p(v_7) = 0$. Let $p(v_8) \geq 2$. If $p(v_9) = 1$ then we move one pebble to v_{15} and then we move another three pebbles to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$, since $p(S_2 - \{v_2, v_4, v_{14}\}) - 2 \geq 17$ and hence we are done. Assume $p(v_9) = 0$ and so $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 20$. Clearly we can move one pebble to v_2 from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$. □

Theorem 2.6 *For the Jahangir graph $J_{2,m}$ where $m \geq 8$, $f(J_{2,m}) = 2m + 10$.*

Proof If m is even, then consider the following configuration C_1 such that $C_1(v_2) = 0$, $C_1(v_{m+2}) = 15$, $C_1(v_{m-2}) = 3$, $C_1(v_{m+6}) = 3$, $C_1(x) = 1$ where $x \notin N[v_2]$, $x \notin N[v_{m+2}]$, $x \notin N[v_{m-2}]$, and $x \notin N[v_{m+6}]$ and $C_1(y) = 0$ for all other vertices of $J_{2,m}$. If m is odd, then consider the following configuration C_2 such that $C_2(v_2) = 0$, $C_2(v_{m+1}) = 15$, $C_2(v_{m-3}) = 3$, $C_2(v_{m+5}) = 3$, $C_2(x) = 1$ where $x \notin N[v_2]$, $x \notin N[v_{m+1}]$, $x \notin N[v_{m-3}]$, and $x \notin N[v_{m+5}]$ and $C_1(y) = 0$ for all other vertices of $J_{2,m}$. Then, we cannot move a pebble to v_2. The total number of pebbles placed in both configurations is $15+2(3)+(m-4)(1)+(m-8)(1) = 2m+9$. Therefore, $f(J_{2,m}) \geq 2m + 10$.

To prove that $f(J_{2,m}) \leq 2m + 10$, for $m \geq 8$, we have the following cases:

Case 1. Let v_{2m+1} be the target vertex.

Clearly, $p(v_{2m+1}) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq m+10$, there exists a vertex, say v_2, such that $p(v_2) \geq 2$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_{2m+1} easily. If $p(v_{2m}) \geq 2$ or $p(v_4) \geq 2$ then also we can move one pebble to v_{2m+1}. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$ and $p(v_{2m}) \leq 1$. Thus, we can move one pebble to v_{2m+1} easily, since $p(v_2) \geq 4$.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $i \in \{2, 2m, 2m+1\}$ by Remark 1.24. Let $p(v_3) \geq 2$. If $p(v_2) = 1$ or $p(v_{2m+1}) = 1$ or a vertex of $S_1 - \{v_1, v_3\}$ has more than one pebble then we can move one pebble to v_1 easily. Otherwise, there exists a vertex, say v_6, of $S_2 - \{v_2, v_{2m}\}$, contains more than one pebble and hence we are done if $p(v_5) = 1$ or $p(v_7) = 1$. Let $p(v_5) = p(v_7) = 0$ and so $p(v_6) \geq 4$ and hence we are done. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1\}$. Clearly, $p(S_2 - \{v_2, v_{2m}\}) \geq m + 8$, and so we can move two pebbles to v_{2m+1} and hence we are done.

Case 3: Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$ by Remark 1.24. Let $p(v_5) \geq 4$. If $p(v_1) = 1$ or

$p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$ or $p(v_{2m+1}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{2m}) \leq 1$ and $p(v_{2m+1}) \leq 1$. Also we assume that $p(S_1 - \{v_1, v_3, v_5\}) \leq m-1$ such that we cannot move two pebbles to v_{2m+1}. Let $p(v_7) \geq 2$ and so $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3, v_5, v_7\}$. If $p(v_{2m+1}) = 1$ or $p(v_5) \geq 6$ then clearly, we are done. Assume $p(v_{2m+1}) = 0$ and $p(v_5) = 4$ or 5. Thus $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+4$ and we can move one pebble to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done. Assume $p(v_7) \leq 1$. In a similar way, we may assume that $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3, v_5, v_7\}$. Let $p(v_5) = 6$ or 7 and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+4$. Thus we can move one pebble to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$. Assume $p(v_5) = 4$ or 5 and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+6$. If $p(v_j) = 1$, a vertex v_j of $S_1 - \{v_1, v_3, v_5, v_{2m-1}\}$ then we can move one pebble to v_2 easily through v_{2m+1}. Let $p(S_1 - \{v_1, v_3, v_5, v_{2m-1}\}) = 0$ and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq 2m+2$. Clearly, we can move two pebbles to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done since $p(v_5) \geq 4$. Assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_k) \leq 3$, for all $v_k \in S_1 - \{v_1, v_3, v_5\}$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move four pebbles to v_{2m+1} and hence one pebble can be moved to v_2 from v_{2m+1}

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_{2m+1}) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$. Assume $p(v_1) = p(v_3) = p(v_{2m+1}) = 0$ and $p(v_4) \leq 1$, $p(v_{2m}) \leq 1$. Clearly, $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+4$ and hence we can move one pebble to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1 - \{v_1, v_3\}$.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$ or $p(v_{2m+1}) \geq 2$. Let $p(v_{2m+1}) = 1$ and so we can move three pebbles to v_{2m+1} from the two vertices of $S_1 - \{v_1, v_3\}$ and the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$. Assume $p(v_{2m+1}) = 0$ and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+6$. Thus we can move two pebbles to v_{2m+1} from the vertices $S_2 - \{v_2, v_4, v_{2m}\}$ and then we move two more pebbles to v_{2m+1} from the two vertices of $S_1 - \{v_1, v_3\}$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$ or $p(v_{2m+1}) \geq 2$. Let $p(v_{2m+1}) = 1$ and so $p(S_2 - \{v_2, v_4, v_{14}\}) \geq m+7$. Thus we can move three pebbles to v_{2m+1} from the vertex of $S_1 - \{v_1, v_3\}$ and the vertices $S_2 - \{v_2, v_4, v_{2m}\}$. Assume $p(v_{2m+1}) = 0$ and let v_5 is the vertex of $S_1 - \{v_1, v_3\}$ contains more than one pebble on it. So $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+8$. If $p(v_j) = 1$, a vertex v_j of $S_1 - \{v_1, v_3, v_5, v_{2m-1}\}$ then we can move three pebbles to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done since $p(v_5) \geq 2$. Assume $p(S_1 - \{v_1, v_3, v_5, v_{2m-1}\}) = 0$ and so we can move three pebbles to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done. In a similar way, we can move one pebble to v_2 if $p(v_i) \geq 2$, where $v_i \in S_1 - \{v_1, v_3, v_5\}$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$ or $p(v_{2m+1}) \geq 2$. Thus we have $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 9$. Let $p(v_{2m+1}) = 1$. Clearly we can move three pebbles to v_{2m+1} if a vertex v_j of $S_1 - \{v_1, v_3, v_5, v_{2m-1}\}$ such that $p(v_j) = 1$. Assume $p(S_1 - \{v_1, v_3, v_5, v_{2m-1}\}) = 0$ and so we can move three pebbles to v_{2m+1} since $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 12$ and hence we are done. Assume $p(v_{2m+1}) = 0$. Without loss of generality, we let $p(v_6) \geq 2$. If $p(v_5) = 1$ or $p(v_7) = 1$ then we can move two pebbles to v_3 and hence we are done. Assume $p(v_5) = p(v_7) = 0$. Let $p(v_8) \geq 2$. If $p(v_9) = 1$ then we move one pebble to v_{2m+1} and then we move another three pebbles to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$, since $p(S_2 - \{v_2, v_4, v_{2m}\}) - 2 \geq m + 11$ and hence we are done. Assume $p(v_9) = 0$ and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 12$. Clearly we can move one pebble to v_2 from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$. □

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