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On the Quaternionic Normal Curves in the Semi-Euclidean Space E_2^4

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Abstract: In this paper, we define the semi-real quaternionic normal curves in four dimensional semi-Euclidean space E_2^4. We obtain some characterizations of semi-real quaternionic normal curves in terms of their curvature functions. Moreover, we give necessary and sufficient condition for a semi-real quaternionic curve to be a semi-real quaternionic normal curves in E_2^4.

Key Words: Normal curves, semi-real quaternion, semi-quaternionic curve, position vector.

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§1. Introduction

In mathematics, the quaternion were discovered by Irish mathematician S. W.R. Hamilton, in 1843, which are more general form of complex number [5]. He defined a quaternion as the quotient of two directed lines in a three-dimensional space. Also, quaternions can be written as sum of a scalar and a vector. A special feature of quaternions is that the product of two quaternions is noncommutative. Quaternions have an important role in diverse areas such as kinematics and mechanics. They provide us opportunity representation for describing finite rotation in space.

In [1], Serret–Frenet formulae for a quaternionic curves in E^3 and E^4 are given by Baharathi and Nagaraj. After them Coken and Tuna defined Serret–Frenet formulae for a quaternionic curves in semi-Euclidean space E_2^4 ([3]).

In analogy with the Euclidean case, Serret–Frenet formulae for a quaternionic curves in semi-Euclidean space E_2^4 is defined in [11]. Moreover, characterization of quaternionic B_2-slant helices in Euclidean space E^4 given in [3] and quaternionic mannheim curves are studied in semi Eucliden space E^4 in [9].

In the Euclidean Space E^3, normal curves defined as the curves whose position vector always lying in their normal plane [2]. Analogously, normal curves in other space are defined as the curves whose normal planes always contain a fixed point. As well, normal curves have same characterization with spherical curves which case has interesting corollaries for curve theory.

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Recently, İlarslan [6], has been studied some characterizations of spacelike normal curves in the Minkowski 3-space E_1^3. Also, Ilarslan and Nesovic [8] have been investigated spacelike and timelike normal curves in Minkowski space-time.

In this paper, we define the semi-real quaternionic normal curves in four dimensional semi-Euclidean E_2^4. We obtain some characterizations of semi-real quaternionic normal curves in terms of their curvature functions. Moreover, we give necessary and sufficient condition for a semi-real quaternionic curve to be a semi-real quaternionic normal curves in E_2^4.

§2. Preliminary

A brief summary of the theory of semi-real quaternions in the semi-Euclidean space and normal curves are presented in this section.

A pseudo-Riemannian manifold is a differentiable manifold equipped with pseudo-Riemannian metric which is nondegenerate, smooth, symmetric metric tensor. This metric tensor need not be positive definite. We denote the pseudo (semi)-Euclidean $(n+1)$-space by E_ν^{n+1}. If $\nu = 0$, E_ν^{n+1} semi-Euclidean spaces reduce to E^{n+1} Euclidean space, that is, semi-Euclidean space is a generalization of Euclidean space. For $\nu = 1$ and $n \geqslant 1$; E_1^{n+1} is called Lorentz–Minkowski $(n+1)$ space. The Lorentz manifold form the most important subclass of semi-Riemannian manifolds because of their physical application to the theory of relativity. Due to semi-Riemannian metric there are three different kind of curves, namely spacelike, timelike, lightlike (null) depending on the casual character of their tangent vectors, that is, the curve α is called a spacelike (resp. timelike and lightlike) if $\alpha'(t)$ is spacelike (resp. timelike and lightlike) for any $t \in I$.

A semi-real quaternion q is an expression of the form

$$q = ae_1 + be_2 + ce_3 + d \tag{1}$$

such that

$$\begin{cases} e_i \times e_i = -\varepsilon_{e_i}, \quad 1 \leq i \leq 3, \\ e_i \times e_j = \varepsilon_{e_i}\varepsilon_{e_j} e_k, \quad \text{in } E_1^3, \\ e_i \times e_j = -\varepsilon_{e_i}\varepsilon_{e_j} e_k, \quad \text{in } E_2^4, \end{cases} \tag{2}$$

where (ijk) is an even permutation of (123) and $a, b, c, d \in R$.

We can write quaternion as $q = S_q + V_q$ where $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$ denote scalar and vector part of q, respectively. For every $p, q \in Q_\nu$, the multiplication of two semi-real quaternions p and q is defined as

$$p \times q = S_p S_q + < V_p, V_q > + S_p V_q + S_q V_p + V_p \wedge V_q, \text{ for every } p, q \in Q_\nu, \tag{3}$$

where $<,>$ and $\wedge$ are scalar and cross product in E_1^3, respectively. The conjugate of the semi-real quaternion q is denoted by γq and defined $\gamma q = S_q - V_q = d - ae_1 - be_2 - ce_3$. This helps to define the symmetric, non-degenerate, bilinear form h as follows.

$$h : Q_\nu \times Q_\nu \to R,$$

$$h(p,q)=\frac{1}{2}\left[\varepsilon_p\varepsilon_{\gamma q}\left(p\times\gamma q\right)+\varepsilon_q\varepsilon_{\gamma p}\left(q\times\gamma p\right)\right]\quad\text{for } E_1^3 \tag{4}$$

$$h(p,q)=\frac{1}{2}\left[\varepsilon_p\varepsilon_{\gamma q}\left(p\times\gamma q\right)+\varepsilon_q\varepsilon_{\gamma p}\left(q\times\gamma p\right)\right]\quad\text{for } E_2^4,$$

the norm of semi-real quaternion $q\in Q_\nu$ is

$$\|q\|^2=-a^2-b^2+c^2+d^2$$

q is called a semi-real spatial quaternion whenever $q\times\gamma q=0$. For $p,q\in Q_\nu$ where if $h(p,q)=0$ then p and q are called h-orthogonal [11]. If $\|q\|^2=1$, the q is called a semi real unit quaternion.

Recall that the pseudosphere, the pseudohyperbolic space and the lightcone are hyperquadrics in E_2^4, respectively defined by

$$\begin{aligned}S_1^3(m,r) &= \left\{x\in E_2^4 : h(x-m,x-m)=r^2\right\}\\ H_0^3(m,r) &= \left\{x\in E_2^4 : h(x-m,x-m)=-r^2\right\}\\ C_3^3(m,r) &= \left\{x\in E_2^4 : h(x-m,x-m)=0\right\}\end{aligned}$$

where $r>0$ is the radius and $m\in E_2^4$ is the center of hyperquadric.

In the Euclidean space E^3, it is well-known that to each unit speed curve $\alpha : I\subset R\rightarrow E^3$ with at least four continuous derivatives has Frenet frame $\{t,n,b\}$. At each point of the curve which is spanned by $\{t,n\}$, $\{t,b\}$ and $\{n,b\}$ are known as the osculating plane, the rectifying plane and the normal plane, respectively. Rectifying curve is introduced by B.Y.Chen, whose position vector always lies its rectifying plane $\{t,b\}$ ([2]). Similarly, a curve is called a osculating curve if its position vector always lies its osculating plane $\{t,n\}$. İlarslan and Nesovic defined normal curve as

$$\alpha(s)=\lambda(s)n(s)+\mu(s)b(s),$$

where λ and μ are arbitrary differentiable functions in terms of the arc length parameter s ([7]). This means that normal curve's position vector always lies its normal plane $\{n,b\}$.

Analogously, in E^4 the normal curve defined by İlarslan whose position vector always lies in orthogonal complement $T^\perp$ of its tangent vector field of the curve. The position vector of a normal curve α in E^4, satisfies the equation

$$\alpha(s)=\lambda(s)N(s)+\mu(s)B_1(s)+\nu(s)B_2(s),$$

where λ, μ and ν are arbitrary differentiable functions in terms of the arc length parameter s, respectively ([8]).

§3. Some Characterization of Quaternionic Normal Curves in Semi Euclidean Space

In this section, the four-dimensional Euclidean space E_2^4 is identified with the space of unit semi-real quaternion. Let

$$\beta : I \subset R \longrightarrow Q, \quad \beta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1 \tag{5}$$

be a smooth curve β in E_2^4 defined over the interval I. Let the parameter s be chosen such that the tangent $T = \beta'(s) = \sum_{i=1}^{4} \gamma_i'(s) e_i$ has unit magnitude. Let $\{T, N, B_1, B_2\}$ be the Frenet apparatus of the differentiable Euclidean space curve in the Euclidean space E_2^4. Then the Frenet equations are

$$\begin{cases} T'(s) = \varepsilon_N K N(s) \\ N'(s) = -\varepsilon_t \varepsilon_N K T(s) + \varepsilon_n k B_1(s) \\ B_1'(s) = -\varepsilon_t k N(s) + \varepsilon_n (r - \varepsilon_t \varepsilon_T \varepsilon_N K) B_2(s) \\ B_2'(s) = -\varepsilon_b (r - \varepsilon_t \varepsilon_T \varepsilon_N K) B_1, \end{cases} \tag{6}$$

where $T(s)$ is the tangent vector of the curve β and $K = \varepsilon_N \|T'(s)\|$ ([3]).

It is obtained the Frenet formulae in [1] and the apparatus for the curve β by making use of the Frenet formulae for a curve γ in R^3. Moreover, there are relationships between curvatures of the curves β and γ. These relations can be explained that the torsion of β is the principal curvature of the curve γ. Also, the bitorsion of β is $(r - \varepsilon_t \varepsilon_T \varepsilon_N K)$, where r is the torsion of γ and K is the principal curvature of β. These relations are only determined for quaternions, [1].

In this section, we characterize the semi-real quaternionic normal curves with the third curvature $(r - \varepsilon_t \varepsilon_T \varepsilon_N K) \neq 0$ for each s.

Let $\beta = \beta(s)$ be a unit speed semi-real quaternionic normal curve, lying fully in Q_ν. Then its position vector satisfies

$$\beta(s) = \lambda(s) N(s) + \mu(s) B_1(s) + \nu(s) B_2(s) \tag{7}$$

By taking the derivative of (7) with respect to s and using the Frenet equations (6), we obtain

$$T = -\varepsilon_t \varepsilon_N K \lambda T + (\lambda' - \varepsilon_t k \mu) N + (\varepsilon_n k \lambda + \mu' - \varepsilon_b (r - \varepsilon_t \varepsilon_T \varepsilon_N K) \upsilon) B_1 + (\varepsilon_n (r - \varepsilon_t \varepsilon_T \varepsilon_N K) \mu + \upsilon') B_2$$

and therefore

$$\begin{cases} -\varepsilon_t \varepsilon_N K \lambda = 1 \ , = 1, \\ \lambda' - \varepsilon_t k \mu = 0, \\ \varepsilon_n k \lambda + \mu' - \varepsilon_b (r - \varepsilon_t \varepsilon_T \varepsilon_N K) \upsilon = 0, \\ \varepsilon_n (r - \varepsilon_t \varepsilon_T \varepsilon_N K) \mu + \upsilon' = 0. \end{cases} \tag{8}$$

From the first three equations we find

$$\begin{cases} \lambda(s) = -\dfrac{\varepsilon_t\varepsilon_N}{K(s)}\ , \quad \mu(s) = -\dfrac{\varepsilon_N}{k(s)}\left(\dfrac{1}{K(s)}\right)', \\ \upsilon(s) = -\dfrac{\varepsilon_b\varepsilon_N}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\varepsilon_n\dfrac{k(s)}{K(s)} + \left(\dfrac{1}{k(s)}\left(\dfrac{1}{K(s)}\right)'\right)'\right] \end{cases} \tag{9}$$

Substituting relation (9) into (7), we get that the position vector of the semi-real quaternionic normal curve β is given by

$$\begin{aligned} \beta(s) &= -\frac{\varepsilon_t\varepsilon_N}{K(s)}N - \frac{\varepsilon_N}{k(s)}\left(\frac{1}{K(s)}\right)' B_1 \\ &\quad -\frac{\varepsilon_b\varepsilon_N}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\varepsilon_n\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right] B_2 \end{aligned} \tag{10}$$

Then we have the following theorem.

Theorem 3.1 *Let $\beta(s)$ be a unit speed semi-real quaternionic curve, lying fully in Q_ν. Then $\beta(s)$ is a semi-real quaternionic normal curve if and only if*

$$-\frac{(r-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}{k(s)}\left(\frac{1}{K(s)}\right)' = \left[\frac{\varepsilon_n\varepsilon_b}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\begin{array}{c}\varepsilon_t\varepsilon_n\frac{k(s)}{K(s)} + \\ \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\end{array}\right]\right]'. \tag{11}$$

Proof Let us first assume that $\beta(s)$ is a semi-real quaternionic normal curve. Then relations (8) and (9) imply that (11) holds.

Conversely, assume that relation (11) holds. Let us consider the vector $m \in Q_\nu$ given by

$$\begin{aligned} m(s) &= \beta(s) + \frac{\varepsilon_t\varepsilon_N}{K(s)}N + \frac{\varepsilon_N}{k(s)}\left(\frac{1}{K(s)}\right)' B_1 \\ &\quad +\frac{\varepsilon_b\varepsilon_N}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\varepsilon_n\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right] B_2. \end{aligned} \tag{12}$$

Differentiating (12) with respect to s and by applying (6), we get

$$\begin{aligned} m'(s) &= \frac{\varepsilon_n\varepsilon_N(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}{k(s)}\left(\frac{1}{K(s)}\right)' B_2 \\ &\quad + \left(\frac{\varepsilon_b\varepsilon_N}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\varepsilon_n\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right]\right)' B_2. \end{aligned}$$

From relation (11) it follows that m is a constant vector, which means that β is congruent to a semi-real quaternionic normal curve. □

Theorem 3.2 *Let $\beta(s)$ be a unit speed semi-real quaternionic curve, lying fully in Q_ν. If β is a semi-real quaternionic normal curve, then the following statements hold:*

(i) the principal normal and the first binormal component of the position vector β are respectively given by

$$\begin{cases} h(\beta, N) = -\dfrac{\varepsilon_t}{K(s)}, \\ h(\beta, B_1) = -\dfrac{\varepsilon_n \varepsilon_T \varepsilon_N}{k(s)} \left(\dfrac{1}{K(s)}\right)'. \end{cases} \tag{13}$$

(ii) the first binormal and the second binormal component of the position vector β are respectively given by

$$\begin{cases} h(\beta, B_1) = -\dfrac{\varepsilon_n \varepsilon_T \varepsilon_N}{k(s)} \left(\dfrac{1}{K(s)}\right)', \\ h(\beta, B_2) = -\dfrac{\varepsilon_T \varepsilon_N}{(r(s) - \varepsilon_t \varepsilon_T \varepsilon_N K(s))} \left[\varepsilon_t \varepsilon_n \dfrac{k(s)}{K(s)} + \left(\dfrac{1}{k(s)} \left(\dfrac{1}{K(s)}\right)'\right)'\right]. \end{cases} \tag{14}$$

Conversely, if $\beta(s)$ is a unit speed semi-real quaternionic curve, lying fully in Q_ν, and one of statements (i) or (ii) holds, then β is a normal curve.

Proof If $\beta(s)$ is a semi-real quaternionic normal curve, it is easy to check that relation (10) implies statements (i) and (ii).

Conversely, if statement (i) holds, differentiating equation $h(\beta, N) = -\frac{\varepsilon_t}{K(s)}$ with respect to s and by applying (6), we find $h(\beta, T) = 0$ which means that β is a semi-real quaternionic normal curve. If statement (ii) holds, in a similar way we conclude that β is a semi-real quaternionic normal curve. □

In the next theorem, we obtain interesting geometric characterization of semi-real quaternionic normal curves.

Theorem 3.3 *Let $\beta(s)$ be a unit speed semi-real quaternionic curve, lying fully in Q_ν. Then β is a semi-real quaternionic normal curve if and only if β lies in some hyperquadrics in Q_ν.*

Proof First assume that β is a semi-real quaternionic normal curve. It follows, by straightforward calculations using Theorem 3.1, we get

$$\begin{aligned} &2\frac{\varepsilon_N}{K}\left(\frac{1}{K}\right)' + 2\frac{\varepsilon_n \varepsilon_T}{k}\left(\frac{1}{K}\right)'\left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)' \\ &+2\frac{\varepsilon_b \varepsilon_T}{(r - \varepsilon_t \varepsilon_T \varepsilon_N K)}\left[\varepsilon_t \varepsilon_n \frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\left(\frac{1}{(r - \varepsilon_t \varepsilon_T \varepsilon_N K)}\left[\varepsilon_t \varepsilon_n \frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\right)' = 0. \end{aligned} \tag{15}$$

On the other hand, the previous equation is differential of the equation

$$\varepsilon_N \left(\frac{1}{K}\right)^2 + \varepsilon_n \varepsilon_T \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)^2 + \left(\frac{1}{(r - \varepsilon_t \varepsilon_T \varepsilon_N K)}\left[\varepsilon_t \varepsilon_n \frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\right)^2 = r, \quad r \in R. \tag{16}$$

By using (12), it is easy to check that

$$h(\beta - m, \beta - m) = \left(\frac{1}{K}\right)^2 + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)^2 + \left(\frac{1}{(r - \varepsilon_t \varepsilon_T \varepsilon_N K)}\left[\varepsilon_t \varepsilon_n \frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\right)^2, \tag{17}$$

which together with (16) gives $h(\beta - m, \beta - m) = r$. Consequently, β lies in some hypersphere in Q_ν.

Conversely, if β lies in some hyperquadric in Q_ν, then

$$h(\beta - m, \beta - m) = r, \; r \in R, \tag{18}$$

where $m \in Q_\nu$ is a constant vector. By taking the derivative of the previous equation with respect to s, we easily obtain $h(\beta - m, T) = 0$ which proves the theorem. □

Recall that arbitrary curve β in Q_ν is called a W-curve (or a helix), if it has constant curvature functions ([10]). The following theorem gives the characterization of semi-real quaternionic W-curve in Q_ν, in terms of semi-real quaternionic normal curves.

Theorem 3.4 *Every unit speed semi-real quaternionic W-curve, lying fully in Q_ν, is to a semi-real quaternionic normal curve.*

Proof By assumption we have $K(s) = c_1$, $k(s) = c_2$, $(r - \varepsilon_t\varepsilon_T\varepsilon_N K)(s) = c_3$, where $c_1, c_2, c_3 \in R - \{0\}$. Since the curvature functions obviously satisfy relation (11), according to Theorem 3.1, β is a semi-real quaternionic normal curve. □

Lemma 3.1 *A unit speed semi-real quaternionic $\beta(s)$, lying fully in Q_ν, is a semi-real quaternionic normal curve if and only if there exists a differentiable function $f(s)$ such that*

$$\begin{cases} f(s)(r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s)) = \varepsilon_t\varepsilon_n \dfrac{k(s)}{K(s)} + \left(\dfrac{1}{k(s)}\left(\dfrac{1}{K(s)}\right)'\right)', \\ f'(s) = -\varepsilon_n\varepsilon_b \dfrac{(r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s))}{k(s)}\left(\dfrac{1}{K(s)}\right)'. \end{cases} \tag{19}$$

By using the similar methods as in [8], as well as Lemma 3.1, we obtain the following theorem which give the necessary and the sufficient conditions for semi-real quaternionic curves in Q_ν to be the semi-real quaternionic normal curves.

Theorem 3.5 *Let $\beta(s)$ be a unit speed semi-real quaternionic curve in Q_ν whose Frenet formulas obtained from spacelike semi-real spatial quaternionic curve with spacelike principal normal n. Then β is a semi-real quaternionic normal curve if and only if there exist constants a_0, $b_0 \in R$ such that*

$$\begin{aligned} \frac{1}{k(s)}\left(\frac{1}{K(s)}\right)' &= \left(a_0 + \varepsilon_t \int \frac{k(s)}{K(s)} \cos\theta(s) ds\right)\cos\theta(s) \\ &\quad + \left(b_0 + \varepsilon_t \int \frac{k(s)}{K(s)} \sin\theta(s) ds\right)\sin\theta(s), \end{aligned} \tag{20}$$

where $\theta(s) = \int_0^s (r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s)) \; ds$.

Proof If $\beta(s)$ is a semi-real quaternionic normal curve, according to Lemma 3.1 there exists a differentiable function $f(s)$ such that relation (19) holds, whereby $\varepsilon_b = -1$. Let us define differentiable functions $\theta(s)$, $a(s)$ and $b(s)$ by

$$\begin{cases} \theta(s) = \int_0^s (r(s) - \varepsilon_t \varepsilon_T \varepsilon_N K(s))ds \\ a(s) = -\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)' \cosh\theta(s) + f(s)\sinh\theta(s) - \varepsilon_t \int \frac{k(s)}{K(s)}\cosh\theta(s)ds \\ b(s) = -\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)' \sinh\theta(s) - f(s)\cosh\theta(s) - \varepsilon_t \int \frac{k(s)}{K(s)}\sinh\theta(s)ds = \end{cases} \tag{21}$$

By using (19), we easily find $\theta'(s) = (r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s))$, $a'(s) = 0$, $b'(s) = 0$ and thus

$$a(s) = a_0 \ , \ \ b(s) = b_0 \ , \ \ a_0, b_0 \in R. \tag{22}$$

Multiplying the second and the third equations in (21), respectively with $\cosh\theta(s)$ and $-\sinh\theta(s)$, adding the obtained equations and using (22), we conclude that relation (20) holds.

Conversely, assume that there exist constants a_0, $b_0 \in R$ such that relation (20) holds. By taking the derivative of (20) with respect to s, we find

$$-\varepsilon_t\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)' = (r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s)) \begin{bmatrix} \left(a_0 + \varepsilon_t \int \frac{k(s)}{K(s)}\cosh\theta(s)ds\right)\sinh\theta(s) \\ -\left(b_0 + \varepsilon_t \int \frac{k(s)}{K(s)}\sinh\theta(s)ds\right)\cosh\theta(s) \end{bmatrix}. \tag{23}$$

Let us define the differentiable function $f(s)$ by

$$f(s) = \frac{1}{(r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right] \tag{24}$$

Next, relations (23) and (24) imply

$$f(s) = \left(a_0 + \varepsilon_t \int \frac{k(s)}{K(s)}\cosh\theta(s)ds\right)\sinh\theta(s) - \left(b_0 + \varepsilon_t \int \frac{k(s)}{K(s)}\sinh\theta(s)ds\right)\cosh\theta(s)$$

By using this and (20), we obtain $f'(s) = \frac{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}{k(s)}\left(\frac{1}{K(s)}\right)'$. Finally, Lemma 3.1 implies that β is congruent to a semi-real quaternionic normal curve. □

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