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Tchebychev and Brahmagupta Polynomials and Golden Ratio: Two New Interconnections

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Abstract: The present paper explores interconnections between sequences related to convergents of generalized golden ratios and four kinds of Tchebychev polynomials. By defining and adding Brahmagupta polynomials of third and fourth kind, the paper also interconnects the four kinds of Brahmagupta polynomials to the four kinds of Tchebychev Polynomials respectively. In this way, the present paper provides two spectacular views of Tchebychev polynomials of all four kinds through golden ratio and Brahmagupta polynomials.

Key Words: Fibonacci and Lucas numbers, Tchebychev polynomials and Brahmagupta polynomials.

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§1. Introduction

The algebraic integer $\Phi = \frac{-1+\sqrt{5}}{2}$ obtained as one of the roots of the quadratic equation $t^2 + t - 1 = 0$ is well known in the literature as golden ratio. Φ is also given by the beautiful continued fraction expansion

$$\Phi = \frac{1}{1} {}_{+} \frac{1}{1} {}_{+} \frac{1}{1} {}_{+} \; \cdots \; {}_{+} \frac{1}{1} {}_{+} \; \cdots \; = \lim_{n \longrightarrow \infty} \frac{F_n}{F_{n+1}}, \tag{1}$$

where F_n is the well known Fibonacci numbers. Approximating Φ by $\dfrac{F_n}{F_{n+1}}$ for a suitable n, ancient Greek architects have constructed what are called golden triangles, golden rectangles and so on, which have enhanced the beauty of architecture of their buildings. An elegant number theoretic result is that (L_n, F_n), where $L_n = F_{n-1} + F_{n+1}$ is well known as Lucas number, satisfies the quadratic Diophantine equation

$$L_n^2 - 5F_n^2 = 4(-1)^n.$$

For more details please refer [6], [9], [10]. Choosing one of the roots of $xt^2 + t - 1 = 0$ and $t^2 + xt - 1 = 0$ one gets the following two generalizations of golden ratio with interesting

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continued fraction expansions ([7], [11])

$$\Phi_1(x) = \frac{-1+\sqrt{1+4x}}{2x} = \frac{1}{1}{}_{+}\frac{x}{1}{}_{+}\frac{x}{1}{}_{+}\frac{x}{1}{}_{+\cdots+}, \qquad x \geq 0, \tag{2}$$

$$\Phi_2(x) = \frac{-x+\sqrt{x^2+4}}{2} = \frac{1}{x}{}_{+}\frac{1}{x}{}_{+}\frac{1}{x}{}_{+}\frac{1}{x}{}_{+\cdots+}, \qquad x > 0. \tag{3}$$

They have a nontrivial interconnection become

$$\Phi_2(x) = \frac{1}{x}\ \Phi_1\left(\frac{1}{x^2}\right). \tag{4}$$

When $x = 1$, $\Phi_1(1) = \Phi_2(1) = \Phi$.

The four kinds of Tchebychev polynomials well studied in the literature ([1], [3], [7]) are described below when $x = \cos\theta$:

$$\begin{aligned} T_n(x) &= \cos n\theta \quad ;\ T_0 = 1,\ \ T_1(x) = x, \cdots\ ,\\ U_n(x) &= \frac{\sin(n+1)\theta}{\sin\theta}\ \ ;\ \ U_0 = 1,\ U_1(x) = 2\ x, \cdots\ ,\\ V_n(x) &= \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\ \theta}\ \ ;\ V_0 = 1,\ \ V_1(x) = 2\ x - 1, \cdots\ ,\\ W_n(x) &= \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\ \ ;\ \ W_0 = 1,\ W_1(x) = 2\ x + 1, \cdots\ . \end{aligned}$$

They satisfy the three term recurrence relations

$$P_{n+1}(x) = 2\ xP_n(x) - P_{n-1}(x)$$

with the above initial condition. Their interrelations are nicely described below in the literature ([1], [3], [7]):

$$\begin{aligned} U_n(x) &= \frac{T'_{n+1}(x)}{n+1},\\ V_n(x) &= \frac{T_{n+1}(x)+T_n(x)}{x+1} = U_{n+1}(x) - U_n(x)\\ \text{and}\qquad W_n(x) &= \frac{T_{n+1}(x)-T_n(x)}{x-1} = U_{n+1}(x) + U_n(x) = (-1)^n V_n(-x). \end{aligned}$$

Their link to trigonometric functions will yield the following worth quoting orthogonality properties ([1], [3], [7]):

$$\int_{-1}^{1} T_m(x)T_n(x)\frac{1}{\sqrt{1-x^2}}dx = \begin{cases} 0, & m \neq n;\\ \pi, & m = n = 0;\\ \frac{\pi}{2}, & m = n \neq 0, \end{cases}$$

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}dx = \begin{cases} 0, & m \neq n; \\ \frac{\pi}{2}, & m = n, \end{cases}$$

$$\int_{-1}^{1} V_m(x)V_n(x)\sqrt{\frac{1+x}{1-x}}dx = \begin{cases} 0, & m \neq n; \\ \pi, & m = n. \end{cases}$$

$$\int_{-1}^{1} W_m(x)W_n(x)\sqrt{\frac{1-x}{1+x}}dx = \begin{cases} 0, & m \neq n; \\ \pi, & m = n. \end{cases}$$

An amazing result on $\{T_{n+1}, U_n\}$ is that the continued fraction expansion ([11])

$$\sqrt{x^2-1} = x - \frac{1}{2x\,-}\,\frac{1}{2x\,-}\,\frac{1}{2x\,-}\,\frac{1}{2x\,-\cdots}, \qquad x > 1, \tag{5}$$

which is constructed using

$$\sqrt{x^2-1} = x - \frac{1}{x+\sqrt{x^2-1}}, \qquad x > 1$$

has the sequence of convergents

$$\left\{\frac{P(x)}{Q(x)}\right\} = \left\{\frac{x}{1}, \frac{2x^2-1}{2x}, \ \ldots\ , \frac{T_{n+1}(x)}{U_n(x)}, \ \ldots\right\}. \tag{6}$$

A related result is that the following continued fraction ([11])

$$\sqrt{\frac{x+1}{x-1}} = 1 + \frac{2}{2x-1\,-}\,\frac{1}{2x\,-}\,\frac{1}{2x\,-}\,\frac{1}{2x\,-\cdots}, \qquad x > 1, \tag{7}$$

which can also be written as

$$\sqrt{\frac{x+1}{x-1}} = 1 + \frac{2}{(x-1)+\sqrt{x^2-1}}, \qquad x > 1$$

has the sequence of convergents

$$\left\{\frac{\widetilde{P}(x)}{\widetilde{Q}(x)}\right\} = \left\{\frac{1}{1}, \frac{2x+1}{2x-1}, \ \ldots\ , \frac{W_n(x)}{V_n(x)}, \ \ldots\right\}. \tag{8}$$

A pair of two variable polynomials with a parameter $(x_n(x,y,t), y_n(x,y,t))$ is said to be Brahmagupta polynomials ([5], [6], [9]) if $x_n(x,y,t)$ and $y_n(x,y,t)$ satisfy

$$(x_n \pm y_n\sqrt{t}) = (x \pm y\sqrt{t})^n, \qquad n = 0,1,2,\ldots$$

or

$$x_n^2 - ty_n^2 = (x^2-ty^2)^n$$

or

$$(x_m^2 - ty_m^2)(x_n^2 - ty_n^2) = (x_m x_n + ty_m y_n)^2 - t(x_m y_n + x_n y_m)^2. \tag{9}$$

The last identity (9) is called Brahmagupta identity ([12]), which is a more general form of Diophantine identity

$$(x_m^2 + y_m^2)(x_n^2 + y_n^2) \quad = \quad (x_m x_n - y_m y_n)^2 + (x_m y_n + x_n y_m)^2.$$

Both x_n and y_n satisfy the following three term recurrence relations:

$$x_{n+1} = 2x\ x_n - (x^2 - ty^2)x_{n-1}, \quad x_0 = 1, \quad x_1 = x, \quad n = 1, 2, 3, \cdots \tag{10}$$

and

$$\frac{y_{n+1}}{y} = 2x\ \frac{y_n}{y} - (x^2 - ty^2)\ \frac{y_{n-1}}{y}, \quad \frac{y_1}{y} = 1, \quad \frac{y_2}{y} = 2x, \quad n = 2, 3, 4, \cdots . \tag{11}$$

They are related to golden ratio as well as Tchebychev polynomials by the following relations [9]:

(1) For $x = \dfrac{1}{2}$, $y = \dfrac{1}{2}$ and $t = 5$, one recovers easily

$$\begin{aligned} -x + y\sqrt{t} &= \Phi, \\ 2x_n &= L_n, \\ \frac{y_{n+1}}{y} &= F_{n+1}. \end{aligned}$$

(2) For $x^2 - ty^2 = 1$, one gets directly

$$x_n = T_n(x), \qquad \frac{y_{n+1}}{y} = U_n(x),\ n = 0, 1, 2, \cdots .$$

In the background of the above curious ideas and results the paper intends to do justice to its title. In the next section, the convergents of $\sqrt{\frac{x+1}{x-1}}$ related to $\Phi_1(x)$ are shown to be related to all the four kinds of Tchebychev polynomials in a rigors manner. The convergents of $\Phi_1(x)$ and $\Phi_2(x)$ are shown to be related to $U_n(x)$ and $V_n(x)$ only. In the third and the last section, first two kinds of Brahmagupta polynomials are shown to be related to $T_n(x)$ and $U_n(x)$.

The new things added are Brahmagupta polynomials of third and fourth kind which are defined with the help of Brahmagupta polynomials of second kind. Of course when $x^2 - ty^2 = 1$, all of them will become respective kinds of Tchebychev polynomials.

§2. Generalization of Golden Ratio and Expressions for Their Convergents Interms of Tchebychev Polynomials

First let us consider the generalization of the golden ratio

$$\Phi_1(x) = \frac{-1 + \sqrt{1 + 4x}}{2x} = \sum_{n=0}^{\infty} (-1)^n C_n x^n$$

valid for $|x| < \frac{1}{4}$, where $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the n^{th} Catalan number. The above series is a

Stieltje's series ([8], [11]) because

$$\begin{aligned}\frac{-1+\sqrt{1+4x}}{2x} &= \frac{1}{4}\int_0^4 \frac{dt}{\sqrt{1+xt}} \\ &= \frac{1}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{x}{1+\cdots}\,, \qquad x>0\end{aligned}$$

and the sequence of convergents is

$$\left\{\frac{P(x)}{Q(x)}\right\} = \left\{\frac{1}{1}, \frac{1}{x+1}, \frac{1+x}{1+2x}, \frac{1+2x}{1+3x+x^2},\ \cdots\ ,\ \frac{A_n(x)}{A_{n+1}(x)},\ \cdots\right\},$$

where

$$\begin{aligned}A_{n+1}(x) &= A_n(x) + xA_{n-1}(x), \\ A_1(x) &= 1, \quad A_2(x) = 1, \quad n = 2, 3, 4, \cdots .\end{aligned}$$

For $x = 1$, as expected one gets

$$A_n = F_n, \quad n = 1, 2, 3, \cdots .$$

In order to express $A_n(x)$ interms of Tchebychev polynomials, we use

$$\begin{aligned}\frac{1+\sqrt{1+4x}}{2} &= \left[\frac{-1+\sqrt{1+4x}}{2x}\right]^{-1} \\ &= 1\ +\ \frac{x}{1+}\,\frac{x}{1+\cdots+}\,\frac{x}{1+\cdots}\,, \qquad x>0\end{aligned} \tag{12}$$

and the sequence of convergents is

$$\left\{\frac{P(x)}{Q(x)}\right\} = \left\{\frac{1}{1}, \frac{1+x}{1}, \frac{1+2x}{1+x},\ \cdots\ ,\ \frac{A_{n+1}(x)}{A_n(x)},\ \cdots\ \right\}.$$

Let us apply the following transformation

$$x = \frac{1}{2(s-1)} \quad \text{or} \quad s-1 = \frac{1}{2x}, \quad x>0,$$

which enables us to wrote

$$\sqrt{1+4x} = \sqrt{\frac{s+1}{s-1}}.$$

Since

$$\begin{aligned}\sqrt{1+4x} &= 1+2x\ \left[\frac{-1+\sqrt{1+4x}}{2x}\right] \\ &= 1\ +\ \frac{2x}{1+}\,\frac{x}{1+}\,\frac{x}{1+\cdots+}\,\frac{x}{1+\cdots}\,, \qquad x>0.\end{aligned}$$

Using the above transformation

$$
\begin{aligned}
\sqrt{\frac{s+1}{s-1}} &= 1+\frac{\frac{1}{s-1}}{1}{}_{+}\frac{\frac{1}{2(s-1)}}{1}{}_{+\cdots} \\
&= 1 + \frac{1}{s-1}{}_{+}\frac{1}{2}{}_{+}\frac{1}{s-1}{}_{+}\frac{1}{2}{}_{+\cdots}, \qquad x>0,
\end{aligned} \tag{13}
$$

which is valid because $s = 1+\frac{1}{2x}$, $x > 0$ and the sequence of convergents is

$$
\left\{\frac{P(x)}{Q(x)}\right\} = \left\{\frac{1}{1}, \frac{s}{(s-1)}, \frac{2s+1}{2s-1}, \frac{2s^2-1}{2(s-1)2s}, \cdots, \frac{P_{2n-1}(s)}{Q_{2n-1}(s)}, \frac{P_{2n}(s)}{Q_{2n}(s)}, \cdots \right\}.
$$

The numerator and denominator polynomials of the continued fraction (13) satisfy the following relations:

(1) $P_{2n+1}(s) = 2P_{2n}(s) + P_{2n-1}(s)$;

(2) $P_{2n}(s) = (s-1)P_{2n-1}(s) + P_{2n-2}(s)$;

(3) $Q_{2n+1}(s) = 2Q_{2n}(s) + Q_{2n-1}(s)$;

(4) $Q_{2n}(s) = (s-1)Q_{2n-1}(s) + Q_{2n-2}(s)$.

Using the above relation, we get the following three term recurrence relation for the odd and the even convergents of the continued fraction (13):

$$
\begin{aligned}
P_{2n+1}(s) &= 2\ [(s-1)P_{2n-1}(s) + P_{2n+2}(s)] + P_{2n-1}(s) \\
&= 2s\ P_{2n-1}(s) + [2\ P_{2n-2}(s) - P_{2n-1}(s)], \\
P_{2n+1}(s) &= 2s\ P_{2n-1}(s) - P_{2n-3}(s)
\end{aligned}
$$

and

$$
\begin{aligned}
P_{2n}(s) &= (s-1)\ [2\ P_{2n-2}(s) + P_{2n-3}(s)] + P_{2n-2}(s) \\
&= 2s\ P_{2n-2}(s) + [(s-1)\ P_{2n-3}(s) - P_{2n-2}(s)], \\
P_{2n}(s) &= 2s\ P_{2n-2}(s) - P_{2n-4}(s).
\end{aligned}
$$

Similarly, we obtain the followings:

$$
Q_{2n+1}(s) = 2s\ Q_{2n-1}(s) - Q_{2n-3}(s)
$$

and

$$
Q_{2n}(s) = 2s\ Q_{2n-2}(s) - Q_{2n-4}(s).
$$

Since

$$
\begin{aligned}
P_1(s) &= 1, P_3(s) = 2s-1, \\
P_{2n-1}(s) &= V_{n-1}(s); \qquad n = 1,2,3,\cdots,
\end{aligned}
$$

$$\begin{aligned} Q_1(s) &= 1, Q_3(s) = 2s+1, \\ Q_{2n-1}(s) &= W_{n-1}(s); \quad n = 1,2,3,\cdots, \end{aligned}$$

$$\begin{aligned} P_2(s) &= s, P_4(s) = 2s^2 - 1, \\ P_{2n}(s) &= T_n(s); \quad n = 1,2,3,\cdots \end{aligned}$$

and

$$\begin{aligned} Q_2(s) &= (s-1), Q_4(s) = (s-1)2s, \\ Q_{2n}(s) &= (s-1)U_{n-1}(s); \quad n = 1,2,3,\cdots. \end{aligned}$$

The odd and even convergents of the continued fraction (12) are:

$$\begin{aligned} \frac{A_{2n}(x)}{A_{2n-1}(x)} = \frac{1}{2}\left[1 + \frac{W_{n-1}(s)}{V_{n-1}(s)}\right] &= \frac{U_{n-1}(s)}{V_{n-1}(s)} \\ &= \frac{x^{n-1}U_{n-1}\left(1+\frac{1}{2x}\right)}{x^{n-1}V_{n-1}\left(1+\frac{1}{2x}\right)} \end{aligned}$$

and

$$\begin{aligned} \frac{A_{2n+1}(x)}{A_{2n}(x)} = \frac{1}{2}\left[1 + \frac{T_n(s)}{(s-1)U_{n-1}(s)}\right] &= \frac{1}{2(s-1)}\left[\frac{(s-1)U_{n-1}(s)+T_n(s)}{U_{n-1}(s)}\right] \\ &= \frac{1}{2(s-1)}\,\frac{V_n(s)}{U_{n-1}(s)} \\ &= \frac{x^n V_n\left(1+\frac{1}{2x}\right)}{x^{n-1}U_{n-1}\left(1+\frac{1}{2x}\right)}. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} V_n\left(1+\frac{x}{2}\right) &= x^n\ A_{2n+1}\left(\frac{1}{x}\right), \\ U_n\left(1+\frac{x}{2}\right) &= x^n\ A_{2n+2}\left(\frac{1}{x}\right). \end{aligned}$$

Now, we obtain the odd and even convergents of the continued fraction (12) interms of second and third kind of Tchebychev polynomials

$$\frac{A_{2n-1}(x)}{A_{2n}(x)} = \frac{x^{n-1}V_{n-1}\left(1+\frac{1}{2x}\right)}{x^{n-1}U_{n-1}\left(1+\frac{1}{2x}\right)}$$

and

$$\frac{A_{2n}(x)}{A_{2n+1}(x)} = \frac{x^{n-1}U_{n-1}\left(1+\frac{1}{2x}\right)}{x^{n}V_{n}\left(1+\frac{1}{2x}\right)}.$$

(Similar results are derived in [7].)

Similarly the following continued fraction

$$\Phi_2(x) = \frac{-x+\sqrt{x^2+4}}{2} = \frac{1}{x}\left[\frac{-1+\sqrt{1+\frac{4}{x^2}}}{\frac{2}{x^2}}\right] = \frac{1}{x+}\,\frac{1}{x+}\cdots+\frac{1}{x+}\cdots, \qquad x>0 \tag{14}$$

has the following odd and even convergents:

$$\frac{B_{2n-1}(x)}{B_{2n}(x)} = \frac{1}{x}\,\frac{A_{2n-1}\left(\frac{1}{x^2}\right)}{A_{2n}\left(\frac{1}{x^2}\right)} = \frac{1}{x}\,\frac{(x^2)^{n-1}A_{2n-1}\left(\frac{1}{x^2}\right)}{(x^2)^{n-1}A_{2n}\left(\frac{1}{x^2}\right)} = \frac{1}{x}\,\frac{V_{n-1}\left(1+\frac{x^2}{2}\right)}{U_{n-1}\left(1+\frac{x^2}{2}\right)}$$

and

$$\frac{B_{2n}(x)}{B_{2n+1}(x)} = \frac{1}{x}\,\frac{A_{2n}\left(\frac{1}{x^2}\right)}{A_{2n+1}\left(\frac{1}{x^2}\right)} = \frac{1}{x}\,\frac{x^2(x^2)^{n-1}A_{2n}\left(\frac{1}{x^2}\right)}{(x^2)^{n}A_{2n+1}\left(\frac{1}{x^2}\right)} = x\,\frac{U_{n-1}\left(1+\frac{x^2}{2}\right)}{V_{n}\left(1+\frac{x^2}{2}\right)}.$$

Hence

$$A_{2n+1}(x) = x^nV_n\left(1+\frac{1}{2x}\right), \qquad A_{2n+2}(x) = x^nU_n\left(1+\frac{1}{2x}\right),$$

$$B_{2n+1}(x) = V_n\left(1+\frac{x^2}{2}\right), \qquad B_{2n+2}(x) = x\,U_n\left(1+\frac{x^2}{2}\right).$$

For $x=1$, we obtain

$$F_{2n+1} = A_{2n+1}(1) = B_{2n+1}(1) = V_n\left(\frac{3}{2}\right),$$

$$F_{2n+2} = A_{2n+2}(1) = B_{2n+2}(1) = U_n\left(\frac{3}{2}\right), \qquad n=0,1,2,3,\cdots.$$

§3. Connections Between Tchebychev Polynomials and Brahmagupta Polynomials of All Four Kinds

Brahmagupta polynomials have the following binet forms ([9]):

$$x_n(x,y;t) = \frac{1}{2}[(x+y\sqrt{t})^n + (x-y\sqrt{t})^n], \qquad n=0,1,2,3,\cdots$$

and

$$\frac{y_{n+1}(x,y;t)}{y} = \frac{1}{2y\sqrt{t}}[(x+y\sqrt{t})^{n+1} - (x-y\sqrt{t})^{n+1}], \qquad n=0,1,2,3,\cdots.$$

Put $\beta = x^2 - ty^2$ or $y\sqrt{t} = \sqrt{x^2-\beta}$, then we obtain

$$\begin{aligned}
x_n(x,y;t) &= \frac{1}{2}\,[(x+\sqrt{x^2-\beta})^n + (x-\sqrt{x^2-\beta})^n] \\
&= \frac{\beta^{\frac{n}{2}}}{2}\left[\left(\frac{x}{\sqrt{\beta}}+\sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2-1}\right)^n + \left(\frac{x}{\sqrt{\beta}}-\sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2-1}\right)^n\right] \\
&= \frac{\beta^{\frac{n}{2}}}{2} T_n\left(\frac{x}{\sqrt{\beta}}\right)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{y_{n+1}(x,y;t)}{y} &= \frac{1}{2\sqrt{x^2-\beta}}[(x+\sqrt{x^2-\beta})^{n+1} - (x-\sqrt{x^2-\beta})^{n+1}] \\
&= \frac{\beta^{\frac{n}{2}}}{2\sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2-1}}\left[\left(\frac{x}{\sqrt{\beta}}+\sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2-1}\right)^{n+1}\right. \\
&\qquad \left. - \left(\frac{x}{\sqrt{\beta}}-\sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2-1}\right)^{n+1}\right] \\
&= \beta^{\frac{n}{2}}\, U_n\left(\frac{x}{\sqrt{\beta}}\right).
\end{aligned}$$

Motivated by Tchebychev polynomials of third and forth kind, we can define Brahmagupta polynomials of third and forth kind respectively as follows:

$$\begin{aligned}
v_n(x,y;t) &= \frac{y_{n+1}(x,y;t)}{y} - \beta\,\frac{y_n(x,y;t)}{y}, \\
w_n(x,y;t) &= \frac{y_{n+1}(x,y;t)}{y} + \beta\,\frac{y_n(x,y;t)}{y}.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
v_0 &= w_0 = \frac{y_1}{y} = 1, \\
v_1 &= 2x-\beta, \quad w_1 = 2x+\beta, \\
v_{n+1}(x,y;t) &= 2x\; v_n(x,y;t) - \beta\; v_{n-1}(x,y;t), \\
w_{n+1}(x,y;t) &= 2x\; w_n(x,y;t) - \beta\; w_{n-1}(x,y;t).
\end{aligned}$$

Hence

$$\begin{aligned}
v_{n+1}(x,y;t) &= \beta^{\frac{n}{2}}\left[U_n\left(\frac{x}{\sqrt{\beta}}\right) - \sqrt{\beta}\,U_{n-1}\left(\frac{x}{\sqrt{\beta}}\right)\right], \\
w_{n+1}(x,y;t) &= \beta^{\frac{n}{2}}\left[U_n\left(\frac{x}{\sqrt{\beta}}\right) + \sqrt{\beta}\,U_{n-1}\left(\frac{x}{\sqrt{\beta}}\right)\right].
\end{aligned}$$

If $\beta = 1$, we get back

$$\begin{aligned} v_n\left(x, y;\ \frac{x^2-1}{y^2}\right) &= U_n(x) - U_{n-1}(x) = V_n(x), \\ w_n\left(x, y;\ \frac{x^2-1}{y^2}\right) &= U_n(x) + U_{n-1}(x) = W_n(x), \end{aligned}$$

which are the Tchebychev polynomials of third and forth kind respectively.

The following are generating functions of $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ ([1], [2], [4]):

(1) $T(s) = 2\displaystyle\sum_{n=1}^{\infty} \frac{T_n(x)}{n} s^n$;

(2) $U(s) = \displaystyle\sum_{n=1}^{\infty} U_n(x) s^n = \frac{1}{1-2xs+s^2}$;

(3) $V(s) = \displaystyle\sum_{n=0}^{\infty} V_n(x) s^n = (1-s)\ U(s)$;

(4) $W(s) = \displaystyle\sum_{n=0}^{\infty} W_n(x) s^n = (1+s)\ U(s)$.

It is shown that $U(s) = e^{T(s)}$ ([2]). One can extend the above results to $x_n(x, y; t)$, $\frac{y_{n+1}(x,y;t)}{y}$, $v_n(x, y; t)$ and $w_n(x, y; t)$ including the results in [9]:

(1) $X(s) = 2\displaystyle\sum_{n=1}^{\infty} \frac{x_n(x, y; t)}{n} s^n$;

(2) $Y(s) = \displaystyle\sum_{n=1}^{\infty} \frac{y_{n+1}(x, y; t)}{y} s^n = \frac{1}{1-2xs+\beta\ s^2}$;

(3) $\widetilde{V}(s) = \displaystyle\sum_{n=0}^{\infty} v_n(x, y; t) s^n = (1-\beta\ s)\ U(s)$;

(4) $\widetilde{W}(s) = \displaystyle\sum_{n=0}^{\infty} w_n(x, y; t) s^n = (1+\beta\ s)\ U(s)$;

(5) $Y(s) = e^{X(s)}$.

In this way, the present paper provides two spectacular views of Tchebychev polynomials of all four kinds through golden ratio and Brahmagupta polynomials.

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