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Lagrange Space and Generalized Lagrange Space Arising From Metric $e^{\sigma(x)}g_{ij}(x,y)+\dfrac{1}{c^2}y_iy_j$

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Abstract: Some properties of Lagrange space with metric tensor $g_{ij}(x,y)+\frac{1}{c^2}y_iy_j$ where $g_{ij}(x,y)$ is metric tensor of Finsler space (M^n,F), and associated generalized Lagrange space has been studied by U. P. Singh in his paper [6]. In the present paper some properties of Lagrange space with metric tensor $e^{\sigma(x)}g_{ij}(x,y)+\frac{1}{c^2}y_iy_j$, where $g_{ij}(x,y)$ is metric tensor of Finsler space (M^n,F), $e^{\sigma(x)}$ is conformal factor and associated generalized Lagrange space has been studied.

Key Words: Lagrange space, generalized Lagrange space, $C-$reducible space.

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§1. Introduction

Various authors like R. Miron, M. Anastasiei, H. Shimada, T. Kawaguchi, U. P. Singh have studied Lagrange space and generalized Lagrange space in their papers [3], [2], [4], [5]. A generalized Lagrange space with metric tensor $\gamma_{ij}(x)+\dfrac{1}{c^2}y_iy_j$, where $\gamma_{ij}(x)$ is metric tensor of Riemannian space and c is velocity of light has been studied by Beil in his paper [1]. In this chapter $\gamma_{ij}(x)$ has been replaced by $e^{\sigma(x)}g_{ij}(x,y)$, where $g_{ij}(x,y)$ is metric tensor of Finsler space (M^n,F).

Let M^n is n-dimensional smooth manifold and F is Finsler function, the metric tensor $g_{ij}(x,y)$ is given by

$$g_{ij}(x,y)=\frac{\partial^2F^2}{\partial y^i\,\partial y^j}. \tag{1.1}$$

Since F is Finsler function of homogeneity one, so $g_{ij}(x,y)$ is homogeneous function of degree zero. The angular metric tensor of Finsler space (M^n,F), $h_{ij}(x,y)$ is given by

$$h_{ij}(x,y)=g_{ij}(x,y)-l_il_j, \tag{1.2}$$

where l_i is unit vector given by

$$l_i=\frac{y_i}{F}. \tag{1.3}$$

§2. Generalized Lagrange Space L^n and Associated Lagrange Space L^{*n}

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Consider a generalized Lagrange space $L^n = (M^n, G_{ij}(x,y))$ with metric tensor

$$G_{ij} = e^{\sigma} g_{ij}(x,y) + \frac{1}{c^2} y_i\, y_j. \tag{2.1}$$

The reciprocal metric tensor G^{ij} of G_{ij} is

$$G^{ij} = e^{-\sigma}\left(g^{ij} - \frac{1}{a_1 c^2} y^i y^j\right), \tag{2.2}$$

where

$$a_1 = e^{\sigma} + \frac{F^2}{C^2}, \qquad F^2 = g_{ij} y^i y^j. \tag{2.3}$$

The d-tensor field $\overline{C}_{ijk}$ of L^n is defined as

$$\overline{C}_{jhk} = \frac{1}{2}\left(\frac{\partial G_{jh}}{\partial y^k} + \frac{\partial G_{hk}}{\partial y^j} - \frac{\partial G_{jk}}{\partial y^h}\right). \tag{2.4}$$

Since $\dfrac{\partial y_i}{\partial y^j} = g_{ij}$ from (2.1) and (2.4), we have

$$\overline{C}_{jhk} = e^{\sigma} C_{jhk} + \frac{1}{c^2} g_{jk} y_h, \tag{2.5}$$

$$\overline{C}^i_{jk} = G^{ih}\overline{C}_{jhk} = C^i_{jk} + \frac{1}{a_1 c^2}\, g_{jk} y^i. \tag{2.6}$$

The metric tensor G_{ij} is used to define the Lagrangian L^* is given by

$$L^{*2} = G_{ij} y^i\, y^j. \tag{2.7}$$

The Lagrangian gives a metric tensor G^*_{ij}, is given by

$$G^*_{ij} = \frac{1}{2}\frac{\partial^2 L^{*2}}{\partial y^i\, \partial y^j}. \tag{2.8}$$

From (2.7) and (2.1), we have

$$L^{*2} = e^{\sigma} F^2 + \frac{F^4}{c^2} = a_1 F^2, \tag{2.9}$$

and from (2.8) and (2.9), we have

$$G^*_{ij} = a_2 g_{ij}(x,y) + \frac{4}{c^2}\, y_i\, y_j, \tag{2.10}$$

$$G^{*ij} = \frac{1}{a_2}\left(g^{ij} - \frac{1}{a_2 c^2} y^i\, y^j\right), \tag{2.11}$$

$$C^*_{jhk} = a_2 C_{jhk} + \frac{2}{c^2}(g_{hk} y_j + g_{jh} y_k + g_{jk} y_h). \tag{2.12}$$

From (2.12) and (2.11), we have

$$C^{*i}_{jk} = C^i_{jk} + \frac{2}{a_2 c^2}\left(\delta^i_j y_k + \delta^i_k y_j + \frac{a_2}{a_6} g_{jk} y^i - \frac{8}{a_6 c^2} y^i y_k y_j\right), \tag{2.13}$$

where $a_2 = e^\sigma + \frac{F^2}{c^2}$ and $a_6 = e^\sigma + \frac{6F^2}{c^2}$. In general,

$$a_\gamma = e^\sigma + \frac{\gamma F^2}{c^2}.$$

Theorem 2.1 *If the metric tensor of generalized Lagrange space given by G_{ij} in (2.1) then the metric tensor of associated Lagrange space G^*_{ij} is given by (2.10) and reciprocal metric tensor of generalized Lagrange space and associated Lagrange space are given by (2.2) and (2.11) respectively.*

§3. Angular Metric Tensor of L^n and L^{*n}

For a Finsler space F^n the angular metric tensor h_{ij} is

$$h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j} = g_{ij} - l_i l_j, \tag{3.1}$$

where $l_i = \frac{y_i}{L}$.

The generalized Lagrange space is not obtained from a Lagrangian therefore its angular metric tensor H_{ij}

$$H_{ij} = G_{ij} - L_i L_j. \tag{3.2}$$

Now,

$$L_i = G_{ij} L^j = \left\{ e^\sigma g_{ij}(x,y) + \frac{1}{c^2} y_i y_j \right\} \frac{y^j}{L^*}. \tag{3.3}$$

From (2.9)

$$L_i = G_{ij} L^j = \left(e^\sigma g_{ij}(x,y) + \frac{1}{c^2} y_i y_j \right) \frac{y^j}{\sqrt{a_1} F} = \left(e^\sigma l_i + \frac{F^2}{c^2}\frac{y_i}{F} \right) \frac{1}{\sqrt{a_1}}$$

$$= \left(e^\sigma + \frac{F^2}{c^2} \right) \frac{l_i}{\sqrt{a_1}} = a_1 \frac{l_i}{\sqrt{a_1}} = \sqrt{a_1}\, l_i. \tag{3.4}$$

From (3.4) and (3.2) and (2.1)

$$H_{ij} = e^\sigma g_{ij}(x,y) + \frac{1}{c^2}\, y_i\, y_j - a_1 l_i l_j. \tag{3.5}$$

Putting the value of a_1 in (3.5), we have

$$H_{ij} = e^\sigma h_{ij}. \tag{3.6}$$

The angular metric tensor of Lagrange space L^{*n} is given by

$$H^*_{ij} = L^* \frac{\partial^2 L^*}{\partial y^i\, \partial y^j}.$$

The successive differentiation of (2.9) w.r.t. y^j and y^i gives

$$L^* \frac{\partial L^*}{\partial y^j} = a_1 y_j + \frac{F^2}{c^2}\, y_j, \tag{3.7}$$

$$L^* \frac{\partial^2 L^*}{\partial y^i\, \partial y^j} + \frac{\partial L^*}{\partial y^i}\frac{\partial L^*}{\partial y^j} = \left(\frac{2}{c^2} y_i \right) y_j + a_1 g_{ij} + \frac{F^2}{c^2} g_{ij} + \frac{2}{c^2}\, y_i y_j, \tag{3.8}$$

or

$$L^*\frac{\partial^2 L^*}{\partial y^i \partial y^j}+\frac{\partial L^*}{\partial y^i}\frac{\partial L^*}{\partial y^j}=\frac{4}{c^2}y_i y_j+a_2 g_{ij},$$

or

$$L^*\frac{\partial^2 L^*}{\partial y^i \partial y^j}=\frac{4F^2}{c^2}l_i l_j-L^*_i L^*_j+a_2 g_{ij}, \tag{3.9}$$

Now, from (3.7)

$$L^* L^*_j=a_2 y_j \qquad \Rightarrow \qquad L^*_j=\frac{a_2 y_j}{L^*}. \tag{3.10}$$

From (3.9) and (3.10), we get

$$H^*_{ij}=(a_4-e^\sigma)\, l_i l_j-\frac{a_2^2}{a_1}\, l_i\, l_j+a_2 g_{ij},$$

$$H^*_{ij}=a_2\, h_{ij}+\left(a_6-\frac{a_2^2}{a_1}\right) l_i\, l_j. \tag{3.11}$$

Theorem 3.1 *If the metric tensor of generalized Lagrange space given by G_{ij} in (2.1), the angular metric tensor of generalized Lagrange space and associated Lagrange space are given by (3.6) and (3.11) respectively.*

§4. C-Reducibility of L^n and L^{*n}‘

Definition 4.1 *A generalized Lagrange space L^n is called C-reducible space if*

$$\overline{C}_{jhk}=(M_j H_{hk}+M_h H_{jk}+M_k H_{jh}), \tag{4.1}$$

where M_j are component of a covariant vector field.

Suppose generalized Lagrange space L^n is C-reducible, then (4.1) holds. Using (2.5) and (3.6) and relation $y_h=Fl_h$, (4.1) can be written as

$$e^\sigma C_{jhk}+\frac{F}{c^2}g_{jk}l_h=(M_j h_{hk}+M_h h_{jk}+M_k h_{jh})\, e^\sigma. \tag{4.2}$$

Contracting (4.2) by $l^h l^j l^k$ and using (4.1), we get

$$\frac{F}{c^2}=0 \qquad \Rightarrow \qquad F=0,$$

which is contradiction.

Theorem 4.1 *The generalized Lagrange space $L^n=(M^n,G_{ij})$ can not be C-reducible.*

Now consider the space L^{*n}, its C-reducibility is given by

$$C^*_{jhk}=(M^*_j H^*_{hk}+M^*_h H^*_{jk}+M^*_k H^*_{jh}), \tag{4.3}$$

where M^*_h are component of covariant vector field using (2.12), (3.11), (4.3) and $y_h=Fl_h$ in (4.3), we get

$$a_2 C_{jhk}+\frac{2F}{c^2}(g_{hk}l_j+g_{jh}l_k+g_{jk}l_h)=a_2(M^*_j h_{hk}+M^*_h h_{jk}+M^*_k h_{jh})$$
$$+\left(a_6-\frac{a_2^2}{a_1}\right)(M^*_j l_h l_k+M^*_h l_j l_k+M^*_k l_h l_j), \tag{4.4}$$

Contracting (4.4) by l^j and putting $\rho^* = M_i^* l^i$, we have

$$\frac{2F^2}{c^2}(g_{hk} + 2l_h l_k) = a_2 \rho^* h_{hk} + \left(a_6 - \frac{a_2^2}{a_1}\right)(\rho^* l_h l_k + M_h^* l_k + M_k^* l_h). \tag{4.5}$$

Contracting (4.5) by l^h, we have

$$\frac{6F^2}{c^2} l_k = \left(a_6 - \frac{a_2^2}{a_1}\right)(\rho^* l_k + \rho^* l_k + M_k^*). \tag{4.6}$$

Again contracting (4.6) by l^k, which gives

$$\rho^* = \frac{2F^2}{c^2}\left(\frac{a_1}{a_1 a_6 - a_2^2}\right). \tag{4.7}$$

From (4.6) and (4.7), we have

$$\frac{2F^2}{c^2} l_k = \left(a_6 - \frac{a_2^2}{a_1}\right) M_k^*. \tag{4.8}$$

From (4.8) and (4.5), we have

$$\frac{2F^2}{c^2} g_{hk} = a_2 \rho^* h_{hk} + \frac{2F^2}{c^2} l_h l_k. \tag{4.9}$$

Using $g_{hk} = h_{hk} + l_h l_k$ in (4.9), we get

$$\left(\frac{2F^2}{c^2} - a_2 \rho^*\right) h_{hk} = 0.$$

It gives $\rho^* = \dfrac{2F^2}{c^2 a_2}$, which contradict (4.7). Hence

Theorem 4.2 *The Lagrange space $L^{*n} = (M^n, L^*)$ can not be C-reducible.*

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