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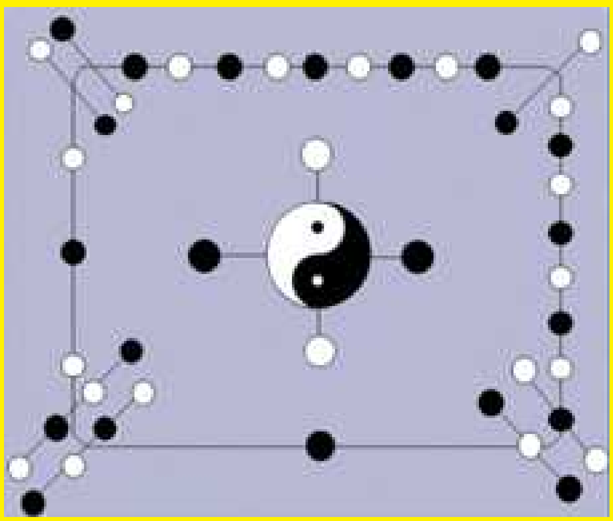

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Famous Words:

Science is unpredictable. If I had known it, I would have found it before.

By Stephen Hawking, a British physicist.

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Spacelike Smarandache Curves of Timelike Curves in Anti de Sitter 3-Space

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Abstract: In this paper, we investigate special spacelike Smarandache curves of timelike curves according to Sabban frame in Anti de Sitter 3-Space. Moreover, we give the relationship between the base curve and its Smarandache curve associated with theirs Sabban Frames. However, we obtain some geometric results with respect to special cases of the base curve. Finally, we give some examples of such curves and draw theirs images under stereographic projections from Anti de Sitter 3-space to Minkowski 3-space.

Key Words: Anti de Sitter space, Minkowski space, Semi Euclidean space, Smarandache curve.

AMS(2010): 53A35, 53C25.

§1. Introduction

It is well known that there are three kinds of Lorentzian space. Minkowski space is a flat Lorentzian space and de Sitter space is a Lorentzian space with positive constant curvature. Lorentzian space with negative constant curvature is called Anti de Sitter space which is quite different from those of Minkowski space and de Sitter space according to causality. The Anti de Sitter space is a vacuum solution of the Einstein's field equation with an attractive cosmological constant in the theory of relativity. The Anti de Sitter space is also important in the string theory and the brane world scenario. Due to this situation, it is a very significant space from the viewpoint of the astrophysics and geometry (Bousso and Randall, 2002; Maldacena, 1998; Witten, 1998).

Smarandache geometry is a geometry which has at least one Smarandachely denied axiom. An axiom is said to be Smarandachely denied, if it behaves in at least two different ways within the same space (Ashbacher, 1997). Smarandache curves are the objects of Smarandache geometry. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve (Turgut and Yılmaz, 2008). Special Smarandache curves are studied in different ambient spaces by some authors (Bektaş and Yüce, 2013; Koc Ozturk et al., 2013; Taşköprü and Tosun, 2014; Turgut and Yımaz, 2008; Yakut et al., 2014).

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This paper is organized as follows. In section 2, we give local diferential geometry of non-dejenerate regular curves in Anti de Sitter 3-space which is denoted by $\mathbb{H}_1^3$. We call that a curve is *AdS curve* in $\mathbb{H}_1^3$ if the curve is immersed unit speed non-dejenerate curve in $\mathbb{H}_1^3$. In section 3, we consider that any spacelike AdS curve $\boldsymbol{\beta}$ whose position vector is composed by Frenet frame vectors on another timelike AdS curve $\boldsymbol{\alpha}$ in $\mathbb{H}_1^3$. The AdS curve $\boldsymbol{\beta}$ is called *AdS Smarandache curve* of $\boldsymbol{\alpha}$ in $\mathbb{H}_1^3$. We define eleven different types of AdS Smarandache curve $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ according to Sabban frame in $\mathbb{H}_1^3$. Also, we give some relations between Sabban apparatus of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for all of possible cases. Moreover, we obtain some corollaries for the spacelike AdS Smarandache curve $\boldsymbol{\beta}$ of AdS timelike curve $\boldsymbol{\alpha}$ which is a planar curve, horocycle or helix, respectively. In subsection 3.1, we define *AdS stereographic projection*, that is, the stereographic projection from $\mathbb{H}_1^3$ to $\mathbb{R}_1^3$. Then, we give an example for base AdS curve and its AdS Smarandache curve, which are helices in $\mathbb{H}_1^3$. Finally, we draw the pictures of some AdS curves by using AdS stereographic projection in Minkowski 3-space.

§2. Preliminary

In this section, we give the basic theory of local differential geometry of non-degenerate curves in Anti de Sitter 3-space which is denoted by $\mathbb{H}_1^3$. For more detail and background about Anti de Sitter space, see (Chen et al., 2014; O'Neill, 1983)..

Let $\mathbb{R}_2^4$ denote the four-dimensional semi Euclidean space with index two, that is, the real vector space $\mathbb{R}^4$ endowed with the scalar product

$$\langle \boldsymbol{x}\,,\boldsymbol{y}\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$$

for all $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ be pseudo-orthonormal basis for $\mathbb{R}_2^4$. Then δ_{ij} is Kronecker-delta function such that $\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \delta_{ij}\varepsilon_j$ for $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = \varepsilon_4 = 1$.

A vector $\boldsymbol{x} \in \mathbb{R}_2^4$ is called *spacelike*, *timelike* and *lightlike (null)* if $\langle \boldsymbol{x}\,,\boldsymbol{x}\,\rangle > 0$ (or $\boldsymbol{x} = 0$), $\langle \boldsymbol{x}\,,\boldsymbol{x}\,\rangle < 0$ and $\langle \boldsymbol{x}\,,\boldsymbol{x}\,\rangle = 0$, respectively. The *norm* of a vector $\boldsymbol{x} \in \mathbb{R}_2^4$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}\,,\boldsymbol{x}\,\rangle|}$. The *signature* of a vector $\boldsymbol{x}$ is denoted by

$$sign(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \text{ is spacelike} \\ 0, & \boldsymbol{x} \text{ is null} \\ -1, & \boldsymbol{x} \text{ is timelike} \end{cases}$$

The sets

$$\begin{aligned} \mathbb{S}_2^3 &= \{\boldsymbol{x} \in \mathbb{R}_2^4 \mid \langle \boldsymbol{x}\,,\boldsymbol{x}\,\rangle = 1\,\} \\ \mathbb{H}_1^3 &= \{\boldsymbol{x} \in \mathbb{R}_2^4 \mid \langle \boldsymbol{x}\,,\boldsymbol{x}\,\rangle = -1\,\} \end{aligned}$$

are called *de Sitter 3-space with index 2* (unit pseudosphere with dimension 3 and index 2 in $\mathbb{R}_2^4$) and *Anti de Sitter 3-space* (unit pseudohyperbolic space with dimension 3 and index 2 in

$\mathbb{R}^4_2$), respectively.

The pseudo vector product of vectors $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3$ is given by

$$\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \boldsymbol{x}^3 = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x^1_1 & x^1_2 & x^1_3 & x^1_4 \\ x^2_1 & x^2_2 & x^2_3 & x^2_4 \\ x^3_1 & x^3_2 & x^3_3 & x^3_4 \end{vmatrix} \tag{1}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis of $\mathbb{R}^4_2$ and $\boldsymbol{x}^i = (x^i_1, x^i_2, x^i_3, x^i_4)$, $i = 1, 2, 3$. Also, it is clear that

$$\langle \boldsymbol{x}, \boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \boldsymbol{x}^3 \rangle = \det(\boldsymbol{x}, \boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$$

for any $\boldsymbol{x} \in \mathbb{R}^4_2$. Therefore, $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \boldsymbol{x}^3$ is pseudo-orthogonal to any $\boldsymbol{x}^i$, $i = 1, 2, 3$.

We give the basic theory of non-degenerate curves in $\mathbb{H}^3_1$. Let $\boldsymbol{\alpha} : I \to \mathbb{H}^3_1$ be regular curve (i.e., an immersed curve) for open subset $I \subset \mathbb{R}$. The regular curve $\boldsymbol{\alpha}$ is said to be spacelike or timelike if $\dot{\boldsymbol{\alpha}}$ is a spacelike or timelike vector at any $t \in I$ where $\dot{\boldsymbol{\alpha}}(t) = d\boldsymbol{\alpha}/dt$. The such curves are called *non-degenerate curve*. Since $\boldsymbol{\alpha}$ is a non-degenerate curve, it admits an arc length parametrization $s = s(t)$. Thus, we can assume that $\boldsymbol{\alpha}(s)$ is a unit speed curve. Then the unit tangent vector of $\boldsymbol{\alpha}$ is given by $\boldsymbol{t}(s) = \boldsymbol{\alpha}'(s)$. Since $\langle \boldsymbol{\alpha}(s), \boldsymbol{\alpha}(s) \rangle = -1$, we have $\langle \boldsymbol{\alpha}(s), \boldsymbol{t}'(s) \rangle = -\delta_1$ where $\delta_1 = sign(\boldsymbol{t}(s))$. The vector $\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s)$ is pseudo-orthogonal to $\boldsymbol{\alpha}(s)$ and $\boldsymbol{t}(s)$. In the case when $\langle \boldsymbol{\alpha}''(s), \boldsymbol{\alpha}''(s) \rangle \neq -1$ and $\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s) \neq 0$, the pirinciple normal vector and the binormal vector of $\boldsymbol{\alpha}$ is given by $\boldsymbol{n}(s) = \frac{\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s)}{\|\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s)\|}$ and $\boldsymbol{b}(s) = \boldsymbol{\alpha}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, respectively. Also, geodesic curvature of $\boldsymbol{\alpha}$ are defined by $\kappa_g(s) = \|\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s)\|$. Hence, we have pseudo-orthonormal frame field $\{\boldsymbol{\alpha}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ of $\mathbb{R}^4_2$ along $\boldsymbol{\alpha}$. The frame is also called the *Sabban frame* of non-dejenerate curve $\boldsymbol{\alpha}$ on $\mathbb{H}^3_1$ such that

$$\begin{aligned} \boldsymbol{t}(s) \wedge \boldsymbol{n}(s) \wedge \boldsymbol{b}(s) &= \delta_3\, \boldsymbol{\alpha}(s), \quad \boldsymbol{n}(s) \wedge \boldsymbol{b}(s) \wedge \boldsymbol{\alpha}(s) = \delta_1 \delta_3\, \boldsymbol{t}(s) \\ \boldsymbol{b}(s) \wedge \boldsymbol{\alpha}(s) \wedge \boldsymbol{t}(s) &= -\delta_2 \delta_3\, \boldsymbol{n}(s), \quad \boldsymbol{\alpha}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s) = \boldsymbol{b}(s). \end{aligned}$$

where $sign(\boldsymbol{t}(s)) = \delta_1$, $sign(\boldsymbol{n}(s)) = \delta_2$, $sign(\boldsymbol{b}(s)) = \delta_3$ and $\det(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}) = -\delta_3$.

Now, if the assumption is $< \boldsymbol{\alpha}''(s), \boldsymbol{\alpha}''(s) > \neq -1$, we can give two different Frenet-Serret formulas of $\boldsymbol{\alpha}$ according to the causal character. It means that if $\delta_1 = 1$ ($\delta_1 = -1$), then $\boldsymbol{\alpha}$ is spacelike (timelike) curve in $\mathbb{H}^3_1$. In that case, the Frenet-Serret formulas are

$$\begin{bmatrix} \boldsymbol{\alpha}'(s) \\ \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \\ \boldsymbol{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \delta_1 & 0 & \kappa_g(s) & 0 \\ 0 & -\delta_1\delta_2\kappa_g(s) & 0 & -\delta_1\delta_3\tau_g(s) \\ 0 & 0 & \delta_1\delta_2\tau_g(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}(s) \\ \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{bmatrix} \tag{2}$$

where the geodesic torsion of $\boldsymbol{\alpha}$ is given by $\tau_g(s) = \frac{\delta_1 \det(\boldsymbol{\alpha}(s), \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}''(s), \boldsymbol{\alpha}'''(s))}{(\kappa_g(s))^2}$.

Remark 2.1 The condition $< \boldsymbol{\alpha}''(s), \boldsymbol{\alpha}''(s) > \neq -1$ is equivalent to $\kappa_g(s) \neq 0$. Moreover, we can show that $\kappa_g(s) = 0$ and $\boldsymbol{t}'(s) - \delta_1\boldsymbol{\alpha}(s) = 0$ if and only if the non-degenerate curve $\boldsymbol{\alpha}$ is a geodesic in $\mathbb{H}^3_1$.

We can give the following definitions by (Barros et al., 2001; Chen et al., 2014).

Definition 2.2 *Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to \mathbb{H}^3_1$ is an immersed spacelike (timelike) curve according to the Sabban frame $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ with geodesic curvature κ_g and geodesic torsion τ_g. Then,*

(1) If $\tau_g \equiv 0$, $\boldsymbol{\alpha}$ is called a planar curve in $\mathbb{H}^3_1$;

(2) If $\kappa_g \equiv 1$ and $\tau_g \equiv 0$, $\boldsymbol{\alpha}$ is called a horocycle in $\mathbb{H}^3_1$;

(3) If τ_g and κ_g are both non-zero constant, $\boldsymbol{\alpha}$ is called a helix in $\mathbb{H}^3_1$.

Remark 2.3 From now on, we call that $\boldsymbol{\alpha}$ is a spacelike (timelike) *AdS curve* if $\boldsymbol{\alpha} : I \subset \mathbb{R} \to \mathbb{H}^3_1$ is an immersed spacelike (timelike) unit speed curve in $\mathbb{H}^3_1$.

§3. Spacelike Smarandache Curves of Timelike Curves in $\mathbb{H}^3_1$

In this section, we consider any timelike AdS curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ and define its spacelike AdS Smarandache curve $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ according to the Sabban frame $\{\boldsymbol{\alpha}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ of $\boldsymbol{\alpha}$ in $\mathbb{H}^3_1$ where s and $s^\star$ is arc length parameter of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

Definition 3.1 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\varphi = \{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. Then the spacelike v_iv_j-Smarandache AdS curve $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ of $\boldsymbol{\alpha}$ is defined by*

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{2}}(av_i(s) + bv_j(s)), \tag{3}$$

where $v_i, v_j \in \varphi$ for $i \neq j$ and $a, b \in \mathbb{R}$ such that

v_iv_j	Condition
$\boldsymbol{\alpha t}$	$a^2 + b^2 = 2$
$\boldsymbol{\alpha n}$	$a^2 - b^2 = 2$
$\boldsymbol{\alpha b}$	$a^2 - b^2 = 2$
$\boldsymbol{tn}$	$a^2 - b^2 = 2$
$\boldsymbol{tb}$	$a^2 - b^2 = 2$
$\boldsymbol{nb}$	$a^2 + b^2 = -2$ (Undefined)

. (4)

Theorem 3.2 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\varphi = \{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. If $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ is spacelike v_iv_j-Smarandache AdS curve with Sabban frame $\{\boldsymbol{\beta}, \boldsymbol{t}_{\boldsymbol{\beta}}, \boldsymbol{n}_{\boldsymbol{\beta}}, \boldsymbol{b}_{\boldsymbol{\beta}}\}$ and geodesic curvature $\widetilde{\kappa_g}$, geodesic torsion $\widetilde{\tau}_g$ where $v_i, v_j \in \varphi$ for $i \neq j$, then the Sabban apparatus of $\boldsymbol{\beta}$ can be constructed by the Sabban apparatus* *of* $\boldsymbol{\alpha}$ *such that*

$v_i v_j$	Condition
$\boldsymbol{\alpha t}$	$b^2\kappa_g(s)^2 - 2 > 0$
$\boldsymbol{\alpha n}$	$b^2\tau_g(s)^2 - (b\kappa_g(s) + a)^2 > 0$
$\boldsymbol{\alpha b}$	$b^2\tau_g(s)^2 - a^2 > 0$
$\boldsymbol{tn}$	$2(\kappa_g(s)^2 - 1) + b^2\left(\tau_g(s)^2 - 1\right) > 0$
$\boldsymbol{tb}$	$(a\kappa_g(s) - b\tau_g(s))^2 - a^2 > 0$
$\boldsymbol{nb}$	(Undefined)

. (5)

Proof We suppose that $v_i v_j = \boldsymbol{\alpha t}$. Now, let $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $\boldsymbol{\alpha t}-$Smarandache AdS curve of timelike AdS curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. Then, $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{2}}(a\boldsymbol{\alpha}(s) + b\boldsymbol{t}(s)) \tag{6}$$

such that $a^2 + b^2 = 2,\ a, b \in \mathbb{R}$ from the Definition 3.1. Differentiating both sides of (6) with respect to s, we get

$$\boldsymbol{\beta}'(s^\star(s)) = \frac{d\boldsymbol{\beta}}{ds^\star}\frac{ds^\star}{ds} = \frac{1}{\sqrt{2}}\left(a\boldsymbol{\alpha}'(s) + b\boldsymbol{t}'(s)\right)$$

and by using (2),

$$\boldsymbol{t}_{\boldsymbol{\beta}}(s^\star(s))\frac{ds^\star}{ds} = \frac{1}{\sqrt{2}}\left(a\,\boldsymbol{t}(s) + b\left(-\boldsymbol{\alpha}(s) + \kappa_g(s)\boldsymbol{n}(s)\right)\right),$$

where

$$\frac{ds^\star}{ds} = \sqrt{\frac{b^2\kappa_g(s)^2 - 2}{2}} \tag{7}$$

with condition $b^2\kappa_g(s)^2 - 2 > 0$.

(From now on, unless otherwise stated, we won't use the parameters "s" and "$s^\star$" in the following calculations for the sake of brevity).

Hence, the tangent vector of spacelike $\boldsymbol{\alpha t}-$Smarandache AdS curve $\boldsymbol{\beta}$ is to be

$$\boldsymbol{t}_{\boldsymbol{\beta}} = \frac{1}{\sqrt{\sigma}}\left(-b\boldsymbol{\alpha} + a\boldsymbol{t} + b\kappa_g\boldsymbol{n}\right), \tag{8}$$

where $\sigma = b^2{\kappa_g}^2 - 2$.

Differentiating both sides of (8) with respect to s, we have

$$\boldsymbol{t}_{\boldsymbol{\beta}}{}' = \frac{\sqrt{2}}{\sigma^2}\left(\lambda_1\boldsymbol{\alpha} + \lambda_2\boldsymbol{t} + \lambda_3\boldsymbol{n} + \lambda_4\boldsymbol{b}\right) \tag{9}$$

by using again (2) and (7), where

$$\begin{array}{lcc} \lambda_1 & = & b^3\kappa_g{\kappa_g}' - a\sigma \\ \lambda_2 & = & -ab^2\kappa_g{\kappa_g}' + b\left({\kappa_g}^2 - 1\right)\sigma \\ \lambda_3 & = & -2b{\kappa_g}' + a\kappa_g\sigma \\ \lambda_4 & = & b\kappa_g\tau_g\sigma\ . \end{array} \tag{10}$$

Now, we can compute

$$\boldsymbol{t_\beta}' - \boldsymbol{\beta} = \frac{1}{\sqrt{2}\sigma^2}\left(\left(2\lambda_1 - a\sigma^2\right)\boldsymbol{\alpha} + \left(2\lambda_2 - b\sigma^2\right)\boldsymbol{t} + 2\lambda_3\boldsymbol{n} + 2\lambda_4\boldsymbol{b}\right) \tag{11}$$

and

$$\left\|\boldsymbol{t_\beta}' - \boldsymbol{\beta}\right\| = \frac{1}{\sigma^2}\sqrt{-\sigma^4 + 2\left(a\lambda_1 + b\lambda_2\right)\sigma^2 + 2\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right)}. \tag{12}$$

From the equations (11) and (12), the principal normal vector of $\boldsymbol{\beta}$ is

$$\boldsymbol{n_\beta} = \frac{1}{\sqrt{2\mu}}\left(\left(2\lambda_1 - a\sigma^2\right)\boldsymbol{\alpha} + \left(2\lambda_2 - b\sigma^2\right)\boldsymbol{t} + 2\lambda_3\boldsymbol{n} + 2\lambda_4\boldsymbol{b}\right) \tag{13}$$

and the geodesic curvature of $\boldsymbol{\beta}$ is

$$\widetilde{\kappa_g} = \frac{\sqrt{\mu}}{\sigma^2}, \tag{14}$$

where

$$\mu = -\sigma^4 + 2\left(a\lambda_1 + b\lambda_2\right)\sigma^2 + 2\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right). \tag{15}$$

Also, from the equations (6), (8) and (13), the binormal vector of $\boldsymbol{\beta}$ as pseudo vector product of $\boldsymbol{\beta}, \boldsymbol{t_\beta}$ and $\boldsymbol{n_\beta}$ is given by

$$\boldsymbol{b_\beta} = \frac{1}{\sqrt{\sigma\mu}}\left(\left(-b^2\kappa_g\lambda_4\right)\boldsymbol{\alpha} + \left(ab\kappa_g\lambda_4\right)\boldsymbol{t} + 2\lambda_4\boldsymbol{n} + \left(-b^2\kappa_g\lambda_1 + ab\kappa_g\lambda_2 - 2\lambda_3\right)\boldsymbol{b}\right). \tag{16}$$

Finally, differentiating both sides of (9) with respect to s, we get

$$\boldsymbol{t_\beta}'' = \frac{-2}{\sigma^{7/2}}\left(\begin{array}{l} \left(2\lambda_1\sigma' - ({\lambda_1}' - \lambda_2)\sigma\right)\boldsymbol{\alpha} + \left(2\lambda_2\sigma' - (\lambda_1 + {\lambda_2}' + \kappa_g\lambda_3)\sigma\right)\boldsymbol{t} \\ + \left(2\lambda_3\sigma' - (\kappa_g\lambda_2 + {\lambda_3}' - \tau_g\lambda_4)\sigma\right)\boldsymbol{n} + \left(2\lambda_4\sigma' - (\tau_g\lambda_3 + {\lambda_4}')\sigma\right)\boldsymbol{b} \end{array}\right) \tag{17}$$

by using again (2) and (7). Hence, from the equations (6), (8), (9), (14) and (17), the geodesic torsion of $\boldsymbol{\beta}$ is

$$\widetilde{\tau_g} = \frac{2}{\sigma\mu}\left(\begin{array}{l} \kappa_g(b\lambda_1 - a\lambda_2)(b\tau_g\lambda_3 + a\lambda_4 + b{\lambda_4}') - b\kappa_g(b{\lambda_1}' - a{\lambda_2}')\lambda_4 \\ +2\tau_g({\lambda_3}^2 + {\lambda_4}^2) + ab{\kappa_g}^2\lambda_3\lambda_4 - 2({\lambda_3}'\lambda_4 - \lambda_3{\lambda_4}') \end{array}\right) \tag{18}$$

under the condition $a^2 + b^2 = 2$. Thus, we obtain the Sabban aparatus of $\boldsymbol{\beta}$ for the choice $v_iv_j = \boldsymbol{\alpha t}$.

It can be easily seen that the other types of v_iv_j-Smarandache curves $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ by using same method as the above. The proof is complete. □

Corollary 3.3 *Let $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ be a timelike AdS curve and $\boldsymbol{\beta}=\boldsymbol{\beta}(s^\star)$ be spacelike v_iv_j-Smarandache AdS curve of $\boldsymbol{\alpha}$, then the following table holds for the special cases of $\boldsymbol{\alpha}$ under the conditions (4) and (5):*

v_iv_j	$\boldsymbol{\alpha}$ *is planar curve*	$\boldsymbol{\alpha}$ *is horocycle*	$\boldsymbol{\alpha}$ *is helix*
$\boldsymbol{\alpha t}$	*planar curve*	*undefined*	*helix*
$\boldsymbol{\alpha n}$	*undefined*	*undefined*	*helix*
$\boldsymbol{\alpha b}$	*undefined*	*undefined*	*helix*
$\boldsymbol{tn}$	*planar curve*	*undefined*	*helix*
$\boldsymbol{tb}$	*planar curve*	*undefined*	*helix*

Definition 3.4 *Let $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\varphi=\{\boldsymbol{\alpha},\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. Then the spacelike $v_iv_jv_k-$Smarandache AdS curve $\boldsymbol{\beta}=\boldsymbol{\beta}(s^\star)$ of $\boldsymbol{\alpha}$ is defined by*

$$\boldsymbol{\beta}(s^\star(s))=\frac{1}{\sqrt{3}}(av_i(s)+bv_j(s)+cv_k(s)), \tag{19}$$

where $v_i,v_j,v_k\in\varphi$ for $i\neq j\neq k$ and $a,b,c\in\mathbb{R}$ such that

$v_iv_jv_k$	*Condition*
$\boldsymbol{\alpha tn}$	$a^2+b^2-c^2=3$
$\boldsymbol{\alpha tb}$	$a^2+b^2-c^2=3$
$\boldsymbol{\alpha nb}$	$a^2-b^2-c^2=3$
$\boldsymbol{tnb}$	$a^2-b^2-c^2=3$

. (20)

Theorem 3.5 *Let $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\varphi=\{\boldsymbol{\alpha},\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. If $\boldsymbol{\beta}=\boldsymbol{\beta}(s^\star)$ is spacelike $v_iv_jv_k-$Smarandache AdS curve with Sabban frame $\{\boldsymbol{\beta},\boldsymbol{t_\beta},\boldsymbol{n_\beta},\boldsymbol{b_\beta}\}$ and geodesic curvature $\widetilde{\kappa_g}$, geodesic torsion $\widetilde{\tau_g}$ where $v_i,v_j,v_k\in\varphi$ for $i\neq j\neq k$, then the Sabban apparatus of $\boldsymbol{\beta}$ can be constructed by the*

Sabban apparatus of $\boldsymbol{\alpha}$ *such that*

$v_iv_jv_k$	*Condition*
$\boldsymbol{\alpha t n}$	$(b^2-c^2)\kappa_g(s)^2-2ac\kappa_g(s)+c^2(\tau_g(s)^2-1)-3>0$
$\boldsymbol{\alpha t b}$	$(b\kappa_g(s)-c\tau_g(s))^2-\left(c^2+3\right)>0$
$\boldsymbol{\alpha n b}$	$\left(b^2+c^2\right)\tau_g(s)^2-(a+b\kappa_g(s))^2>0$
$\boldsymbol{t n b}$	$(a\kappa_g(s)-c\tau_g(s))^2+b^2\left(\tau_g(s)^2-\kappa_g(s)^2\right)-a^2>0$

. (21)

Proof We suppose that $v_iv_jv_k=\boldsymbol{\alpha t b}$. Now, let $\boldsymbol{\beta}=\boldsymbol{\beta}(s^\star)$ be spacelike $\boldsymbol{\alpha t b}-$Smarandache AdS curve of timelike AdS curve $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$. Then, $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\beta}(s^\star(s))=\frac{1}{\sqrt{3}}(a\boldsymbol{\alpha}(s)+b\boldsymbol{t}(s)+c\boldsymbol{b}(s)) \tag{22}$$

such that $a^2+b^2-c^2=3,\ a,b,c\in\mathbb{R}$ from the Definition 3.4. Differentiating both sides of (22) with respect to s, we get

$$\boldsymbol{\beta'}(s^\star(s))=\frac{d\boldsymbol{\beta}}{ds^\star}\frac{ds^\star}{ds}=\frac{1}{\sqrt{3}}\left(a\boldsymbol{\alpha'}(s)+b\boldsymbol{t'}(s)+c\boldsymbol{b'}(s)\right)$$

and by using (2),

$$\boldsymbol{t_\beta}(s^\star(s))\frac{ds^\star}{ds}=\frac{1}{\sqrt{3}}\left(a\boldsymbol{t}(s)+b\left(-\boldsymbol{\alpha}(s)+\kappa_g(s)\boldsymbol{n}(s)\right)+c\left(-\tau_g(s)\boldsymbol{n}(s)\right)\right)$$

where

$$\frac{ds^\star}{ds}=\sqrt{\frac{(b\,\kappa_g(s)-c\,\tau_g(s))^2-(c^2+3)}{3}} \tag{23}$$

with the condition $(b\kappa_g(s)-c\tau_g(s))^2-\left(c^2+3\right)>0$.

(From now on, unless otherwise stated, we won't use the parameters "s" and "$s^\star$" in the following calculations for the sake of brevity).

Hence, the tangent vector of spacelike $\boldsymbol{\alpha t b}-$Smarandache AdS curve $\boldsymbol{\beta}$ is to be

$$\boldsymbol{t_\beta}=\frac{1}{\sqrt{\sigma}}\left(-b\boldsymbol{\alpha}+a\boldsymbol{t}+\left(b\kappa_g-c\tau_g\right)\boldsymbol{n}\right), \tag{24}$$

where $\sigma=\left(b\,\kappa_g-c\,\tau_g\right)^2-\left(c^2+3\right)$.

Differentiating both sides of (24) with respect to s, we have

$$\boldsymbol{t_\beta}'=\frac{\sqrt{3}}{\sigma^2}\left(\lambda_1\boldsymbol{\alpha}+\lambda_2\boldsymbol{t}+\lambda_3\boldsymbol{n}+\lambda_4\boldsymbol{b}\right) \tag{25}$$

by using again (2) and (23), where

$$\begin{cases} \lambda_1 &= b\left(b\kappa_g - c\tau_g\right)\left(b{\kappa_g}' - c{\tau_g}'\right) - a\sigma \\ \lambda_2 &= -a\left(b\kappa_g - c\tau_g\right)\left(b{\kappa_g}' - c{\tau_g}'\right) + \left(b\left(-1+{\kappa_g}^2\right) - c\kappa_g\tau_g\right)\sigma \\ \lambda_3 &= -\left(3+c^2\right)\left(b{\kappa_g}' - c{\tau_g}'\right) + a\kappa_g\sigma \\ \lambda_4 &= \tau_g\left(b\kappa_g - c\tau_g\right)\sigma \end{cases} \tag{26}$$

Now, we can compute

$$\boldsymbol{t_\beta}' - \boldsymbol{\beta} = \frac{1}{\sqrt{3}\sigma^2}\left(\left(3\lambda_1 - a\sigma^2\right)\boldsymbol{\alpha} + \left(3\lambda_2 - b\sigma^2\right)\boldsymbol{t} + 3\lambda_3\boldsymbol{n} + \left(3\lambda_4 - c\sigma^2\right)\boldsymbol{b}\right) \tag{27}$$

and

$$\|\boldsymbol{t_\beta}' - \boldsymbol{\beta}\| = \frac{1}{\sigma^2}\sqrt{-\sigma^4 + 2\left(a\lambda_1 + b\lambda_2 - c\lambda_4\right)\sigma^2 + 3\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right)}. \tag{28}$$

From the equations (27) and (28), the principal normal vector of $\boldsymbol{\beta}$ is

$$\boldsymbol{n_\beta} = \frac{1}{\sqrt{3\mu}}\left(\left(3\lambda_1 - a\sigma^2\right)\boldsymbol{\alpha} + \left(3\lambda_2 - b\sigma^2\right)\boldsymbol{t} + 3\lambda_3\boldsymbol{n} + \left(3\lambda_4 - c\sigma^2\right)\boldsymbol{b}\right) \tag{29}$$

and the geodesic curvature of $\boldsymbol{\beta}$ is

$$\widetilde{\kappa_g} = \frac{\sqrt{\mu}}{\sigma^2}, \tag{30}$$

where

$$\mu = -\sigma^4 + 2\left(a\lambda_1 + b\lambda_2 - c\lambda_4\right)\sigma^2 + 3\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right). \tag{31}$$

Also, from the equations (22), (24) and (29), the binormal vector of $\boldsymbol{\beta}$ as pseudo vector product of $\boldsymbol{\beta}, \boldsymbol{t_\beta}$ and $\boldsymbol{n_\beta}$ is given by

$$\boldsymbol{b_\beta} = \frac{1}{\sqrt{\sigma\mu}}\begin{pmatrix} (c(b\kappa_g - c\tau_g)\lambda_2 - (ac)\lambda_3 - b(b\kappa_g - c\tau_g)\lambda_4)\,\boldsymbol{\alpha} \\ -\,(c(b\kappa_g - c\tau_g)\lambda_1 + (bc)\lambda_3 - a(b\kappa_g - c\tau_g)\lambda_4)\,\boldsymbol{t} \\ -\left((ac)\lambda_1 + (bc)\lambda_2 - (c^2+3)\lambda_4\right)\boldsymbol{n} \\ -\left((b\kappa_g - c\tau_g)(b\lambda_1 - a\lambda_2) + (c^2+3)\lambda_3\right)\boldsymbol{b} \end{pmatrix}. \tag{32}$$

Finally, differentiating both sides of (25) with respect to s, we get

$$\boldsymbol{t_\beta}'' = \frac{-3}{\sigma^{7/2}}\begin{pmatrix} \left(2\lambda_1\sigma' - ({\lambda_1}' - \lambda_2)\sigma\right)\boldsymbol{\alpha} + \left(2\lambda_2\sigma' - (\lambda_1 + {\lambda_2}' + \kappa_g\lambda_3)\sigma\right)\boldsymbol{t} \\ + \left(2\lambda_3\sigma' - (\kappa_g\lambda_2 + {\lambda_3}' - \tau_g\lambda_4)\sigma\right)\boldsymbol{n} + \left(2\lambda_4\sigma' - (\tau_g\lambda_3 + {\lambda_4}')\sigma\right)\boldsymbol{b} \end{pmatrix} \tag{33}$$

by using again (2) and (23). Hence, from the equations (22), (24), (25), (30) and (33), the geodesic torsion of $\boldsymbol{\beta}$ is

$$\widetilde{\tau_g} = \frac{3}{\sigma\mu}\begin{pmatrix} c\left(a\lambda_3 - \lambda_2\left(b\kappa_g - c\tau_g\right)\right)\left(\lambda_2 - {\lambda_1}'\right) - c\left(b\lambda_3 + \lambda_1\left(b\kappa_g - c\tau_g\right)\right)\left(\lambda_1 + \kappa_g\lambda_3 + {\lambda_2}'\right) \\ +\lambda_4\left(b\kappa_g - c\tau_g\right)\left(b\left(\lambda_2 - {\lambda_1}'\right) + a\left(\lambda_1 + \kappa_g\lambda_3 + {\lambda_2}'\right)\right) + c\left(a\lambda_1 + b\lambda_2\right)\left(\kappa_g\lambda_2 - \tau_g\lambda_4 + {\lambda_3}'\right) \\ -\left(3+c^2\right)\lambda_4\left(\kappa_g\lambda_2 - \tau_g\lambda_4 + {\lambda_3}'\right) + \left(\left(3+c^2\right)\lambda_3 + \left(b\lambda_1 - a\lambda_2\right)\left(b\kappa_g - c\tau_g\right)\right)\left(\tau_g\lambda_3 + {\lambda_4}'\right) \end{pmatrix} \tag{34}$$

under the condition $a^2 + b^2 - c^2 = 3$. Thus, we obtain the Sabban aparatus of $\boldsymbol{\beta}$ for the choice

$v_i v_j v_k = \boldsymbol{\alpha t b}$.

It can be easily seen that the other types of $v_i v_j v_k-$Smarandache curves $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ by using same method as the above. The proof is complete. □

Corollary 3.6 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve and $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $v_i v_j v_k-$Smarandache AdS curve of $\boldsymbol{\alpha}$, then the following table holds for the special cases of $\boldsymbol{\alpha}$ under the conditions (20) and (21):*

$v_i v_j v_k$	$\boldsymbol{\alpha}$ *is planar curve*	$\boldsymbol{\alpha}$ *is horocycle*	$\boldsymbol{\alpha}$ *is helix*
$\boldsymbol{\alpha t n}$	*planar curve*	*undefined*	*helix*
$\boldsymbol{\alpha t b}$	*planar curve*	*undefined*	*helix*
$\boldsymbol{\alpha n b}$	*undefined*	*undefined*	*helix*
$\boldsymbol{t n b}$	*planar*	*undefined*	*helix*

Definition 3.7 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. Then the spacelike $\boldsymbol{\alpha t n b}-$Smarandache AdS curve $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ of $\boldsymbol{\alpha}$ is defined by*

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{4}}(a_0\boldsymbol{\alpha}(s) + b_0\boldsymbol{t}(s) + c_0\boldsymbol{n}(s) + d_0\boldsymbol{b}(s)), \tag{35}$$

where $a_0, b_0, c_0, d_0 \in \mathbb{R}$ such that

$${a_0}^2 + b_0^2 - {c_0}^2 - {d_0}^2 = 4. \tag{36}$$

Theorem 3.8 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. If $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ is spacelike $\boldsymbol{\alpha t n b}-$Smarandache AdS curve with Sabban frame $\{\boldsymbol{\beta}, \boldsymbol{t_\beta}, \boldsymbol{n_\beta}, \boldsymbol{b_\beta}\}$ and geodesic curvature $\widetilde{\kappa_g}$, geodesic torsion $\widetilde{\tau_g}$, then the Sabban apparatus of $\boldsymbol{\beta}$ can be constructed by the Sabban apparatus of $\boldsymbol{\alpha}$ under the condition*

$$(b_0\kappa_g(s) - d_0\tau_g(s))^2 - (a_0 + c_0\kappa_g(s))^2 + {c_0}^2\tau_g(s)^2 - {b_0}^2 > 0. \tag{37}$$

Proof Let $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $\boldsymbol{\alpha t n b}-$Smarandache AdS curve of timelike AdS curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. Then, $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{4}}(a_0\boldsymbol{\alpha}(s) + b_0\boldsymbol{t}(s) + c_0\boldsymbol{n}(s) + d_0\boldsymbol{b}(s)) \tag{38}$$

such that ${a_0}^2 + {b_0}^2 - {c_0}^2 - {d_0}^2 = 4,\ a_0, b_0, c_0, d_0 \in \mathbb{R}$ from the Definition 3.7. Differentiating both sides of (38) with respect to s, we get

$$\boldsymbol{\beta'}(s^\star(s)) = \frac{d\boldsymbol{\beta}}{ds^\star}\frac{ds^\star}{ds} = \frac{1}{\sqrt{4}}\left(a_0\boldsymbol{\alpha'}(s) + b_0\boldsymbol{t'}(s) + c_0\boldsymbol{n'} + d_0\boldsymbol{b'}(s)\right)$$

and by using (2),

$$\boldsymbol{t_\beta}(s^\star(s))\frac{ds^\star}{ds} = \frac{1}{\sqrt{4}}\left(a_0\boldsymbol{t}(s) + b_0\left(-\boldsymbol{\alpha}(s) + \kappa_g(s)\boldsymbol{n}(s)\right) + c_0\left(\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{b}(s)\right) + d_0\left(-\tau_g(s)\boldsymbol{n}(s)\right)\right)$$

where

$$\frac{ds^\star}{ds} = \sqrt{\frac{\left(b_0\kappa_g(s) - d_0\tau_g(s)\right)^2 - \left(a_0 + c_0\kappa_g(s)\right)^2 + {c_0}^2\tau_g(s)^2 - {b_0}^2}{4}} \tag{39}$$

with the condition $\left(b_0\kappa_g(s) - d_0\tau_g(s)\right)^2 - \left(a_0 + c_0\kappa_g(s)\right)^2 + {c_0}^2\tau_g(s)^2 - {b_0}^2 > 0$.

(From now on, unless otherwise stated, we won't use the parameters "s" and "$s^\star$" in the following calculations for the sake of brevity).

Hence, the tangent vector of spacelike $\boldsymbol{\alpha t n b}$−Smarandache AdS curve $\boldsymbol{\beta}$ is to be

$$\boldsymbol{t_\beta} = \frac{1}{\sqrt{\sigma}}\left(-b_0\boldsymbol{\alpha} + \left(a_0 + c_0\kappa_g\right)\boldsymbol{t} + \left(b_0\kappa_g - d_0\tau_g\right)\boldsymbol{n} + c_0\tau_g\boldsymbol{b}\right), \tag{40}$$

where $\sigma = \left(b_0\kappa_g - d_0\tau_g\right)^2 - \left(a_0 + c_0\kappa_g\right)^2 + {c_0}^2{\tau_g}^2 - {b_0}^2$.

Differentiating both sides of (40) with respect to s, we have

$$\boldsymbol{t_\beta}' = \frac{2}{\sigma^2}\left(\lambda_1\boldsymbol{\alpha} + \lambda_2\boldsymbol{t} + \lambda_3\boldsymbol{n} + \lambda_4\boldsymbol{b}\right) \tag{41}$$

by using again (2) and (39) where

$$\begin{aligned}
\lambda_1 &= -b_0\left(a_0c_0 + c_0^2\kappa_g - b_0\left(b_0\kappa_g - d_0\tau_g\right)\right){\kappa_g}' + b_0\left(c_0^2\tau_g - d_0\left(b_0\kappa_g - d_0\tau_g\right)\right){\tau_g}' - \left(a_0 + c_0\kappa_g\right)\sigma \\
\lambda_2 &= \begin{array}{l}\left(-b_0^2\left(c_0 + a_0\kappa_g\right) + b_0d_0\left(a_0 - c_0\kappa_g\right)\tau_g + c_0\left({c_0}^2 + {d_0}^2\right)\tau_g^2\right){\kappa_g}' \\ + \left(b_0d_0\kappa_g\left(a_0 + c_0\kappa_g\right) - \left(c_0^2 + d_0^2\right)\left(a_0 + c_0\kappa_g\right)\tau_g\right){\tau_g}' + \left(b_0\left(\kappa_g^2 - 1\right) - d_0\kappa_g\tau_g\right)\sigma\end{array} \\
\lambda_3 &= \begin{array}{l}-\left(a_0c_0\left(b_0\kappa_g + d_0\tau_g\right) + c_0^2\left(d_0\kappa_g\tau_g - b_0\left(\tau_g^2 - 1\right)\right) + b_0\left(4 + d_0^2\right)\right){\kappa_g}' \\ + \left(2a_0c_0d_0\kappa_g + c_0^2\left(d_0\left(1 + \kappa_g^2\right) - b_0\kappa_g\tau_g\right) + d_0\left(4 + d_0^2\right)\right){\tau_g}' + \left(a_0\kappa_g + c_0\left(\kappa_g^2 - \tau_g^2\right)\right)\sigma\end{array} \\
\lambda_4 &= \begin{array}{l}c_0\left(c_0\left(a_0 + c_0\kappa_g\right) - b_0\left(b_0\kappa_g - d_0\tau_g\right)\right)\tau_g{\kappa_g}' + \\ c_0\left(\tau_g\left(b_0d_0\kappa_g - \left(c_0^2 + d_0^2\right)\tau_g\right) + \sigma\right){\tau_g}' + \left(b_0\kappa_g - d_0\tau_g\right)\tau_g\sigma\end{array}
\end{aligned} \tag{42}$$

Now, we can compute

$$\boldsymbol{t_\beta}' - \boldsymbol{\beta} = \frac{1}{2\sigma^2}\left(\left(4\lambda_1 - a_0\sigma^2\right)\boldsymbol{\alpha} + \left(4\lambda_2 - b_0\sigma^2\right)\boldsymbol{t} + \left(4\lambda_3 - c_0\sigma^2\right)\boldsymbol{n} + \left(4\lambda_4 - d_0\sigma^2\right)\boldsymbol{b}\right) \tag{43}$$

and

$$\left\|\boldsymbol{t_\beta}' - \boldsymbol{\beta}\right\| = \frac{1}{\sigma^2}\sqrt{-\sigma^4 + 2\left(a_0\lambda_1 + b_0\lambda_2 - c_0\lambda_3 - d_0\lambda_4\right)\sigma^2 + 4\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right)}. \tag{44}$$

From the equations (43) and (44), the principal normal vector of $\boldsymbol{\beta}$ is

$$\boldsymbol{n_\beta} = \frac{1}{2\sqrt{\mu}}\left(\left(4\lambda_1 - a_0\sigma^2\right)\boldsymbol{\alpha} + \left(4\lambda_2 - b_0\sigma^2\right)\boldsymbol{t} + \left(4\lambda_3 - c_0\sigma^2\right)\boldsymbol{n} + \left(4\lambda_4 - d_0\sigma^2\right)\boldsymbol{b}\right) \tag{45}$$

and the geodesic curvature of $\boldsymbol{\beta}$ is

$$\widetilde{\kappa_g} = \frac{\sqrt{\mu}}{\sigma^2}, \tag{46}$$

where

$$\mu = -\sigma^4 + 2\left(a_0\lambda_1 + b_0\lambda_2 - c_0\lambda_3 - d_0\lambda_4\right)\sigma^2 + 4\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right) \tag{47}$$

Also, from the equations (38),(40) and (45), the binormal vector of $\boldsymbol{\beta}$ as pseudo vector product of $\boldsymbol{\beta}, \boldsymbol{t_\beta}$ and $\boldsymbol{n_\beta}$ is given by

$$\begin{aligned}\boldsymbol{b_\beta} \;=\;& \frac{1}{\sqrt{\mu}\sigma}\big((-b_0^2\kappa_g\lambda_4 + c_0(-d_0\kappa_g\lambda_3 + a_0\lambda_4) - c_0^2(\tau_g\lambda_2 - \kappa_g\lambda_4) - d_0(d_0\tau_g\lambda_2 + a_0\lambda_3) \\ & + b_0(c_0\tau_g\lambda_3 + d_0(\kappa_g\lambda_2 + \tau_g\lambda_4)))\boldsymbol{\alpha} + (b_0(-d_0(\kappa_g\lambda_1 + \lambda_3) + (c_0 + a_0\kappa_g)\lambda_4) \\ & + (c_0^2\lambda_1 - a_0c_0\lambda_3 + d_0(d_0\lambda_1 - a_0\lambda_4))\tau_g)\boldsymbol{t} + (a_0^2\lambda_4 - b_0(d_0\lambda_2 - b_0\lambda_4) \\ & - c_0\lambda_1(d_0\kappa_g - b_0\tau_g) - a_0(d_0\lambda_1 + c_0(\tau_g\lambda_2 - \kappa_g\lambda_4)))\boldsymbol{n} + (c_0^2\kappa_g\lambda_1 - a_0^2\lambda_3 \\ & - b_0^2(\kappa_g\lambda_1 + \lambda_3) + b_0(c_0\lambda_2 + d_0\tau_g\lambda_1) + a_0(c_0(\lambda_1 - \kappa_g\lambda_3) + (b_0\kappa_g - d_0\tau_g)\lambda_2 t))\boldsymbol{b}\big)\end{aligned} \tag{48}$$

Finally, differentiating both sides of (41) with respect to s, we get

$$\boldsymbol{t_\beta}'' = \frac{-4}{\sigma^{7/2}}\left(\begin{array}{c}\left(2\lambda_1\sigma' - \left({\lambda_1}' - \lambda_2\right)\sigma\right)\boldsymbol{\alpha} + \left(2\lambda_2\sigma' - (\lambda_1 + {\lambda_2}' + \kappa_g\lambda_3)\sigma\right)\boldsymbol{t} \\ + \left(2\lambda_3\sigma' - (\kappa_g\lambda_2 + {\lambda_3}' - \tau_g\lambda_4)\sigma\right)\boldsymbol{n} + \left(2\lambda_4\sigma' - (\tau_g\lambda_3 + {\lambda_4}')\sigma\right)\boldsymbol{b}\end{array}\right) \tag{49}$$

by using again (2) and (49). Hence, from the equations (38), (40), (41), (46) and (49), the geodesic torsion of $\boldsymbol{\beta}$ is

$$\begin{aligned}\widetilde{\tau_g} \;=\;& \frac{4}{\mu\sigma}\big((b_0^2\kappa_g\lambda_4 + (a_0 + c_0\kappa_g)(d_0\lambda_3 - c_0\lambda_4) + (c_0^2 + d_0^2)\tau_g\lambda_2 \\ & - b_0(c_0\tau_g\lambda_3 + d_0(\kappa_g\lambda_2 + \tau_g\lambda_4)))(\lambda_2 - \lambda_1') \\ & + (b_0(-d_0(\kappa_g\lambda_1 + \lambda_3) + (c_0 + a_0\kappa_g)\lambda_4) + (c_0^2\lambda_1 - a_0c_0\lambda_3 \\ & + d_0\tau_g(d_0\lambda_1 - a_0\lambda_4)))(\lambda_1 + \kappa_g\lambda_3 + \lambda_2') + (d_0((a_0 + c_0\kappa_g)\lambda_1 \\ & + b_0\lambda_2) - (a_0(a_0 + c_0\kappa_g) + b_0^2)\lambda_4 - c_0\tau_g(b_0\lambda_1 - a_0\lambda_2))(\kappa_g\lambda_2 - \lambda_4\tau_g + \lambda_3') \\ & + (-c_0^2\kappa_g\lambda_1 + a_0^2\lambda_3 + b_0^2(\kappa_g\lambda_1 + \lambda_3) - b_0(c_0\lambda_2 + d_0\lambda_1\tau_g) + a_0(c_0(-\lambda_1 + \kappa_g\lambda_3) \\ & + \lambda_2(-b_0\kappa_g + d_0\tau_g)))(\lambda_3\tau_g + \lambda_4'))\end{aligned} \tag{50}$$

under the condition (36). The proof is complete. □

Corollary 3.9 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve and $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $\boldsymbol{\alpha tnb}-$Smarandache AdS curve of $\boldsymbol{\alpha}$, then the following table holds for the special cases of $\boldsymbol{\alpha}$ under the conditions (36) and (37):*

	$\boldsymbol{\alpha}$ *is planar curve*	$\boldsymbol{\alpha}$ *is horocycle*	$\boldsymbol{\alpha}$ *is helix*
$\boldsymbol{\alpha tnb}$	*planar curve*	*undefined*	*helix*

Consequently, we can give the following corollaries by Corollary 3.3, Corollary 3.6, Corollary 3.9.

Corollary 3.10 *Let $\boldsymbol{\alpha}$ be a timelike horocycle in $\mathbb{H}_1^3$. Then, there exist no spacelike Smarandache AdS curve of $\boldsymbol{\alpha}$ in $\mathbb{H}_1^3$.*

Corollary 3.11 *Let $\boldsymbol{\alpha}$ be a timelike AdS curve and $\boldsymbol{\beta}$ be any spacelike Smarandache AdS curve of $\boldsymbol{\alpha}$. Then, $\boldsymbol{\alpha}$ is helix if and only if $\boldsymbol{\beta}$ is helix.*

§4. Examples and AdS Stereographic Projection

Let $\mathbb{R}_1^3$ denote Minkowski 3-space (three-dimensional semi Euclidean space with index one), that is, the real vector space $\mathbb{R}^3$ endowed with the scalar product

$$\langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}} \rangle_\star = -\overline{x_1}\,\overline{y_1} + \overline{x_2}\,\overline{y_2} + \overline{x_3}\,\overline{y_3}$$

for all $\overline{\boldsymbol{x}} = (\overline{x_1}, \overline{x_2}, \overline{x_3})$, $\overline{\boldsymbol{y}} = (\overline{y_1}, \overline{y_2}, \overline{y_3}) \in \mathbb{R}^3$. The set

$$\mathbb{S}_1^2 = \left\{ \overline{\boldsymbol{x}} \in \mathbb{R}_1^3 \mid \langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star = 1 \right\}$$

is called *de Sitter plane* (unit pseudosphere with dimension 2 and index 1 in $\mathbb{R}_1^3$). Then, the stereographic projection Φ from $\mathbb{H}_1^3$ to $\mathbb{R}_1^3$ and its inverse is given by

$$\Phi : \mathbb{H}_1^3 \backslash \Gamma \to \mathbb{R}_1^3 \backslash \mathbb{S}_1^2\,, \;\; \Phi(\boldsymbol{x}) = \left(\frac{x_2}{1+x_1}, \frac{x_3}{1+x_1}, \frac{x_4}{1+x_1} \right)$$

and

$$\Phi^{-1} : \mathbb{R}_1^3 \backslash \mathbb{S}_1^2 \to \mathbb{H}_1^3 \backslash \Gamma\,, \;\; \Phi^{-1}(\overline{\boldsymbol{x}}) = \left(\frac{1+\langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star}{1-\langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star}, \frac{2\overline{x_1}}{1-\langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star}, \frac{2\overline{x_2}}{1-\langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star}, \frac{2\overline{x_3}}{1-\langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star} \right)$$

according to set $\Gamma = \left\{ \boldsymbol{x} \in \mathbb{H}_1^3 \mid x_1 = -1 \right\}$, respectively. It is easily seen that Φ is conformal map.

Hence, the stereographic projection Φ of $\mathbb{H}_1^3$ is called *AdS stereographic projection*. Now, we can give the following important proposition about projection regions of any AdS curve.

Proposition 4.1 *Let Φ be AdS stereographic projection. Then the following statements are satisfied for all $\boldsymbol{x} \in \mathbb{H}_1^3$:*

(a) $x_1 > -1 \Leftrightarrow \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}) \rangle_\star < 1$;

(b) $x_1 < -1 \Leftrightarrow \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}) \rangle_\star > 1$.

Now, we give an example for timelike AdS curve as helix and some spacelike Smarandache AdS curves of the base curve. Besides, we draw pictures of these curves by using Mathematica.

Example 4.2 Let AdS curve $\boldsymbol{\alpha}$ be

$$\begin{aligned}\boldsymbol{\alpha}(s) &= \Big(\sqrt{2}\cosh(\sqrt{2}s), 2^{1/4}\cosh(\sqrt{5}s)+\sqrt{1+\sqrt{2}}\sinh(\sqrt{5}s),\\ &\qquad \sqrt{2}\sinh(\sqrt{2}s), \sqrt{1+\sqrt{2}}\cosh(\sqrt{5}s)+2^{1/4}\sinh\sqrt{5}s\Big).\end{aligned}$$

Then the tangent vector of $\boldsymbol{\alpha}$ is given by

$$\begin{aligned}\boldsymbol{t}(s) &= \Big(2\sinh(\sqrt{2}s), \sqrt{5\left(1+\sqrt{2}\right)}\cosh\sqrt{5}s+2^{1/4}\sqrt{5}\sinh(\sqrt{5}s),\\ &\qquad 2\cosh(\sqrt{2}s), 2^{1/4}\sqrt{5}\cosh(\sqrt{5}s)+\sqrt{5\left(1+\sqrt{2}\right)}\sinh(\sqrt{5}s)\Big),\end{aligned}$$

and since

$$\langle \boldsymbol{t}(s)\,,\,\boldsymbol{t}(s)\rangle = -1,$$

$\boldsymbol{\alpha}$ is timelike AdS curve. By direct calculations, we get easily the following rest of Sabban frame's elements of $\boldsymbol{\alpha}$:

$$\begin{aligned}\boldsymbol{n}(s) &= \Big(\cosh(\sqrt{2}s), 2^{3/4}\cosh(\sqrt{5}s)+\sqrt{2\left(1+\sqrt{2}\right)}\sinh(\sqrt{5}s),\\ &\qquad \sinh(\sqrt{2}s), \sqrt{2\left(1+\sqrt{2}\right)}\cosh(\sqrt{5}s)+2^{3/4}\sinh(\sqrt{5}s)\Big),\\ \boldsymbol{b}(s) &= \Big(\sqrt{5}\sinh(\sqrt{2}s), 2\sqrt{1+\sqrt{2}}\cosh(\sqrt{5}s)+2^{5/4}\sinh(\sqrt{5}s),\\ &\qquad \sqrt{5}\cosh(\sqrt{2}s), 2^{5/4}\cosh(\sqrt{5}s)+2\sqrt{1+\sqrt{2}}\sinh(\sqrt{5}s)\Big).\end{aligned}$$

and the geodesic curvatures of $\boldsymbol{\alpha}$ are obtained by

$$\kappa_g = 3\sqrt{2}\,,\ \tau_g = -\sqrt{10}.$$

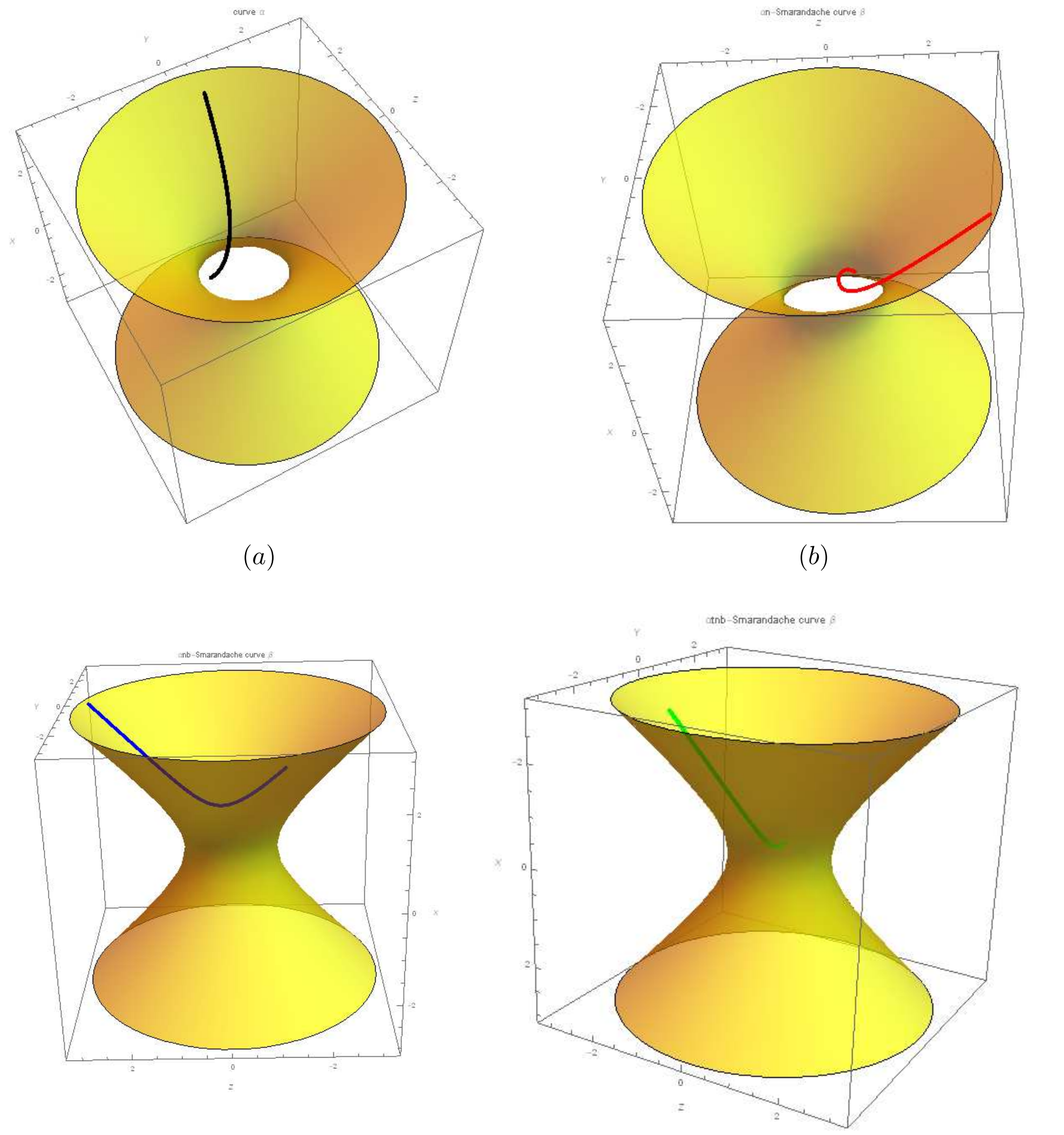
Thus, $\boldsymbol{\alpha}$ is a helix in $\mathbb{H}^3_1$. Now, we can define some spacelike Smarandache AdS curves of $\boldsymbol{\alpha}$ as the following:

$$\begin{aligned}{}_{\boldsymbol{\alpha n}}\boldsymbol{\beta}(s^\star(s)) &= \tfrac{1}{\sqrt{2}}\left(\sqrt{3}\,\boldsymbol{\alpha}(s)-\boldsymbol{n}(s)\right)\\ {}_{\boldsymbol{\alpha nb}}\boldsymbol{\beta}(s^\star(s)) &= \tfrac{1}{\sqrt{3}}\left(\sqrt{6}\,\boldsymbol{\alpha}(s)-\sqrt{2}\,\boldsymbol{n}(s)+\boldsymbol{b}(s)\right)\\ {}_{\boldsymbol{\alpha tnb}}\boldsymbol{\beta}(s^\star(s)) &= \tfrac{1}{2}\left(\tfrac{\sqrt{71}}{6}\boldsymbol{\alpha}(s)-\tfrac{3}{2}\boldsymbol{t}(s)+\tfrac{1}{3}\boldsymbol{n}(s)+\tfrac{1}{3}\boldsymbol{b}(s)\right)\end{aligned}$$

and theirs geodesic curvatures are obtained by

$$\boldsymbol{\alpha n}\kappa_g = 1.9647\,,\ \boldsymbol{\alpha n}\tau_g = -0.0619$$
$$\boldsymbol{\alpha nb}\kappa_g = 1.9773\,,\ \boldsymbol{\alpha nb}\tau_g = -0.0126$$
$$\boldsymbol{\alpha tnb}\kappa_g = 2.0067\,,\ \boldsymbol{\alpha tnb}\tau_g = -0.0044$$

in numeric form, respectively. Hence, the above spacelike Smarandache AdS curves of $\boldsymbol{\alpha}$ are also helix in $\mathbb{H}^3_1$, seeing Figure 1.



(a) (b)

(c) (d)

Figure 1

where, (a) is the timelike AdS helix $\boldsymbol{\alpha}$, (b) the spacelike $\boldsymbol{\alpha n}$-Smarandache AdS helix of $\boldsymbol{\alpha}$, (c) the spacelike $\boldsymbol{\alpha nb}$-Smarandache AdS helix of $\boldsymbol{\alpha}$ and (d) the spacelike $\boldsymbol{\alpha tnb}$-Smarandache AdS helix of $\boldsymbol{\alpha}$.

§5. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Conformal Ricci Soliton in Almost $C(\lambda)$ Manifold

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Abstract: In this paper we have studied conformal curvature tensor, Ricci curvature tensor, projective curvature tensor in almost $C(\lambda)$ manifold admitting conformal Ricci soliton. We have studied conformally semi symmetric almost $C(\lambda)$ manifold admitting conformal Ricci soliton. We have found that a Ricci conharmonically symmetric almost $C(\lambda)$ manifold admitting conformal Ricci soliton is Einstein manifold. Similarly we have proved that a conformally symmetric almost $C(\lambda)$ manifold M with respect to projective curvature tensor admitting conformal Ricci soliton is η-Einstein manifold. We have studied Ricci projectively symmetric almost $C(\lambda)$ manifold also.

Key Words: Almost $C(\lambda)$ manifold, Ricci flow, conformal Ricci soliton, conformal curvature tensor, Ricci curvature tensor, projective curvature tensor.

AMS(2010): 53C15, 53C20, 53C25, 53D10.

§1. Introduction

The concept of Ricci flow was first introduced by R. S. Hamilton [5] in 1982. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifold admits a geometric decomposition. Hamilton also [6]classified all compact manifolds with positive curvature operator in dimension four. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric g.

A self-similar solution to the Ricci flow [6], [10] is called a Ricci soliton [5] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\pounds_X g + 2S = 2\lambda g, \tag{1.2}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero and negative respectively.

In 2004, A.E. Fischer [4] introduced the concept of conformal Ricci flow which is a variation

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of the classical Ricci flow equation. In classical Ricci flow equation the unit volume constraint plays an important role but in conformal Ricci flow equation scalar curvature r is considered as constraint. As the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. The conformal Ricci flow equation on M is defined by the equation [4],

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \tag{1.3}$$

and $r = -1$, where p is a scalar non-dynamical field(time dependent scalar field), r is the scalar curvature of the manifold and n is the dimension of manifold.

The notion of conformal Ricci soliton was introduced by N. Basu and A. Bhattacharyya [1] in 2015 and the conformal Ricci soliton equation is given by

$$\pounds_X g + 2S = [2\sigma - (p + \frac{2}{n})]g. \tag{1.4}$$

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

In 1981, the notion of almost $C(\lambda)$ manifold was first introduced by D. Janssen and L. Vanhecke [7]. After that Z. Olszak and R. Rosca [9] have also studied such manifolds. Our present paper is motivated by this work.

In this paper we have studied conformal curvature tensor, conharmonic curvature tensor, Ricci curvature tensor, projective curvature tensor in almost $C(\lambda)$ manifold admitting conformal Ricci soliton. We have studied conformally semi symmetric almost $C(\lambda)$ manifold admitting conformal Ricci soliton. We have found that a Ricci conharmonically symmetric almost $C(\lambda)$ manifold admitting conformal Ricci soliton is Einstein manifold. Similarly we have proved that a conformally symmetric almost $C(\lambda)$ manifold M with respect to projective curvature tensor admitting conformal Ricci soliton is η-Einstein manifold. We have also studied Ricci projectively symmetric almost $C(\lambda)$ manifold.

\$2. Preliminaries

Let M be a $(2n+1)$ dimensional connected almost contact metric manifold with an almost contact metric structure(ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ is a covariant vector field, η is a 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta o \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.5}$$

$$\nabla_X \xi = -\phi X, \tag{2.6}$$

for all $X, Y \in \chi(M)$.

If an almost contact Riemannian manifold M satisfies the condition

$$S = ag + b\eta \otimes \eta,$$

for some functions $a, b \in C^\infty(M)$ and S is the Ricci tensor, then M is said to be an η- Einstein manifold.

An almost contact manifold is called an almost $C(\lambda)$ manifold if the Riemann curvature R satisfies the following relations [8]

$$R(X,Y)Z = R(\phi X, \phi Y)Z - \lambda[Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y, Z)$$
$$+g(\phi X, Z)\phi Y], \tag{2.7}$$

where $X, Y, Z \in TM$ and λ is a real number.

From (2.7) we have

$$\left.\begin{aligned} R(X,Y)\xi &= R(\phi X, \phi Y)\xi - \lambda[X\eta(Y) - Y\eta(X)], \\ R(\xi, X)Y &= \lambda[\eta(Y)X - g(X,Y)\xi]. \end{aligned}\right\} \tag{2.8}$$

Now from definition of Lie derivative we have

$$(\pounds_\xi g)(X,Y) = (\nabla_\xi g)(X,Y) + g(-\phi X, Y) + g(X, -\phi Y) = 0 \tag{2.9}$$
$$(\because g(X, \phi Y) = -g(\phi X, Y)).$$

Now applying (2.9) in conformal Ricci soliton equation (1.4) we get

$$S(X,Y) = Ag(X,Y), \tag{2.10}$$

where $A = \frac{1}{2}[2\sigma - (p + \frac{2}{n})]$. Hence the manifold becomes an Einstein manifold.

Also we have,

$$QX = AX. \tag{2.11}$$

If we put $Y = \xi$ in (2.10) we get

$$S(X,\xi) = A\eta(X). \tag{2.12}$$

Again if we put $X = \xi$ in (2.12) we get

$$S(\xi,\xi) = A. \tag{2.13}$$

Using these results we shall prove some important results of almost $C(\lambda)$ manifold in the following sections.

§3. Almost $C(\lambda)$ Manifold Admitting Conformal Ricci Soliton and $R(\xi, X).C = 0$

Let M be a $(2n+1)$ dimensional almost $C(\lambda)$ manifold admitting conformal Ricci soliton (g, V, σ). Conformal curvature tensor C on M is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$
$$+[\frac{r}{2n(2n-1)}][g(Y,Z)X - g(X,Z)Y], \tag{3.1}$$

where r is scalar curvature.

Since the manifold satisfies conformal Ricci soliton so we have $r = -1$ ([4]).

After putting $r = -1$ and $Z = \xi$ in (3.1) we have

$$C(X,Y)\xi = R(X,Y)\xi - \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY]$$
$$-\frac{1}{2n(2n-1)}[g(Y,\xi)X - g(X,\xi)Y]. \tag{3.2}$$

Using $(2.4), (2.8), (2.11), (2.12)$ in (3.2) we get

$$C(X,Y)\xi = R(\phi X, \phi Y)\xi - \lambda[X\eta(Y) - Y\eta(X)] - \frac{1}{2n-1}[A\eta(Y)X - A\eta(X)Y$$
$$+\eta(Y)AX - \eta(X)AY] - \frac{1}{2n(2n-1)}[\eta(Y)X - \eta(X)Y].$$

After a brief simplification we get

$$C(X,Y)\xi = R(\phi X, \phi Y)\xi - B(\eta(Y)X - \eta(X)Y), \tag{3.3}$$

where $B = \lambda + \frac{2A}{2n-1} + \frac{1}{2n(2n-1)}$, and

$$\eta(C(X,Y)Z) = \eta(R(\phi X, \phi Y)Z) + B[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]. \tag{3.4}$$

Now we assume that $R(\xi, X).C = 0$ holds in M i.e. the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ ([11]), which implies

$$R(\xi, X)(C(Y,Z)W) - C(R(\xi, X)Y, Z)W - C(Y, R(\xi, X)Z)W$$
$$-C(Y,Z)R(\xi, X)W = 0, \tag{3.5}$$

for all vector fields X, Y, Z, W on M.

Using (2.8) in (3.5) and putting $W=\xi$ we get

$$\eta(C(Y,Z)\xi)X-g(X,C(Y,Z)\xi)\xi-\eta(Y)C(X,Z)\xi+g(X,Y)C(\xi,Z)\xi-\eta(Z)C(Y,X)\xi$$
$$+g(X,Z)C(Y,\xi)\xi-\eta(\xi)C(Y,Z)X+g(X,\xi)C(Y,Z)\xi=0. \tag{3.6}$$

Using $(2.1),(3.3),(3.4)$ in (3.6) we have

$$\eta(R(\phi Y,\phi Z)\xi))X-g(X,R(\phi Y,\phi Z)\xi-B[Y\eta(Z)-Z\eta(Y)])\xi-\eta(Y)C(X,Z)\xi$$
$$+g(X,Y)C(\xi,Z)\xi-\eta(Z)C(Y,X)\xi+g(X,Z)C(Y,\xi)\xi-C(Y,Z)X$$
$$+\eta(X)C(Y,Z)\xi=0. \tag{3.7}$$

Operating with η and putting $Z=\xi$ in (3.7) we get

$$Bg(X,Y)-B\eta(X)\eta(Y)-\eta(C(Y,\xi)X)-\eta(R(\phi Y,\phi X)\xi)=0. \tag{3.8}$$

Now,

$$C(Y,\xi)X=R(Y,\xi)X-\frac{1}{2n-1}[S(\xi,X)Y-S(Y,X)\xi+g(\xi,X)QY-g(Y,X)Q\xi]$$
$$-\frac{1}{2n(2n-1)}[g(\xi,X)Y-g(Y,X)\xi]. \tag{3.9}$$

Using $(2.1),(2.8),(2.12)$ in (3.9) and operating with η we get

$$\eta(C(Y,\xi)X)=(\lambda+\frac{A}{2n-1}+\frac{1}{2n(2n-1)})g(X,Y)-(\lambda+\frac{2A}{2n-1}$$
$$+\frac{1}{2n(2n-1)})\eta(X)\eta(Y)+\frac{1}{2n-1}S(X,Y). \tag{3.10}$$

Putting (3.10) in (3.8) we obtain

$$\frac{A}{2n-1}g(X,Y)+\eta(R(\phi X,\phi Y)\xi)-\frac{1}{2n-1}S(X,Y)=0. \tag{3.11}$$

In view of (2.8) we get from (3.11)

$$\frac{A}{2n-1}g(X,Y)+\eta(R(X,Y)\xi)-\frac{1}{2n-1}S(X,Y)=0,$$

which can be written as

$$\frac{A}{2n-1}g(X,Y)-\frac{1}{2n-1}S(X,Y)=-g(R(X,Y)\xi,\xi). \tag{3.12}$$

Then we have

$$S(X,Y)=Ag(X,Y),$$

since $g(R(X,Y)\xi,\xi)=0$, where $A=\frac{1}{2}[2\sigma-(p+\frac{2}{n})]$.

Theorem 3.1 *If an almost $C(\lambda)$ manifold admitting conformal Ricci soliton is conformally semi symmetric i.e. $R(\xi,X).C=0$, then the manifold is Einstein manifold where C is Conformal curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§4. Almost $C(\lambda)$ Manifold Admitting Conformal Ricci Soliton and $K(\xi,X).S=0$

Let M be a $(2n+1)$ dimensional almost $C(\lambda)$ manifold admitting conformal Ricci soliton (g,V,σ). The conharmonic curvature tensor K on M is defined by [3]

$$K(X,Y)Z=R(X,Y)Z-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y$$
$$+g(Y,Z)QX-g(X,Z)QY], \tag{4.1}$$

for all $X,Y,Z\in\chi(M)$, R is the curvature tensor and Q is the Ricci operator.

Also the equation (4.1) can be written in the form

$$K(\xi,X)Y=R(\xi,X)Y-\frac{1}{2n-1}[S(X,Y)\xi-S(\xi,Y)X+g(X,Y)Q\xi$$
$$-g(\xi,Y)QX]. \tag{4.2}$$

Using (2.8), (2.11), (2.12) in (4.2) we have

$$K(\xi,X)Y=\lambda[\eta(Y)X-g(X,Y)\xi]-\frac{1}{2n-1}[S(X,Y)\xi-A\eta(Y)X-\eta(Y)AX$$
$$+g(X,Y)A\xi]. \tag{4.3}$$

Similarly from (4.2) we get

$$K(\xi,X)Z=\lambda[\eta(Z)X-g(X,Z)\xi]-\frac{1}{2n-1}[S(X,Z)\xi-A\eta(Z)X-\eta(Z)AX$$
$$+g(X,Z)A\xi]. \tag{4.4}$$

Now we assume that the tensor derivative of S by $K(\xi,X)$ is zero i.e. $K(\xi,X).S=0$ (the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ ([11]). It follows that

$$S(K(\xi,X)Y,Z)+S(Y,K(\xi,X)Z)=0, \tag{4.5}$$

which implies

$$S(\lambda\eta(Y)X-\lambda g(X,Y)\xi-\frac{1}{2n-1}S(X,Y)\xi+\frac{A}{2n-1}\eta(Y)X$$
$$-\frac{A}{2n-1}g(X,Y)\xi+\frac{A}{2n-1}\eta(Y)X,Z)+S(Y,\lambda\eta(Z)X-\lambda g(X,Z)\xi$$

$$-\frac{1}{2n-1}S(X,Z)\xi + \frac{A}{2n-1}\eta(Z)X - \frac{A}{2n-1}g(X,Z)\xi + \frac{A}{2n-1}\eta(Z)X) = 0. \tag{4.6}$$

Putting $Z = \xi$ in (4.6), using $(2.1), (2.4), (2.12), (2.13)$ and after a long calculation we obtain

$$S(X,Y) = Ag(X,Y),$$

where $A = \frac{1}{2}[2\sigma - (p + \frac{2}{n})]$.

Theorem 4.1 *If an almost $C(\lambda)$ manifold admitting conformal Ricci soliton and the manifold is Ricci conharmonically symmetric i.e. $K(\xi,X).S = 0$, then the manifold is Einstein manifold where K is conharmonic curvature tensor and S is a Ricci tensor.*

§5. Almost $C(\lambda)$ Manifold Admitting Conformal Ricci Soliton and $P(\xi,X).C = 0$

Let M be a $(2n+1)$ dimensional almost $C(\lambda)$ manifold admitting conformal Ricci soliton (g,V,σ). The Weyl projective curvature tensor P on M is given by [2]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{5.1}$$

(5.1) can be written as

$$P(\xi,X)Y = R(\xi,X)Y - \frac{1}{2n}[S(X,Y)\xi - S(\xi,Y)X].$$

Using $(2.8), (2.12)$ in the above equation we get

$$P(\xi,X)Y = \lambda[\eta(Y)X - g(X,Y)\xi] - \frac{1}{2n}[S(X,Y)\xi - A\eta(Y)X]. \tag{5.2}$$

Now we consider that the tensor derivative of C by $P(\xi,X)$ is zero i.e. $P(\xi,X).C = 0$ holds in M (the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ [11]). So

$$P(\xi,X)C(Y,Z)W - C(P(\xi,X)Y,Z)W - C(Y,P(\xi,X)Z)W$$

$$-C(Y,Z)P(\xi,X)W = 0 \tag{5.3}$$

for all vector fields X,Y,Z,W on M.

Using (5.2) in (5.3) and putting $W = \xi$ we get

$$\lambda\eta(C(Y,Z)\xi)X - \lambda g(X,C(Y,Z)\xi)\xi - \frac{1}{2n}S(X,C(Y,Z)\xi)\xi + \frac{A}{2n}\eta(C(Y,Z)\xi)X$$

$$-\lambda\eta(Y)C(X,Z)\xi - \frac{A}{2n}\eta(Y)C(X,Z)\xi - \lambda\eta(Z)C(Y,X)\xi - \frac{A}{2n}\eta(Z)C(Y,X)\xi$$

$$-\lambda\eta(\xi)C(Y,Z)X + \lambda g(X,\xi)C(Y,Z)\xi + \frac{1}{2n}S(X,\xi)C(Y,Z)\xi - \frac{A}{2n}\eta(\xi)C(Y,Z)X = 0. \tag{5.4}$$

Operating with η, using $(2.4), (2.12), (3.3)$ and putting $Z = \xi$ we get after a lengthy calculation that

$$(\lambda B-(\lambda+\frac{A}{2n})(\lambda+\frac{A}{2n-1}+\frac{1}{2n(n-1)}))g(X,Y)$$
$$+((\lambda+\frac{A}{2n})(\lambda+\frac{A}{2n-1}+\frac{1}{2n(n-1)})-\lambda B-\frac{AB}{2n})\eta(X)\eta(Y)$$
$$=((\lambda+\frac{A}{2n})(\frac{1}{2n-1})-\frac{B}{2n})S(X,Y),$$

which clearly shows that the manifold is η-Einstein.

Theorem 5.1 *If an almost $C(\lambda)$ manifold admitting conformal Ricci soliton and $P(\xi,X).C=0$ holds i.e. the manifold is conformally symmetric with respect to projective curvature tensor, then the manifold becomes η-Einstein manifold, where P is projective curvature tensor and C is conformal curvature tensor.*

§6. Almost $C(\lambda)$ Manifold Admitting Conformal Ricci Soliton and $R(\xi,X).P=0$

Let M be a $(2n+1)$ dimensional almost $C(\lambda)$ manifold admitting conformal Ricci soliton (g,V,σ). We assume that the manifold is projectively semi-symmetric i.e. $R(\xi,X).P=0$ holds in M, which implies

$$R(\xi,X)(P(Y,Z)W)-P(R(\xi,X)Y,Z)W-P(Y,R(\xi,X)Z)W$$
$$-P(Y,Z)R(\xi,X)W=0 \tag{6.1}$$

for all vector fields X,Y,Z,W on M.

Using (2.8) in (6.1) and putting $W=\xi$ we get

$$\lambda\eta(R(Y,Z)\xi-\frac{1}{2n}S(Z,\xi)Y+\frac{1}{2n}S(Y,\xi)Z)X-\lambda g(X,R(Y,Z)\xi-\frac{1}{2n}S(Z,\xi)Y$$
$$+\frac{1}{2n}S(Y,\xi)Z)\xi-\lambda\eta(Y)P(X,Z)\xi+\lambda g(X,Y)P(\xi,Z)\xi-\lambda\eta(Z)P(Y,X)\xi$$
$$+\lambda g(X,Z)P(Y,\xi)\xi-\lambda P(Y,Z)X+\lambda\eta(X)P(Y,Z)\xi=0.$$

Using $(2.4),(2.8),(2.12)$ and operating with η in the above equation we get

$$-\lambda^2\eta(Y)g(X,Z)+\frac{\lambda A}{2n}\eta(Z)g(X,Y)-\frac{\lambda A}{2n}\eta(Y)g(X,Z)$$
$$+\lambda^2 g(X,Y)\eta(Z)-\lambda\eta(P(Y,Z)X)=0. \tag{6.2}$$

Putting $Z=\xi$ in (6.2) we get

$$S(X,Y)=Ag(X,Y),$$

which implies that the manifold is an Einstein manifold.

Theorem 6.1 *If an almost $C(\lambda)$ manifold admitting conformal Ricci soliton and $R(\xi, X).P = 0$ holds i.e. the manifold is projectively semi-symmetric, then the manifold is an Einstein manifold, where P is projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§7. Almost $C(\lambda)$ Manifold Admitting Conformal Ricci Soliton and $P(\xi, X).S = 0$

Let M be a $(2n + 1)$ dimensional almost $C(\lambda)$ manifold admitting conformal Ricci soliton (g, V, σ). Now the equation (5.1) can be written as

$$P(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n}[S(X, Y)\xi - S(\xi, Y)X] \tag{7.1}$$

and

$$P(\xi, X)Z = R(\xi, X)Z - \frac{1}{2n}[S(X, Z)\xi - S(\xi, Z)X]. \tag{7.2}$$

Now we assume that the manifold is Ricci projectively symmetric i.e. $P(\xi, X).S = 0$ holds in M, which gives

$$S(P(\xi, X)Y, Z) + S(Y, P(\xi, X)Z) = 0. \tag{7.3}$$

Using $(2.10), (2.12), (7.1), (7.2)$ in (7.3) we have

$$Ag(R(\xi, X)Y - \frac{1}{2n}S(X, Y)\xi + \frac{A}{2n}\eta(Y)X, Z) + Ag(Y, R(\xi, X)Z$$
$$-\frac{1}{2n}S(X, Z)\xi + \frac{A}{2n}\eta(Z)X) = 0. \tag{7.4}$$

Using $(2.4), (2.8)$ in (7.4) and putting $Z = \xi$ we get

$$S(X, Y) = Ag(X, Y),$$

which proves that the manifold is an Einstein manifold.

Theorem 7.1 *If an almost $C(\lambda)$ manifold admitting conformal Ricci soliton and $P(\xi, X).S = 0$ holds i.e. the manifold is Ricci projectively symmetric, then the manifold is an Einstein manifold, where P is projective curvature tensor and S is the Ricci tensor.*

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Labeled Graph — A Mathematical Element

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Abstract: The universality of contradiction and connection of things in nature implies that a thing is nothing else but a labeled topological graph G^L with a labeling map $L : V(G)\bigcup E(G) \to \mathscr{L}$ in space, which concludes also that labeled graph should be an element for understanding things in the world. This fact proposes 2 directions on labeled graphs: (1) verify a graph family $\mathscr{G}$ whether or not they can be labeled by a labeling L constraint on special conditions, and (2) establish mathematical systems such as those of groups, rings, linear spaces or Banach spaces over graph G, i.e., view labeled graphs G^L as elements of that system. However, all results on labeled graphs are nearly concentrated on the first in past decades, which is in fact searching structure G of the labeling set $\mathscr{L}$. The main purpose of this survey is to show the role of labeled graphs in extending mathematical systems over graphs G, particularly graphical tensors and $\overrightarrow{G}$-flows with conservation laws and applications to physics and other sciences such as those of labeled graphs with sets or Euclidean spaces $\mathbb{R}^n$ labeling, labeled graph solutions of non-solvable systems of differential equations with global stability and extended Banach or Hilbert $\overrightarrow{G}$-flow spaces. All of these makes it clear that holding on the reality of things by classical mathematics is partial or local, only on the coherent behaviors of things for itself homogenous without contradictions, but the mathematics over graphs G is applicable for contradictory systems over G because contradiction is universal in the nature, which can turn a contradictory system to a compatible one, i.e., mathematical combinatorics.

Key Words: Topological graph, labeling, group, linear space, Banach space, Smarandache multispace, non-solvable equation, graphical tensor, $\overrightarrow{G}$-flow, mathematical combinatorics.

AMS(2010): 03A10,05C15,20A05, 34A26,35A01,51A05,51D20,53A35.

§1. Introduction

Just as the philosophical question on human beings: *where we come from, and where to go*? There is also a question on our world: *Is our world continuous or discrete*? Different peoples with different world views will answer this question differently, particularly for researchers on continuous or discrete sciences, for instance, the fluid mechanics or elementary particles with

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interactions. Actually, a natural thing T is complex, ever hybrid with other things on the eyes of human beings sometimes. Thus, holding on the true face of thing T is difficult, maybe result in disputation for persons standing on different views or positions for T, which also implies that all contradictions are man made, not the nature of things. For this fact, a typical example was shown once by the famous fable "the blind men with an elephant". In this fable, there are six blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, such as those shown in Fig.1 following. Each of them insisted on his own and not accepted others. They then entered into an endless argument.

Fig.1

All of you are right! A wise man explains to them: *why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.*

Thus, the best result on an elephant for these blind men is

$$\begin{aligned}\text{An elephant} &= \{4 \text{ pillars}\}\bigcup\{1 \text{ rope}\}\bigcup\{1 \text{ tree branch}\}\\ &\bigcup \{2 \text{ hand fans}\}\bigcup\{1 \text{ wall}\}\bigcup\{1 \text{ solid pipe}\}\end{aligned}$$

A thing T is usually identified with known characters on it at one time, and this process is advanced gradually by ours. For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be the known and $\nu_i, i \geq 1$ the unknown characters at time t. Then, the thing T is understood by

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right)\bigcup\left(\bigcup_{k\geq 1}\{\nu_k\}\right) \tag{1.1}$$

in logic and with an approximation $T^\circ = \bigcup\limits_{i=1}^{n}\{\mu_i\}$ at time t. Particularly, *how can the wise man tell these blind men the visual image of an elephant in fable of the blind men with an elephant*? If the wise man is a discrete mathematician, he would tell the blind men that an elephant looks like nothing else but a labeled tree shown in Fig.2.

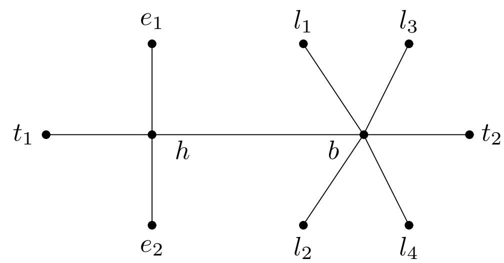

Fig.2

where, $\{t_1\}$ =tusk, $\{e_1, e_2\}$ =ears, $\{h\}$ =head, $\{b\}$ =belly, $\{l_1, l_2, l_3, l_4\}$ =legs and $\{t_2\}$ =tail. Hence, labeled graphs are elements for understanding things of the world in our daily life. *What is the philosophical meaning of this fable for understanding things in the world*? It lies in that the situation of human beings knowing things in the world is analogous to these blind men. We can only hold on things by canonical model (1.1), or the labeled tree in Fig.2.

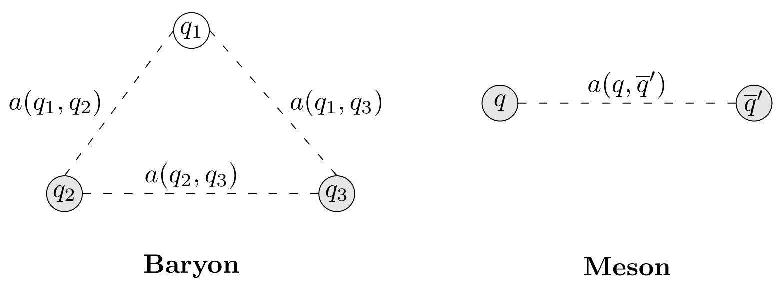

Fig.3

Notice that the elementary particle theory is indeed a discrete notion on matters in the nature. For example, a baryon is predominantly formed from three quarks, and a meson is mainly composed of a quark and an antiquark in the quark models of Sakata, or Gell-Mann and Ne'eman ([27], [32]) such as those shown in Fig.3, which are nothing else but both multiverses ([3]), or graphs labeled by quark $q_i \in \{\mathbf{u}, \mathbf{d}, \mathbf{c}, \mathbf{s}, \mathbf{t}, \mathbf{b}\}$ for $i = 1, 2, 3$ and antiquark $\overline{q}' \in \left\{\overline{\mathbf{u}}, \overline{\mathbf{d}}, \overline{\mathbf{c}}, \overline{\mathbf{s}}, \overline{\mathbf{t}}, \overline{\mathbf{b}}\right\}$, where $a(q, q')$ denotes the strength between quarks q and q'.

Certainly, a natural thing can not exist out of the live space, the universe. Thus, the labeled graphs in Fig.2 and 3 are actually embedded in the Euclidean space $\mathbb{R}^3$, i.e. a labeled topological graph. Generally, a topological graph $\varphi(G)$ in a space $\mathscr{S}$ is an embedding of $\varphi: G \to \varphi(G) \subset \mathscr{S}$ with $\varphi(p) \neq \varphi(q)$ if $p \neq q$ for $\forall p, q \in G$, i.e., edges of G only intersect at vertices in $\mathscr{S}$. There is a well-known result on embedding of graphs without loops and multiple edges in $\mathbb{R}^n$ for $n \geq 3$ ([10]), i.e., *there always exists an embedding of G that all edges are straight segments in $\mathbb{R}^n$.*

Mathematically, a labeling on a graph G is a mapping $L: V(G) \bigcup E(G) \to \mathscr{L}$ with a labeling set $\mathscr{L}$ such as two labeled graphs on K_4 with integers in $\{1, 2, 3, 4\}$ shown in Fig.4, and they have been concentrated more attentions of researchers, particularly, the dynamical survey paper [4] first published in 1998. Usually, $\mathscr{L}$ is chosen to be a segment of integers $\mathbb{Z}$ and a labeling $L: V(G) \to \mathscr{L}$ with constraints on edges in $E(G)$. Only on the journal: *International*

Journal of Mathematical Combinatorics in the past 9 years, we searched many papers on labeled graphs. For examples, the graceful, harmonic, Smarandache edge m-mean labeling ([29]) and quotient cordial labeling ([28]) are respectively with edge labeling $|L(u)-L(v)|$, $|L(u)+L(v)|$, $\left\lceil \frac{f(u)+f(v)}{m} \right\rceil$ for $m \geq 2$, $\left\lfloor \frac{f(u)}{f(v)} \right\rfloor$ or $\left\lfloor \frac{f(v)}{f(u)} \right\rfloor$ according $f(u) \geq f(v)$ or $f(v) > f(u)$ for $\forall uv \in E(G)$, and a Smarandache-Fibonacci or Lucas graceful labeling is such a labeling $L: V(G) \to \{S(0), S(1), S(2), \cdots, S(q)\}$ that the induced edge labeling is $\{S(1), S(2), \cdots, S(q)\}$ by $L(uv) = |L(u) - L(v)|$ for $\forall uv \in E(G)$ for a Smarandache-Fibonacci or Lucas sequence $\{S(i),\ i \geq 1\}$ ([23]).

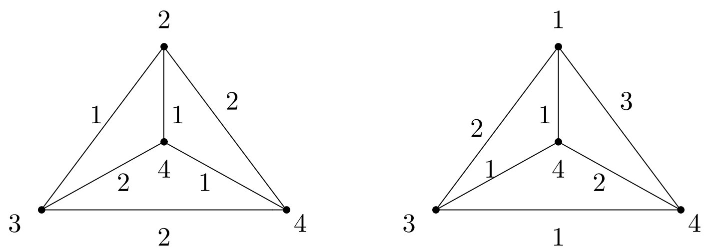

Fig.4

Similarly, an n-signed labeling is a n-tuple of $\{-1,+1\}^n$ or $\{0,1\}$-vector labeling on edges of graph G with $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labeled with even integer or odd integer([26]), and a graceful set labeling is a labeling $L:\ V(G) \to 2^X$ on vertices of G by subsets of a finite set X with induced edge labeling $L(uv) = L(u) \oplus L(v)$ for $\forall uv \in E(G)$, where " $\oplus$ " denotes the binary operation of taking the symmetric difference of the sets in 2^X ([30]). As a result, the combinatorial structures on $\mathscr{L}$ were partially characterized.

However, for understanding things in the world we should ask ourself: *what are labels on a labeled graph, is it just different symbols*? And *are such labeled graphs a mechanism for understanding the reality of things, or only a labeling game*? Clearly, labeled graphs G considered by researchers are graphs mainly with number labeling, vector symbolic labeling without operation, or finite set labeling, and with an additional assumption that *each vertex of G is mapped exactly into one point of space $\mathscr{S}$* in topology. However, labels all are space objects in Fig.2 and 3. If we put off this assumption, i.e., labeling a topological graph by geometrical spaces, or elements with operations in a linear space, *what will happens*? *Are these resultants important for understanding things in the world*? The answer is certainly YES because this step will enable one to pullback more characters of things, characterize more precisely and then hold on the reality of things in the world, i.e., combines continuous mathematics with the discrete, which is nothing else but the *mathematical combinatorics.*

The main purpose of this report is to survey the role of labeled graphs in extending mathematical systems over graphs G, particularly graphical tensors and $\overrightarrow{G}$-flows with conservation laws and applications to mathematics, physics and other sciences such as those of labeled graphs with sets or Euclidean spaces $\mathbb{R}^n$ labeling, labeled graph solutions of non-solvable systems of differential equations with global stability, labeled graph with elements in a linear space, and extended Banach or Hilbert $\overrightarrow{G}$-flow spaces, $\cdots$, etc. All of these makes it clear that holding on the reality of things by classical mathematics is partial, only on the coherent behaviors of things for itself homogenous without contradictions but the extended mathematics over graphs G can characterize contradictory systems, and accordingly can be applied to hold on the reality of things because contradiction is universal in the nature.

For terminologies and notations not mentioned here, we follow references [5] for functional analysis, [9]-[11] for graphs and combinatorial geometry, [2] for differential equations, [27] for elementary particles, and [1],[10] for Smarandache multispaces or multisystems.

§2. Graphs Labeled by Sets

Notice that the understanding form (1.1) of things is in fact a Smarandache multisystem following, which shows the importance of labeled graphs for things.

Definition 2.1([1],[10]) *Let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be m mathematical systems, different two by two. A Smarandache multisystem $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m}\Sigma_i$ with rules $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ on $\widetilde{\Sigma}$, denoted by $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$.*

Definition 2.2([9]-[11]) *For an integer $m\geq 1$, let $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ be a Smarandache multi- system consisting of m mathematical systems $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$. An inherited combinatorial structure $G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]$ of $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ is a labeled topological graph defined following:*

$$V\left(G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]\right)=\{\Sigma_1,\Sigma_2,\cdots,\Sigma_m\},$$
$$E\left(G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]\right)=\{(\Sigma_i,\Sigma_j)|\Sigma_i\bigcap\Sigma_j\neq\emptyset,\ 1\leq i\neq j\leq m\}\ \textit{with labeling}$$
$$L:\ \Sigma_i\to L(\Sigma_i)=\Sigma_i\quad \textit{and}\quad L:\ (\Sigma_i,\Sigma_j)\to L(\Sigma_i,\Sigma_j)=\Sigma_i\bigcap\Sigma_j$$

for integers $1\leq i\neq j\leq m$.

For example, let $\Sigma_1=\{a,b,c\}$, $\Sigma_2=\{a,b,e\}$, $\Sigma_3=\{b,c,e\}$, $\Sigma_4=\{a,c,e\}$ and $\mathcal{R}_i=\emptyset$ for integers $1\leq i\leq 4$. The multisystem $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ with $\widetilde{\Sigma}=\bigcup\limits_{i=1}^{4}\Sigma_i=\{a,b,c,d,e\}$ and $\widetilde{\mathscr{R}}=\emptyset$ is characterized by the labeled topological graph $G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]$ shown in Fig.5.

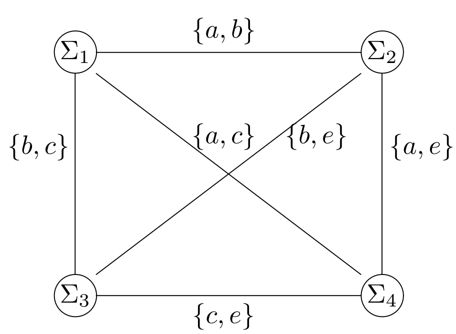

Fig.5

2.1 Exact Labeling

A multiset $\widetilde{S}=\bigcup\limits_{i=1}^{m} S_i$ is *exact* if $S_i=\bigcup\limits_{j=1,j\neq i}^{m}(S_j\bigcap S_i)$ for any integer $1\leq i\leq m$, i.e., for any vertex $v\in V\left(G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]\right)$, $S_v=\bigcup\limits_{u\in N_{G^L}(v)}(S_v\bigcap S_u)$ such as those shown in Fig.5. Clearly, a multiset $\widetilde{S}$ uniquely determines a labeled graph G^L by Definition 2.2, and conversely, if G^L is a graph labeled by sets, we are easily get an exact multiset

$$\widetilde{S}=\bigcup_{v\in V(G^L)} S_v \quad \text{with} \quad S_v=\bigcup_{u\in N_{G^L}(v)}\left(S_v\bigcap S_u\right).$$

This concludes the following result.

Theorem 2.3([10]) *A multiset $\widetilde{S}$ uniquely determine a labeled graph $G^L[\widetilde{S}]$, and conversely, any graph G^L labeled by sets uniquely determines an exact multiset $\widetilde{S}$.*

All labeling sets on edges of graph in Fig.4 are 2-sets. Generally, we know

Theorem 2.4 *For any graph G, if $|S|\geq k\chi(G)\geq\Delta(G)\chi(G)$ or $\begin{pmatrix}|S|\\ k\end{pmatrix}\geq\chi'(G)$, there is a labeling L with k-subset labels of S on all vertices or edges on G, where $\varepsilon(G)$, $\Delta(G)$ $\chi(G)$ and $\chi'(G)$ are respectively the size, the maximum valence, the chromatic number and the edge chromatic number of G.*

Furthermore, if G is an s-regular graph, there exist integers k,l such that there is a labeling L on G with k-set, l-set labels on its vertices and edges, respectively.

Proof Clearly, if $\begin{pmatrix}|S|\\ k\end{pmatrix}\geq\chi'(G)$, we are easily find $\chi'(G)$ different k-subsets $C_1,C_2,\cdots,C_{\chi'(G)}$ of S labeled on edges in G, and if $|S|\geq k\chi(G)\geq\Delta(G)\chi(G)$, there are $\chi(G)$ different k-subsets $C_1,C_2,\cdots,C_{\chi(G)}$ of S labeled on vertices in G such that $S_i\bigcap S_j=\emptyset$ or not if and only if $uv\notin E(G)$ or not, where u and v are labeled by S_i and S_j, respectively.

Furthermore, if G is an s-regular graph, we can always allocate $\chi'(G)$ l-sets $\{C_1,C_2,\cdots,C_{\chi'(G)}\}$ with $C_i\bigcap C_j=\emptyset$ for integers $1\leq i\neq j\leq\chi'(G)$ on edges in $E(G)$ such that colors on adjacent edges are different, and then label vertices v in G by $\bigcup\limits_{u\in N_G(v)} C(vu)$, which is a sl-set. The proof is complete for integer $k=sl$. $\square$

2.2 Linear Space Labeling

Let $(\widetilde{V};F)$ be a multilinear space consisting of subspaces $V_i,\ 1\leq i\leq|G|$ of linear space V over a field F. Such a multilinear space $(\widetilde{V};F)$ is said to be *exact* if $V_i=\bigoplus\limits_{j\neq i}(V_i\bigcap V_j)$ holds for integers $1\leq i\leq n$. According to linear algebra, two linear spaces V and V' over a field F are isomorphic if and only if $\dim V=\dim V'$, which enables one to characterize a vector V space by its basis $\mathscr{B}(V)$ and label edges of $G[\widetilde{V};F]$ by $L:\ V_uV_v\to\mathscr{B}\left(V_u\bigcap V_v\right)$ for $\forall V_uV_v\in E\left(G[\widetilde{V};F]\right)$

in Definition 2.2 such as those shown in Fig.6.

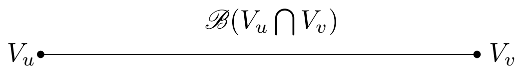

Fig.6

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Clearly, if $(\widetilde{V}; F)$ is exact, i.e., $V_i = \bigoplus\limits_{j \neq i}(V_i \bigcap V_j)$, then it is clear that

$$\mathscr{B}(V) = \bigcup_{VV' \in E\left(G[\widetilde{V};F]\right)} \mathscr{B}(V \bigcap V') \quad \text{and} \quad (\mathscr{B}(V \bigcap V')) \bigcap (\mathscr{B}(V \bigcap V'') = \emptyset$$

by definition. Conversely, if

$$\mathscr{B}(V) = \bigcup_{VV' \in E\left(G[\widetilde{V};F]\right)} \mathscr{B}(V \bigcap V') \quad \text{and} \quad \mathscr{B}\left(V \bigcap V'\right) \bigcap \mathscr{B}\left(V \bigcap V''\right) = \emptyset$$

for $V', V'' \in N_{G[\widetilde{V};F]}(V)$. Notice also that $VV' \in E\left(G[\widetilde{V};F]\right)$ if and only if $V \bigcap V' \neq \emptyset$, we know that

$$V_i = \bigoplus_{j \neq i}(V_i \bigcap V_j)$$

for integers $1 \leq i \leq n$. This concludes the following result.

Theorem 2.5([10]) *Let $(\widetilde{V}; F)$ be a multilinear space with $\widetilde{V} = \bigcup\limits_{i=1}^{n} V_i$. Then it is exact if and only if*

$$\mathscr{B}(V) = \bigcup_{VV' \in E\left(G[\widetilde{V};F]\right)} \mathscr{B}\left(V \bigcap V'\right) \quad \textit{and} \quad \mathscr{B}\left(V \bigcap V'\right) \bigcap \mathscr{B}\left(V \bigcap V''\right) = \emptyset$$

for $V', V'' \in N_{G[\widetilde{V};F]}(V)$.

2.3 Euclidean Space Labeling

Let $\mathbf{R}^n$ be a Euclidean space with normal basis $\mathscr{B}(\mathbf{R}^n) = \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\}$, where $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, \cdots, 0, 1)$ and let $(\widetilde{V}; F)$ be a multilinear space with $\widetilde{V} = \bigcup\limits_{i=1}^{m} \mathbb{R}^{n_i}$ in Theorem 2.5, where $\mathbb{R}^{n_i} \bigcap \mathbb{R}^{n_j} \neq \mathbb{R}^{\min\{i,j\}}$ for integers $1 \leq i \neq j \leq n_m$. If the labeled graph $G[\widetilde{V}; F]$ is known, we are easily determine the dimension of $\dim\widetilde{V}$. For example, let G^L be a labeled graph shown in Fig.7. We are easily finding that $\mathscr{B}(\widetilde{\mathbf{R}}) = \{\overline{\epsilon}_1, \overline{\epsilon}_2, \overline{\epsilon}_3, \overline{\epsilon}_4, \overline{\epsilon}_5, \overline{\epsilon}_6\}$, i.e., $\dim\widetilde{V} = 6$.

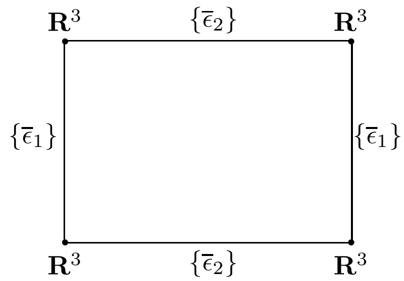

Fig.7

Notice that $\widetilde{V}$ is not exact in Fig.7 because basis $\overline{\epsilon}_3, \overline{\epsilon}_4, \overline{\epsilon}_5, \overline{\epsilon}_6$ are additional. Generally, we are easily know the result by the inclusion-exclusion principle.

Theorem 2.6([8]) *Let G^L be a graph labeled by $\mathbf{R}^{n_{v_1}}, \mathbf{R}^{n_{v_2}}, \cdots, \mathbf{R}^{n_{v_{|G|}}}$. Then*

$$\dim G^L = \sum_{\langle v_i \in V(G) | 1 \leq i \leq s \rangle \in CL_s(G)} (-1)^{s+1} \dim(\mathbf{R}^{n_{v_1}} \bigcap \mathbf{R}^{n_{v_2}} \bigcap \cdots \bigcap \mathbf{R}^{n_{v_s}}),$$

where $CL_s(G)$ consists of all complete graphs of order s in G^L.

However, if edge labelings $\mathscr{B}(\mathbb{R}^{n_u} \bigcap \mathbb{R}^{n_v})$ are not known for $uv \in E\left(G^L\right)$, *can we still determine the dimension* $\dim G^L$? In fact, we only get the maximum and minimum dimensions $\dim_{max} G^L$, $\dim_{min} G^L$ in case.

Theorem 2.7([8]) *Let G^L be a graph labeled by $\mathbf{R}^{n_{v_1}}, \mathbf{R}^{n_{v_2}}, \cdots, \mathbf{R}^{n_{v_{|G|}}}$ on vertices. Then its maximum dimension* $\dim_{max} G^L$ *is*

$$\dim_{max} G^L = 1 - m + \sum_{v \in V(G^L)} n_v$$

with conditions $\dim(\mathbf{R}^{n_u} \cap \mathbf{R}^{n_v}) = 1$ *for* $\forall uv \in E\left(G^L\right)$.

However, for determining the minimum value $\dim_{min} G^L$ of graph G^L labeled by Euclidean spaces is a difficult problem in general. We only know the following result on labeled complete graphs $K_m, m \geq 3$.

Theorem 2.8([8]) *For any integer $r \geq 2$, let $K_m^L(r)$ be a complete graph K_m labeled by Euclidean space $\mathbb{R}^r$ on its vertices, and there exists an integer s, $0 \leq s \leq r-1$ such that*

$$\begin{pmatrix} r+s-1 \\ r \end{pmatrix} < m \leq \begin{pmatrix} r+s \\ r \end{pmatrix}.$$

Then

$$\dim_{min} K_m^L(r) = r + s.$$

Particularly,

$$\dim_{min} K_m^L(3) = \begin{cases} 3, & \text{if} \quad m = 1, \\ 4, & \text{if} \quad 2 \le m \le 4, \\ 5, & \text{if} \quad 5 \le m \le 10, \\ 2 + \lceil \sqrt{m} \rceil, & \text{if} \quad m \ge 11. \end{cases}$$

All of these results presents a combinatorial model for characterizing things in the space $R^n, n \ge 4$, particularly, the G^L solution of equations in the next subsection.

2.4 G^L-Solution of Equations

Let $\mathbb{R}^m$, $\mathbb{R}^n$ be Euclidean spaces of dimensional $m,\ n \ge 1$ and let $T:\ \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a $\mathbb{C}^k,\ 1 \le k \le \infty$ mapping such that $T(\overline{x}_0, \overline{y}_0) = \overline{0}$ for $\overline{x}_0 \in \mathbb{R}^n$, $\overline{y}_0 \in \mathbb{R}^m$ and the $m \times m$ matrix $\partial T^j / \partial y^i (\overline{x}_0, \overline{y}_0)$ is non-singular, i.e.,

$$\det\left(\frac{\partial T^j}{\partial y^i}\right)\Big|_{(\overline{x}_0, \overline{y}_0)} \ne 0, \text{ where } 1 \le i, j \le m.$$

Then the implicit mapping theorem concludes that there exist opened neighborhoods $V \subset \mathbb{R}^n$ of $\overline{x}_0$, $W \subset \mathbb{R}^m$ of $\overline{y}_0$ and a $\mathbb{C}^k$ mapping $\phi : V \to W$ such that $T(\overline{x}, \phi(\overline{x})) = \overline{0}$. Thus there always exists solution $\overline{y}$ for the equation $T(\overline{x}, \overline{y}) = \overline{0}$ in case.

By the implicit function theorem, we can always choose mappings $T_1, T_2, \cdots, T_m$ and subsets $S_{T_i} \subset \mathbb{R}^n$ where $S_{T_i} \ne \emptyset$ such that $T_i :\ S_{T_i} \to 0$ for integers $1 \le i \le m$. Consider the system of equations

$$\begin{cases} T_1(x_1, x_2, \cdots, x_n) = 0 \\ T_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T_m(x_1, x_2, \cdots, x_n) = 0 \end{cases} \qquad (ES_m)$$

in Euclidean space $\mathbb{R}^n, n \ge 1$. Clearly, the system (ES_m) is non-solvable or not dependent on

$$\bigcap_{i=1}^{m} S_{T_i} = \emptyset \quad \text{or} \quad \ne \emptyset.$$

This fact implies the following interesting result.

Theorem 2.9 *A system (ES_m) of equations is solvable if and only if $\bigcap_{i=1}^{m} S_{T_i} \ne \emptyset$.*

Furthermore, if (ES_m) is solvable, it is obvious that $G^L[ES_m] \simeq K_m^L$. We conclude that (ES_m) is non-solvable if $G^L[ES_m] \not\simeq K_m^L$. Thus the case of solvable is special respect to the general case, non-solvable. However, the understanding on non-solvable case was abandoned in classical for a wrongly thinking, i.e., meaningless for hold on the reality of things.

By Definition 2.2, all spaces $S_{T_i}, 1 \le i \le m$ exist for the system (ES_m) and we are easily get a labeled graph $G^L[ES_m]$, which is in fact a combinatorial space, a really geometrical figure in $\mathbb{R}^n$. For example, in cases of linear algebraic equations, we can further determine $G^L[ES_m]$ whatever the system (ES_m) is solvable or not as follows.

A *parallel family* $\mathscr{C}$ of system (ES_m) of linear equations consists of linear equations in (ES_m) such that they are parallel two by two but there are no other linear equations parallel to any one in $\mathscr{C}$. We know a conclusion following on $G^L[ES_m]$ for linear algebraic systems.

Theorem 2.10([12]) *Let (ES_m) be a linear equation system for integers $m, n \geq 1$. Then*

$$G^L[ES_m] \simeq K^L_{n_1,n_2,\cdots,n_s}$$

with $n_1 + n + 2 + \cdots + n_s = m$, where $\mathscr{C}_i$ is the parallel family with $n_i = |\mathscr{C}_i|$ for integers $1 \leq i \leq s$ in (ES_m) and it is non-solvable if $s \geq 2$.

Similarly, let

$$\dot{X} = A_1X, \cdots, \dot{X} = A_kX, \cdots, \dot{X} = A_mX \qquad (LDES^1_m)$$

be a linear ordinary differential equation system of first order with

$$A_k = \begin{bmatrix} a^{[k]}_{11} & a^{[k]}_{12} & \cdots & a^{[k]}_{1n} \\ a^{[k]}_{21} & a^{[k]}_{22} & \cdots & a^{[k]}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a^{[k]}_{n1} & a^{[k]}_{n2} & \cdots & a^{[k]}_{nn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a^{[k]}_{ij}$ is a real number for integers $0 \leq k \leq m,\ 1 \leq i, j \leq n$.

Notice that the solution space of the ith in $(LDES^1_m)$ is a linear space. We know the result following.

Theorem 2.11([13], [14]) *Every linear system $(LDES^1_m)$ of homogeneous differential equations uniquely determines a labeled graph $G^L[LDES^1_m]$, and conversely, every graph G^L labeled by basis of linear spaces uniquely determines a homogeneous differential equation system $(LDES^1_m)$ such that $G^L[LDES^1_m] \simeq G^L$.*

For example, let $(LDES^1_m)$ be the system of linear homogeneous differential equations

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$ and $\dot{x} = \dfrac{dx}{dt}$. Then the solution basis of equations $(1) - (6)$ are respectively $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ with a labeled graph shown in Fig.8.

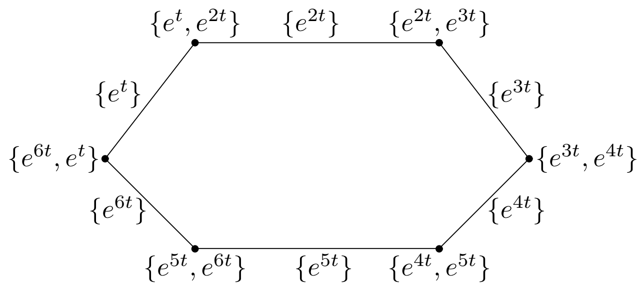

Fig.8

An integral labeled graph G^{L^I} is such a labeling $L^I: G \to \mathbb{Z}^+$ that $L^I(uv) \leq \min\{L^I(u), L^I(v)\}$ for $\forall uv \in E(G)$, and two integral labeled graphs $G_1^{L^I}$ and $G_2^{L'^I}$ are said to be identical, denoted by $G_1^{L^I} = G_2^{L'^I}$ if $G_1 \overset{\varphi}{\simeq} G_2$ and $L_1^I(x) = L_2^I(\varphi(x))$ for graph isomorphisms φ and $\forall x \in V(G_1)\bigcup E(G_1)$. For example, these labeled graphs shown in Fig.9 are all integral on $K_4 - e$, we know $G_1^{L_1^I} = G_2^{L_2^I}$ but $G_1^{L_1^I} \neq G_3^{L_3^I}$ by definition.

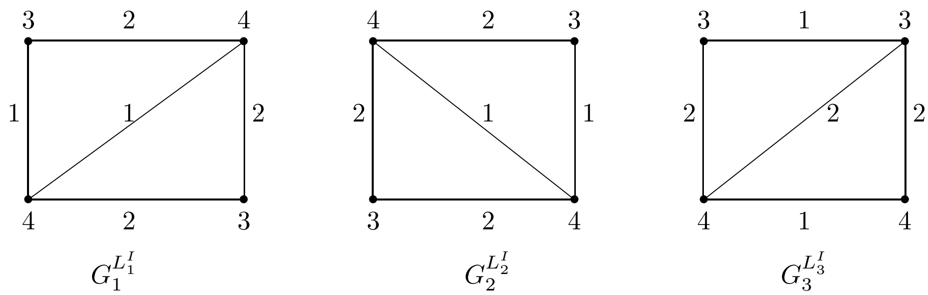

Fig.9

For 2 linear systems $(LDES_m^1)$, $(LDES_m^1)'$ of ordinary differential equations, they are called *combinatorially equivalent*, denoted by $(LDES_m^1) \overset{\varphi}{\simeq} (LDES_m^1)'$ if there is an isomorphism $\varphi: G^L[LDES_m^1] \to G^{L'}[LDES_m^1]$ of graph, linear isomorphisms $\xi: x \to \xi(x)$ of spaces and labelings L_1, L_2 such that $\varphi L_1(x) = L_2\varphi(\xi(x))$ for $\forall x \in V(G^L[LDES_m^1])\bigcup E(G^L[LDES_m^1])$, which are completely characterized by the integral labeled graphs.

Theorem 2.12([13], [14]) *Let $(LDES_m^1)$, $(LDES_m^1)'$ be two linear system of ordinary differential equations with integral labeled graphs $G^{L^I}[LDES_m^1]$, $G^{L'^I}[LDES_m^1]'$. Then $(LDES_m^1) \overset{\varphi}{\simeq} (LDES_m^1)'$ if and only if $G^{L^I}[LDES_m^1] = G^{L'^I}[LDES_m^1]'$.*

§3. Graphical Tensors

As shown in last section, labeled graphs by sets, particularly, geometrical sets such as those of Euclidean spaces $\mathbb{R}^n, n \geq 1$ are useful for holding on things characterized by non-solvable systems of equations. A further question on labeled graphs is

For labeled graphs G_1^L, G_2^L, G_3^L, is there a binary operation $o: (G_1^L, G_2^L) \to G_3^L$? And can we established algebra on labeled graphs?

Answer these questions enables one to extend linear spaces over graphs G hold with conservation laws on its each vertex and establish tensors underlying graphs.

3.1 Action Flows

Let $(\mathscr{V}; +, \cdot)$ be a linear space over a field $\mathscr{F}$. An *action* flow $\left(\overrightarrow{G}; L, A\right)$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L : (v,u) \to L(v,u)$, 2 end-operators $A_{vu}^+ : L(v,u) \to L^{A_{vu}^+}(v,u)$ and $A_{uv}^+ : L(u,v) \to L^{A_{uv}^+}(u,v)$ on $\mathscr{V}$ with $L(v,u) = -L(u,v)$ and $A_{vu}^+(-L(v,u)) = -L^{A_{vu}^+}(v,u)$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$

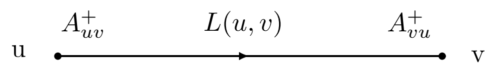

Fig.10

holding with conservation laws

$$\sum_{u \in N_G(v)} L^{A_{vu}^+}(v,u) = \mathbf{0} \quad \text{for} \quad \forall v \in V\left(\overrightarrow{G}\right)$$

such as those shown for vertex v in Fig.11 following

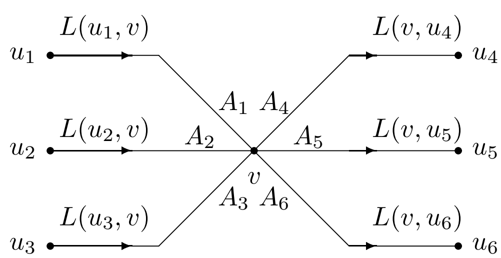

Fig.11

with a conservation law

$$-L^{A_1}(v,u_1) - L^{A_2}(v,u_2) - L^{A_4}(v,u_3) + L^{A_4}(v,u_4) + L^{A_5}(v,u_5) + L^{A_6}(v,u_6) = \mathbf{0},$$

and such a set $\{-L^{A_i}(v,u_i), 1 \le i \le 3\} \bigcup \{L^{A_j}, 4 \le j \le 6\}$ is called a conservation family at vertex v.

Action flow is a useful model for holding on natural things. It combines the discrete with that of analytical mathematics and therefore, it can help human beings understanding the nature.

For example, let $L : (v,u) \to L(v,u) \in \mathbb{R}^n \times \mathbb{R}^+$ with action operators $A_{vu}^+ = a_{vu}\dfrac{\partial}{\partial t}$ and $a_{vu} : \mathbb{R}^n \to \mathbb{R}$ for any edge $(v,u) \in E\left(\overrightarrow{G}\right)$ in Fig.12.

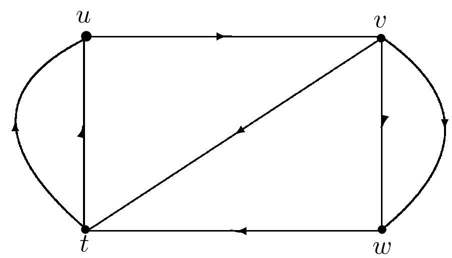

Fig.12

Then the conservation laws are partial differential equations

$$\begin{cases} a_{tu^1}\dfrac{\partial L(t,u)^1}{\partial t}+a_{tu^2}\dfrac{\partial L(t,u)^2}{\partial t}=a_{uv}\dfrac{\partial L(u,v)}{\partial t} \\ a_{uv}\dfrac{\partial L(u,v)}{\partial t}=a_{vw^1}\dfrac{\partial L(v,w)^1}{\partial t}+a_{vw^2}\dfrac{\partial L(v,w)^2}{\partial t}+a_{vt}\dfrac{\partial L(v,t)}{\partial t} \\ a_{vw^1}\dfrac{\partial L(v,w)^1}{\partial t}+a_{vw^2}\dfrac{\partial L(v,w)^2}{\partial t}=a_{wt}\dfrac{\partial L(w,t)}{\partial t} \\ a_{wt}\dfrac{\partial L(w,t)}{\partial t}+a_{vt}\dfrac{\partial L(v,t)}{\partial t}=a_{tu^1}\dfrac{\partial L(t,u)^1}{\partial t}+a_{tu^2}\dfrac{\partial L(t,u)^2}{\partial t} \end{cases},$$

which maybe solvable or not but characterizes behavior of natural things.

If $A=\mathbf{1}_{\mathscr{V}}$, an action flows $\left(\overrightarrow{G};L,\mathbf{1}_{\mathscr{V}}\right)$ is called $\overrightarrow{G}$-flow and denoted by $\overrightarrow{G}^L$ for simplicity. We naturally define

$$\overrightarrow{G}^{L_1}+\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_1+L_2} \ \text{ and } \ \lambda\cdot\overrightarrow{G}^{L}=\overrightarrow{G}^{\lambda\cdot L}$$

for $\forall\lambda\in\mathscr{F}$. All $\overrightarrow{G}$-flows $\overrightarrow{G}^{\mathscr{V}}$ on $\overrightarrow{G}$ naturally form a linear space $\left(\overrightarrow{G}^{\mathscr{V}};+,\cdot\right)$ because it hold with:

(1) A field $\mathscr{F}$ of scalars;

(2) A set $\overrightarrow{G}^{\mathscr{V}}$ of objects, called extended vectors;

(3) An operation "+", called extended vector addition, which associates with each pair of vectors $\overrightarrow{G}^{L_1},\overrightarrow{G}^{L_2}$ in $\overrightarrow{G}^{\mathscr{V}}$ a extended vector $\overrightarrow{G}^{L_1+L_2}$ in $\overrightarrow{G}^{\mathscr{V}}$, called the sum of $\overrightarrow{G}^{L_1}$ and $\overrightarrow{G}^{L_2}$, in such a way that

(a) Addition is commutative, $\overrightarrow{G}^{L_1}+\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}+\overrightarrow{G}^{L_1}$;
(b) Addition is associative, $\left(\overrightarrow{G}^{L_1}+\overrightarrow{G}^{L_2}\right)+\overrightarrow{G}^{L_3}=\overrightarrow{G}^{L_1}+\left(\overrightarrow{G}^{L_2}+\overrightarrow{G}^{L_3}\right)$;

(c) There is a unique extended vector $\overrightarrow{G}^{\mathbf{0}}$, i.e., $\mathbf{0}(v,u)=\mathbf{0}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ in $\overrightarrow{G}^{\mathscr{V}}$, called zero vector such that $\overrightarrow{G}^{L}+\overrightarrow{G}^{\mathbf{0}}=\overrightarrow{G}^{L}$ for all $\overrightarrow{G}^{L}$ in $\overrightarrow{G}^{\mathscr{V}}$;

(d) For each extended vector $\overrightarrow{G}^{L}$ there is a unique extended vector $\overrightarrow{G}^{-L}$ such that $\overrightarrow{G}^{L}+\overrightarrow{G}^{-L}=\overrightarrow{G}^{\mathbf{0}}$ in $\overrightarrow{G}^{\mathscr{V}}$;

(4) An operation "·, called scalar multiplication, which associates with each scalar k in F and an extended vector $\overrightarrow{G}^{L}$ in $\overrightarrow{G}^{\mathscr{V}}$ an extended vector $k\cdot\overrightarrow{G}^{L}$ in $\mathscr{V}$, called the product of k with $\overrightarrow{G}^{L}$, in such a way that

(a) $1\cdot\overrightarrow{G}^{L}=\overrightarrow{G}^{L}$ for every $\overrightarrow{G}^{L}$ in $\overrightarrow{G}^{\mathscr{V}}$;
(b) $(k_1k_2)\cdot\overrightarrow{G}^{L}=k_1(k_2\cdot\overrightarrow{G}^{L})$;

(c) $k\cdot(\overrightarrow{G}^{L_1}+\overrightarrow{G}^{L_2})=k\cdot\overrightarrow{G}^{L_1}+k\cdot\overrightarrow{G}^{L_2}$;

(d) $(k_1+k_2)\cdot\overrightarrow{G}^{L}=k_1\cdot\overrightarrow{G}^{L}+k_2\cdot\overrightarrow{G}^{L}$.

3.2 Dimension of Action Flow Space

Theorem 3.1 *Let $\mathscr{G}$ be all action flows $\left(\overrightarrow{G};L,A\right)$ with $A\in\mathbf{O}(\mathscr{V})$. Then*

$$\dim\mathscr{G}=(\dim\mathbf{O}(\mathscr{V})\times\dim\mathscr{V})^{\beta(\overrightarrow{G})}$$

if both $\mathscr{V}$ and $\mathbf{O}(\mathscr{V})$ are finite. Otherwise, $\dim\mathscr{G}$ is infinite.

Particularly, if operators $A\in\mathscr{V}^*$, the dual space of $\mathscr{V}$ on graph $\overrightarrow{G}$, then

$$\dim\mathscr{G}=(\dim\mathscr{V})^{2\beta(\overrightarrow{G})},$$

where $\beta\left(\overrightarrow{G}\right)=\varepsilon\left(\overrightarrow{G}\right)-\left|\overrightarrow{G}\right|+1$ is the Betti number of $\overrightarrow{G}$.

Proof The infinite case is obvious. Without loss of generality, we assume $\overrightarrow{G}$ is connected with dimensions of $\mathscr{V}$ and $\mathbf{O}(\mathscr{V})$ both finite. Let $L(v)=\{L^{A^+_{vu}}(v,u)\in\mathscr{V}$ for some $u\in V\left(\overrightarrow{G}\right)\}$, $v\in V\left(\overrightarrow{G}\right)$ be the conservation families in $\mathscr{V}$ associated with $\left(\overrightarrow{G};L,A\right)$ such that $L^{A^+_{vu}}(v,u)=-A^+_{uv}(L(u,v))$ and $L(v)\bigcap(-L(u))=L^{A^+_{vu}}(v,u)$ or $\emptyset$. An edge $(v,u)\in E\left(\overrightarrow{G}\right)$ is *flow freely* or not in $\overrightarrow{G}^{\mathscr{V}}$ if $L^{A^+_{vu}}(v,u)$ can be any vector in $\mathscr{V}$ or not. Notice that $L(v)=\{L^{A^+_{vu}}(v,u)\in\mathscr{V}$ for some $u\in V\left(\overrightarrow{G}\right)\}$, $v\in V\left(\overrightarrow{G}\right)$ are the conservation families associated with action flow $\left(\overrightarrow{G};L,A\right)$. There is one flow non-freely edges for any vertex in $\overrightarrow{G}$ at least and $\dim\mathscr{G}$ is nothing else but the number of independent vectors $L(v,u)$ and independent end-operators A^+_{vu} on edges in $\overrightarrow{G}$ which can be chosen freely in $\mathscr{V}$.

We claim that all flow non-freely edges form a connected subgraph T in $\overrightarrow{G}$. If not, there are two components $C_1(T)$ and $C_2(T)$ in T such as those shown in Fig.13.

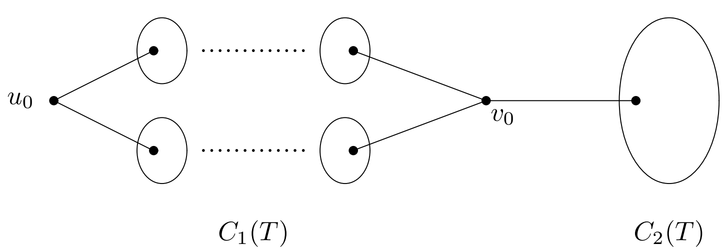

Fig.13

In this case, all edges between $C_1(T)$ and $C_2(T)$ are flow freely in $\overrightarrow{G}$. Let v_0 be such a vertex in $C_1(T)$ adjacent to a vertex in $C_2(T)$. Beginning from the vertex v_0 in $C_1(T)$, we choose vectors on edges in

$$E_G\left(v_0, N_G(v_0)\right) \bigcap \langle C_1(T)\rangle_G\,,$$
$$E_G\left(N_G(v_0)\setminus\{v_0\}, N_G(N_G(v_0))\setminus N_G(v_0)\right) \bigcap \langle C_1(T)\rangle_G\,,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

in $\langle C_1(T)\rangle_G$ by conservation laws, and then finally arrive at a vertex $u_0 \in V(C_2(T))$ such that all flows from $V(C_1(T))\setminus\{u_0\}$ to u_0 are fixed by conservation laws of vertices $N_G(u_0)$, which result in that there are no conservation law of flows on the vertex u_0, a contradiction. Hence, all flow freely edges form a connected subgraph in $\overrightarrow{G}$. Hence, we get that

$$\begin{aligned}\dim\mathscr{G} &\leq \dim\mathbf{O}(\mathscr{V}))^{|E(\overrightarrow{G})-E(T)|} \times (\dim\mathscr{V})^{|E(\overrightarrow{G})-E(T)|}\\ &= (\dim\mathbf{O}(\mathscr{V}) \times \dim\mathscr{V})^{\beta(\overrightarrow{G})}.\end{aligned}$$

We can indeed determine a flow non-freely tree T in $\overrightarrow{G}$ by programming following:

STEP 1. Define $X_1 = \{v_1\}$ for $\forall v_1 \in V\left(\overrightarrow{G}\right)$;

STEP 2. If $V\left(\overrightarrow{G}\right)\setminus X_1 \neq \emptyset$, choose $v_2 \in N_G\left(v_1\right)\setminus X_1$ and let (v_1, v_2) be a flow non-freely edge by conservation law on v_1 and define $X_2 = \{v_1, v_2\}$. Otherwise, $T = v_0$.

STEP 3. If $V\left(\overrightarrow{G}\right)\setminus X_2 \neq \emptyset$, choose $v_3 \in N_G\left(X_1\right)\setminus X_2$. Without loss of generality, assume v_3 adjacent with v_2 and let (v_2, v_3) be a flow non-freely edge by conservation law on v_2 with $X_3 = \{v_1, v_2, v_3\}$. Otherwise, $T = v_1v_2$.

STEP 4. For any integer $k \geq 2$, if X_k has been defined and $V\left(\overrightarrow{G}\right)\setminus X_k \neq \emptyset$, choose $v_{k+1} \in N_G\left(X_k\right)\setminus X_k$. Assume v_{k_1} adjacent with $v^k \in X_k$ and let (v^k, v_{k+1}) be a flow non-freely edge by conservation law on v^k with $X_{k+1} = X_k \bigcup \{v_{k+1}\}$. Otherwise, T is the flow non-freely tree spanned by $\langle X_k\rangle$ in $\overrightarrow{G}$.

STEP 5. The procedure is ended if $X_{|\overrightarrow{G}|}$ has been defined which enable one get a spanning flow non-freely tree T of $\overrightarrow{G}$.

Clearly, all edges in $E\left(\overrightarrow{G}\right)\setminus E(T)$ are flow freely in $\mathscr{V}$. We therefore know

$$\begin{aligned}\dim\mathscr{G} &\geq (\dim\mathbf{O}(\mathscr{V}))^{\varepsilon\left(\overrightarrow{G}\right)-\varepsilon(T)} \times (\dim\mathscr{V})^{\varepsilon\left(\overrightarrow{G}\right)-\varepsilon(T)}\\ &= (\dim O(\mathscr{V}) \times \dim\mathscr{V})^{\varepsilon\left(\overrightarrow{G}\right)-\left|\overrightarrow{G}\right|+1} = (\dim\mathbf{O}(\mathscr{V}) \times \dim\mathscr{V})^{2\beta(\overrightarrow{G})}.\end{aligned}$$

Thus,

$$\dim\mathscr{G} = (\dim\mathbf{O}(\mathscr{V}) \times \dim\mathscr{V})^{2\beta(\overrightarrow{G})}. \tag{3.1}$$

If operators $A \in \mathscr{V}^*$, $\dim\mathscr{V}^* = \dim\mathscr{V}$. We are easily get

$$\dim\mathscr{G} = (\dim\mathscr{V})^{2\beta(\overrightarrow{G})}$$

by the equation (3.1). This completes the proof. $\square$

Particularly, for action flows $\left(\overrightarrow{G};L,\mathbf{1}_{\mathscr{V}}\right)$, i.e., $\overrightarrow{G}$-flow space we have a conclusion on its dimension following

Corollary 3.2 *$\dim\overrightarrow{G}^{\mathscr{V}}=(\dim\mathscr{V})^{\beta(\overrightarrow{G})}$ if $\mathscr{V}$ is finite. Otherwise, $\dim\mathscr{H}$ is infinite.*

3.3 Graphical Tensors

Definition 3.3 *Let $\left(\overrightarrow{G}_1;L_1,A_1\right)$ and $\left(\overrightarrow{G}_2;L_2,A_2\right)$ be action flows on linear space $\mathscr{V}$. Their tensor product $\left(\overrightarrow{G}_1;L_1,A_1\right)\otimes\left(\overrightarrow{G}_2;L_2,A_2\right)$ is defined on graph $\overrightarrow{G}_1\otimes\overrightarrow{G}_2$ with mapping*

$$L:\ ((v_1,u_1),(v_2,u_2))\to(L_1(v_1,u_1),L_2(v_2,u_2))$$

on edge $((v_1,u_1),(v_2,u_2))\in E\left(\overrightarrow{G}_1\otimes\overrightarrow{G}_2\right)$ and end-operators $A^+_{(v_1,u_1)(v_2,u_2)}=A^+_{v_1u_1}\otimes A^+_{v_2u_2}$, $A^+_{(v_2,u_2)(v_1,u_1)}=A^+_{u_1v_1}\otimes A^+_{u_2v_2}$, such as those shown in Fig.14.

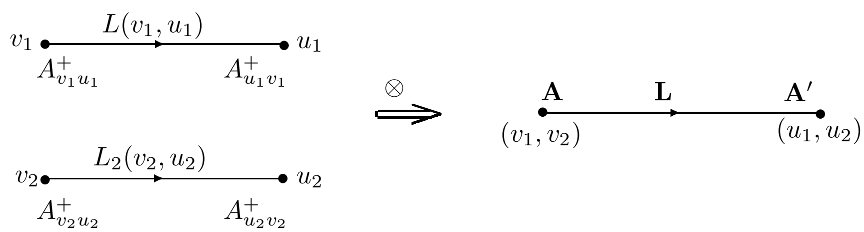

Fig.14

with $\mathbf{L}=(L_1(v_1,u_1),L_2(v_2,u_2))$ and $\mathbf{A}=A^+_{v_1u_1}\otimes A^+_{v_2u_2}$, $\mathbf{A}'=A^+_{u_1v_1}\otimes A^+_{u_2v_2}$, where $\overrightarrow{G}_1\otimes\overrightarrow{G}_2$ is the tensor product of $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ with

$$V\left(\overrightarrow{G}_1\otimes\overrightarrow{G}_2\right)=V\left(\overrightarrow{G}_1\right)\times V\left(\overrightarrow{G}_2\right)$$

and
$$\begin{aligned}E\left(\overrightarrow{G}_1\otimes\overrightarrow{G}_2\right)&=\{((v_1,v_2),(u_1,u_2))\,|\,if\ and\ only\ if\\&\quad(v_1,u_1)\in E\left(\overrightarrow{G}_1\right)\ and\ (v_2,u_2)\in E\left(\overrightarrow{G}_2\right)\}\end{aligned}$$

with an orientation $O^+:\ (v_1,v_2)\to(u_1,u_2)$ on $((v_1,v_2),(u_1,u_2))\in E\left(\overrightarrow{G}_1\otimes\overrightarrow{G}_2\right)$.

Indeed, $\left(\overrightarrow{G}_1;L_1,A_1\right)\otimes\left(\overrightarrow{G}_2;L_2,A_2\right)$ is an action flow with conservation laws on each vertex in $\overrightarrow{G}_1\otimes\overrightarrow{G}_2$ because

$$\begin{aligned}&\sum_{(u_1,u_2)\in N_{G_1\otimes G_2}(v_1,v_2)}A^+_{v_1u_1}\otimes A^+_{v_2u_2}\left(L_1(v_1,u_1),L_2(v_2,u_2)\right)\\=&\sum_{(u_1,u_2)\in N_{G_1\otimes G_2}(v_1,v_2)}A^+_{v_1u_1}\left(L_1(v_1,u_1)\right)A^+_{v_2u_2}\left(L_2(v_2,u_2)\right)\\=&\left(\sum_{u_1\in N_{G_1}(v_1)}\left(L_1(v_1,u_1)\right)^{A^+_{v_1u_1}}\right)\times\left(\sum_{u_2\in N_{G_2}(v_2)}\left(L_2(v_2,u_2)\right)^{A^+_{v_2u_2}}\right)=\mathbf{0}\end{aligned}$$

for $\forall(v_1, v_2) \in V\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2\right)$ by definition.

Theorem 3.4 *The tensor operation is associative, i.e.,*

$$\left(\left(\overrightarrow{G}_1; L_1, A_1\right) \bigotimes \left(\overrightarrow{G}_2; L_2, A_2\right)\right) \bigotimes \left(\overrightarrow{G}_3; L_3, A_3\right)$$
$$= \left(\overrightarrow{G}_1; L_1, A_1\right) \bigotimes \left(\left(\overrightarrow{G}_2; L_2, A_2\right) \bigotimes \left(\overrightarrow{G}_3; L_3, A_3\right)\right).$$

Proof By definition, $\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2\right) \otimes \overrightarrow{G}_3 = \overrightarrow{G}_1 \otimes \left(\overrightarrow{G}_2 \otimes \overrightarrow{G}_3\right)$. Let $(v_1, u_1) \in E\left(\overrightarrow{G}_1\right)$, $(v_2, u_2) \in E\left(\overrightarrow{G}_2\right)$ and $(v_3, u_3) \in E\left(\overrightarrow{G}_3\right)$. Then, $((v_1, v_2, v_3), (u_1, u_2, u_3)) \in E\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2 \otimes \overrightarrow{G}_3\right)$ with flows $(L_1(v_1, u_1), L_2(v_2, u_2), L_3(v_3, u_3))$, and end-operators $\left(A^+_{v_1u_1} \otimes A^+_{v_2,u_2}\right) \otimes A^+_{v_3u_3}$ in $\left(\left(\overrightarrow{G}_1; L_1, A_1\right) \otimes \left(\overrightarrow{G}_2; L_2, A_2\right)\right) \otimes \left(\overrightarrow{G}_3; L_3, A_3\right)$ but $A^+_{v_1u_1} \otimes \left(A^+_{v_2,u_2} \otimes A^+_{v_3u_3}\right)$ in $\left(\overrightarrow{G}_1; L_1, A_1\right) \otimes \left(\left(\overrightarrow{G}_2; L_2, A_2\right) \otimes \left(\overrightarrow{G}_3; L_3, A_3\right)\right)$ on the vertex (v_1, v_2, v_3). However,

$$\left(A^+_{v_1u_1} \otimes A^+_{v_2,u_2}\right) \otimes A^+_{v_3u_3} = A^+_{v_1u_1} \otimes \left(A^+_{v_2,u_2} \otimes A^+_{v_3u_3}\right)$$

for tensors. This completes the proof. □

Theorem 3.4 enables one to define the product $\bigotimes_{i=1}^{n}\left(\overrightarrow{G}_i; L_i, A_i\right)$. Clearly, if $\left\{\overrightarrow{G}_i^{L_{i1}}, \overrightarrow{G}_i^{L_{i2}}, \cdots, \overrightarrow{G}_i^{L_{in_i}}\right\}$ is a base of $\overrightarrow{G}_i^{\mathscr{V}}$, then $\overrightarrow{G}_1^{L_{1i_1}} \otimes \overrightarrow{G}_2^{L_{2i_2}} \otimes \cdots \otimes \overrightarrow{G}_n^{L_{ni_n}}$, $1 \le i_j \le n_i, 1 \le i \le n$ form a base of $\overrightarrow{G}_1^{\mathscr{V}_1} \otimes \overrightarrow{G}_2^{\mathscr{V}_2} \otimes \cdots \otimes \overrightarrow{G}_n^{\mathscr{V}_n}$. This implies a result by Theorem 3.1 and Corollary 3.2.

Theorem 3.5 $$\dim\left(\bigotimes_{i=1}^{m}\left(\overrightarrow{G}_i; L_i, A_i\right)\right) = \prod_{i=1}^{m} \dim \mathscr{V}_i^{2\beta(\overrightarrow{G}_i)}.$$

Particularly, $\dim\left(\bigotimes_{i=1}^{m}\overrightarrow{G}_i^{\mathscr{V}_i}\right) = \prod_{i=1}^{m} \dim \mathscr{V}_i^{\beta(\overrightarrow{G}_i)}$ *and furthermore, if* $\mathscr{V}_i = \mathscr{V}$ *for integers* $1 \le i \le m$*, then*

$$\dim\left(\bigotimes_{i=1}^{m}\overrightarrow{G}_i^{\mathscr{V}}\right) = \dim \mathscr{V}^{\sum\limits_{i=1}^{m}\beta(\overrightarrow{G}_i)},$$

and if each $\overrightarrow{G}_i$ *is a circuit* $\overrightarrow{C}_{n_i}$*, or each* $\overrightarrow{G}_i$ *is a bouquet* $\overrightarrow{B}_{n_i}$ *for integers* $1 \le i \le m$*, then*

$$\dim\left(\bigotimes_{i=1}^{n}\overrightarrow{G}_i^{\mathscr{V}}\right) = \dim \mathscr{V}^{n} \quad \textit{or} \quad \dim\left(\bigotimes_{i=1}^{n}\overrightarrow{G}_i^{\mathscr{V}}\right) = \dim \mathscr{V}^{n_1+n_2+\cdot+n_m}.$$

§4. Banach $\overrightarrow{G}$-Flow Spaces

The Banach and Hilbert spaces are linear space $\mathscr{V}$ over a field $\mathbb{R}$ or $\mathbb{C}$ respectively equipped with a complete norm $\|\cdot\|$ or inner product $\langle\cdot,\cdot\rangle$, i.e., for every Cauchy sequence $\{x_n\}$ in $\mathscr{V}$, there exists an element x in $\mathscr{V}$ such that

$$\lim_{n\to\infty}\|x_n - x\|_{\mathscr{V}} = 0 \quad \text{or} \quad \lim_{n\to\infty}\langle x_n - x, x_n - x\rangle_{\mathscr{V}} = 0.$$

We extend Banach or Hilbert spaces over graph $\overrightarrow{G}$ by a kind of edge labeled graphs, i.e., $\overrightarrow{G}$-flows in this section.

4.1 Banach $\overrightarrow{G}$-Flow Spaces

Let $\mathscr{V}$ be a Banach space over a field $\mathscr{F}$ with $\mathscr{F}=\mathbb{R}$ or $\mathbb{C}$. For any $\overrightarrow{G}$-flow $\overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{V}}$, define

$$\left\|\overrightarrow{G}^L\right\| = \sum_{(v,u)\in E\left(\overrightarrow{G}\right)} \|L(v,u)\|,$$

where $\|L(v,u)\|$ is the norm of $L(v,u)$ in $\mathscr{V}$. Then it is easily to check that

(1) $\left\|\overrightarrow{G}^L\right\| \geq 0$ and $\left\|\overrightarrow{G}^L\right\| = 0$ if and only if $\overrightarrow{G}^L = \overrightarrow{G}^{\mathbf{0}}$.

(2) $\left\|\overrightarrow{G}^{\xi L}\right\| = \xi \left\|\overrightarrow{G}^L\right\|$ for any scalar ξ.

(3) $\left\|\overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2}\right\| \leq \left\|\overrightarrow{G}^{L_1}\right\| + \left\|\overrightarrow{G}^{L_2}\right\|$.

Whence, $\|\cdot\|$ is a norm on linear space $\overrightarrow{G}^{\mathscr{V}}$. Furthermore, if $\mathscr{V}$ is an inner space, define

$$\left\langle \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \right\rangle = \sum_{(u,v)\in E\left(\overrightarrow{G}\right)} \langle L_1(v,u), L_2(v,u)\rangle .$$

Then

(4) $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle \geq 0$ and $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle = 0$ if and only if $L(v,u) = \mathbf{0}$ for $\forall (v,u) \in E(\overrightarrow{G})$, i.e., $\overrightarrow{G}^L = \overrightarrow{G}^{\mathbf{0}}$.

(5) $\left\langle \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \right\rangle = \overline{\left\langle \overrightarrow{G}^{L_2}, \overrightarrow{G}^{L_1} \right\rangle}$ for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$.

(6) For $\overrightarrow{G}^L, \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$, there is

$$\begin{aligned}\left\langle \lambda\overrightarrow{G}^{L_1} \right. &+ \left. \mu\overrightarrow{G}^{L_2}, \overrightarrow{G}^L \right\rangle \\ &= \lambda \left\langle \overrightarrow{G}^{L_1}, \overrightarrow{G}^L \right\rangle + \mu \left\langle \overrightarrow{G}^{L_2}, \overrightarrow{G}^L \right\rangle .\end{aligned}$$

Thus, $\overrightarrow{G}^{\mathscr{V}}$ is an inner space. As the usual, let

$$\left\|\overrightarrow{G}^L\right\| = \sqrt{\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle}$$

for $\overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{V}}$. Then it is also a normed space.

If the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ are complete, then $\left\|\overrightarrow{G}^L\right\|$ and $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle$ are too also, i.e., any Cauchy sequence in $\overrightarrow{G}^{\mathscr{V}}$ is converges. In fact, let $\left\{\overrightarrow{G}^{L_n}\right\}$ be a Cauchy sequence in $\overrightarrow{G}^{\mathscr{V}}$. Then for any number $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\left\|\overrightarrow{G}^{L_n} - \overrightarrow{G}^{L_m}\right\| < \varepsilon$$

if $n, m \geq N(\varepsilon)$. By definition,

$$\|L_n(v,u) - L_m(v,u)\| \leq \left\|\overrightarrow{G}^{L_n} - \overrightarrow{G}^{L_m}\right\| < \varepsilon$$

i.e., $\{L_n(v,u)\}$ is also a Cauchy sequence for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$, which is converges in $\mathscr{V}$ by definition.

Now let $L(v,u) = \lim\limits_{n\to\infty} L_n(v,u)$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$. Clearly,

$$\lim_{n\to\infty} \overrightarrow{G}^{L_n} = \overrightarrow{G}^{L}.$$

Even so, we are needed to show that $\overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{V}}$. By definition,

$$\sum_{u\in N_G(v)} L_n(v,u) = \mathbf{0}, \qquad v \in V\left(\overrightarrow{G}\right)$$

for any integer $n \geq 1$. If $n \to \infty$ on both sides, we are easily knowing that

$$\begin{aligned}\lim_{n\to\infty}\left(\sum_{u\in N_G(v)} L_n(v,u)\right) &= \sum_{u\in N_G(v)} \lim_{n\to\infty} L_n(v,v) \\ &= \sum_{u\in N_G(v)} L(v,u) = \mathbf{0}.\end{aligned}$$

Thus, $\overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{V}}$, which implies that the norm is complete, which can be also applied to the case of Hilbert space. Thus we get the following result.

Theorem 4.1([18], [22]) *For any graph $\overrightarrow{G}$, $\overrightarrow{G}^{\mathscr{V}}$ is a Banach space, and furthermore, if $\mathscr{V}$ is a Hilbert space, $\overrightarrow{G}^{\mathscr{V}}$ is a Hilbert space also.*

An operator $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$ is a *contractor* if

$$\left\|\mathbf{T}\left(\overrightarrow{G}^{L_1}\right) - \mathbf{T}\left(\overrightarrow{G}^{L_2}\right)\right\| \leq \xi \left\|\overrightarrow{G}^{L_1} - \overrightarrow{G}^{L_2})\right\|$$

for $\forall\overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_1} \in \overrightarrow{G}^{\mathscr{V}}$ with $\xi \in [0,1)$. The next result generalizes the fixed point theorem of Banach to Banach $\overrightarrow{G}$-flow space.

Theorem 4.2([18]) *Let $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$ be a contractor. Then there is a uniquely G-flow $\overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{V}}$ such that $\mathbf{T}\left(\overrightarrow{G}^{L}\right) = \overrightarrow{G}^{L}$.*

An operator $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$ is *linear* if

$$\mathbf{T}\left(\lambda\overrightarrow{G}^{L_1} + \mu\overrightarrow{G}^{L_2}\right) = \lambda\mathbf{T}\left(\overrightarrow{G}^{L_1}\right) + \mu\mathbf{T}\left(\overrightarrow{G}^{L_2}\right)$$

for $\forall\overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$ and $\lambda, \mu \in \mathscr{F}$. The following result generalizes the representation theorem of Fréchet and Riesz on linear continuous functionals to Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{V}}$.

Theorem 4.3([18], [22]) *Let* $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \mathbb{C}$ *be a linear continuous functional. Then there is a unique* $\overrightarrow{G}^{\widehat{L}} \in \overrightarrow{G}^{\mathscr{V}}$ *such that* $\mathbf{T}\left(\overrightarrow{G}^{L}\right) = \left\langle \overrightarrow{G}^{L}, \overrightarrow{G}^{\widehat{L}} \right\rangle$ *for* $\forall \overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{V}}$.

4.3 Examples of Linear Operator on Banach $\overrightarrow{G}$-Flow Spaces

Let $\mathscr{H}$ be a Hilbert space consisting of measurable functions $f(x_1, x_2, \cdots, x_n)$ on a set

$$\Delta = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | a_i \le x_i \le b_i, 1 \le i \le n\},$$

which is a functional space $L^2[\Delta]$, with inner product

$$\langle f(\mathbf{x}), g(\mathbf{x}) \rangle = \int_{\Delta} \overline{f(\mathbf{x})} g(\mathbf{x}) d\mathbf{x} \quad \text{for} \quad f(\mathbf{x}), g(\mathbf{x}) \in L^2[\Delta],$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\overrightarrow{G}$ an oriented graph embedded in a topological space. As we shown in last section, we can extended $\mathscr{H}$ on graph $\overrightarrow{G}$ to get Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{H}}$.

The *differential* and *integral operators*

$$D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \int_{\Delta}$$

on $\mathscr{H}$ are extended respectively by

$$D\overrightarrow{G}^{L} = \overrightarrow{G}^{DL(u^v)}$$

and

$$\int_{\Delta} \overrightarrow{G}^{L} = \int_{\Delta} K(\mathbf{x}, \mathbf{y}) \overrightarrow{G}^{L[\mathbf{y}]} d\mathbf{y} = \overrightarrow{G}^{\int_{\Delta} K(\mathbf{x}, \mathbf{y}) L(u^v)[\mathbf{y}] d\mathbf{y}},$$

for $\forall (u, v) \in E\left(\overrightarrow{G}\right)$, where a_i, $\dfrac{\partial a_i}{\partial x_j} \in \mathbb{C}^0(\Delta)$ for integers $1 \le i, j \le n$ and $K(\mathbf{x}, \mathbf{y}) : \Delta \times \Delta \to \mathbb{C} \in L^2(\Delta \times \Delta, \mathbb{C})$ with

$$\int_{\Delta \times \Delta} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} < \infty.$$

Clearly,

$$\begin{aligned}
& D\left(\lambda \overrightarrow{G}^{L_1(v,u)} + \mu \overrightarrow{G}^{L_2(v,u)}\right) = D\left(\overrightarrow{G}^{\lambda L_1(v,u) + \mu L_2(v,u)}\right) \\
& = \overrightarrow{G}^{D(\lambda L_1(v,u) + \mu L_2(v,u))} = \overrightarrow{G}^{D(\lambda L_1(v,u)) + D(\mu L_2(v,u))} \\
& = \overrightarrow{G}^{D(\lambda L_1(v,u))} + \overrightarrow{G}^{D(\mu L_2(v,u))} = D\left(\overrightarrow{G}^{(\lambda L_1(v,u))} + \overrightarrow{G}^{(\mu L_2(v,u))}\right) \\
& = \lambda D\left(\overrightarrow{G}^{L_1(v,u)}\right) + D\left(\mu \overrightarrow{G}^{L_2(v,u)}\right)
\end{aligned}$$

for $\overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{H}}$ and $\lambda, \mu \in \mathbb{R}$, i.e.,

$$D\left(\lambda\overrightarrow{G}^{L_1} + \mu\overrightarrow{G}^{L_2}\right) = \lambda D\overrightarrow{G}^{L_1} + \mu D\overrightarrow{G}^{L_2}.$$

Similarly, we can show also that

$$\int_{\Delta}\left(\lambda\overrightarrow{G}^{L_1} + \mu\overrightarrow{G}^{L_2}\right) = \lambda\int_{\Delta}\overrightarrow{G}^{L_1} + \mu\int_{\Delta}\overrightarrow{G}^{L_2},$$

i.e., the operators D and $\int_{\Delta}$ are linear.

Notice that $\overrightarrow{G}^{L(v,u)} \in \overrightarrow{G}^{\mathscr{H}}$, there must be

$$\sum_{u\in N_G(v)} L(v,u) = \mathbf{0} \quad \text{for} \quad \forall v \in V\left(\overrightarrow{G}\right),$$

We therefore know that

$$\mathbf{0} = D\left(\sum_{u\in N_G(v)} L(v,u)\right) = \sum_{u\in N_G(v)} DL(v,u)$$

and

$$\mathbf{0} = \int_{\Delta}\left(\sum_{u\in N_G(v)} L(v,u)\right) = \sum_{u\in N_G(v)}\int_{\Delta} L(v,u)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$. Consequently,

$$D:\ \overrightarrow{G}^{\mathscr{H}} \to \overrightarrow{G}^{\mathscr{H}}, \text{ and } \int_{\Delta}:\ \overrightarrow{G}^{\mathscr{H}} \to \overrightarrow{G}^{\mathscr{H}}$$

are linear operators on $\overrightarrow{G}^{\mathscr{H}}$.

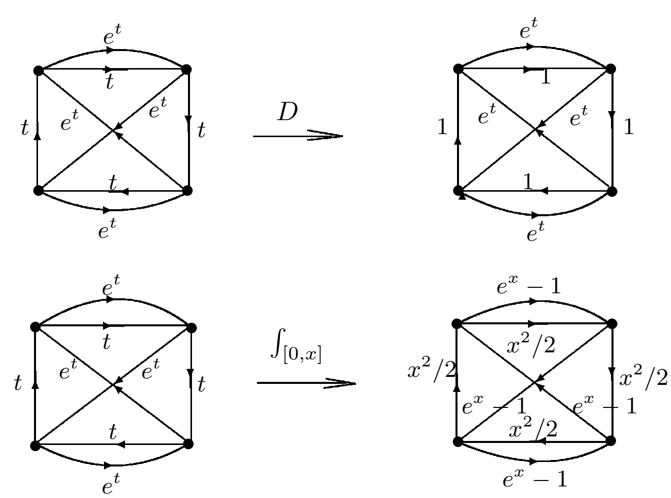

Fig.15

For example, let $f(t) = t$, $g(t) = e^t$, $K(t,\tau) = 1$ on $\Delta = [0, x]$ and let $\overrightarrow{G}^L$ be the $\overrightarrow{G}$-flow shown on the left side in Fig.15. Calculation shows that $Df = 1, Dg = e^t$,

$$\int_0^x K(t,\tau)f(\tau)d\tau = \int_0^x \tau d\tau = \frac{x^2}{2}, \quad \int_0^x K(t,\tau)g(\tau)d\tau = \int_0^x e^\tau d\tau = e^x - 1$$

and the actions $D\overrightarrow{G}^L$, $\int_{[0,1]} \overrightarrow{G}^L$ are shown on the right in Fig.15.

Particularly, the Cauchy problem on heat equation

$$\frac{\partial u}{\partial t} = c^2 \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$$

is solvable in $\mathbb{R}^n \times \mathbb{R}$ if $u(\mathbf{x}, t_0) = \varphi(\mathbf{x})$ is continuous and bounded in $\mathbb{R}^n$, and c is a non-zero constant in $\mathbb{R}$. Certainly, we can also consider the Cauchy problem in $\overrightarrow{G}^{\mathscr{H}}$, i.e.,

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^n \frac{\partial^2 X}{\partial x_i^2}$$

with initial values $X|_{t=t_0}$, and get the following result.

Theorem 4.4([18]) *For $\forall \overrightarrow{G}^{L'} \in \overrightarrow{G}^{\mathbb{R}^n \times \mathbb{R}}$ and a non-zero constant c in $\mathbb{R}$, the Cauchy problems on differential equations*

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^n \frac{\partial^2 X}{\partial x_i^2}$$

with initial value $X|_{t=t_0} = \overrightarrow{G}^{L'} \in \overrightarrow{G}^{\mathbb{R}^n \times \mathbb{R}}$ is solvable in $\overrightarrow{G}^{\mathbb{R}^n \times \mathbb{R}}$ if $L'(v,u)$ is continuous and bounded in $\mathbb{R}^n$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

Fortunately, if the graph $\overrightarrow{G}$ is prescribed with special structures, for instance the circuit decomposable, we can always solve the Cauchy problem on an equation in Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{H}}$ if this equation is solvable in $\mathscr{H}$.

Theorem 4.5([18], [22]) If the graph $\overrightarrow{G}$ is strong-connected with circuit decomposition

$$\overrightarrow{G} = \bigcup_{i=1}^{l} \overrightarrow{C}_i$$

such that $L(v,u) = L_i(\mathbf{x})$ for $\forall (v,u) \in E\left(\overrightarrow{C}_i\right)$, $1 \le i \le l$ and the Cauchy problem

$$\begin{cases} \mathscr{F}_i(\mathbf{x}, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots) = 0 \\ u|_{\mathbf{x}_0} = L_i(\mathbf{x}) \end{cases}$$

is solvable in a Hilbert space $\mathscr{H}$ on domain $\Delta \subset \mathbb{R}^n$ for integers $1 \le i \le l$, then the Cauchy problem

$$\begin{cases} \mathscr{F}_i\left(\mathbf{x}, X, X_{x_1}, \cdots, X_{x_n}, X_{x_1x_2}, \cdots\right) = 0 \\ X|_{\mathbf{x}_0} = \overrightarrow{G}^L \end{cases}$$

such that $L(v,u) = L_i(\mathbf{x})$ for $\forall (v,u) \in X\left(\overrightarrow{C}_i\right)$ is solvable for $X \in \overrightarrow{G}^{\mathscr{H}}$.

§5. Applications

Notice that labeled graph combines the discrete with that of analytic mathematics. This character implies that it can be used as a model for living things in the nature and contributes to system control, gravitational field, interaction fields, economics, traffic flows, ecology, epidemiology and other sciences. But we only mention 2 applications of labeled graphs for limitation of the space, i.e., global stability and spacetime in this section. More its applications can be found in references [6]-[7], [13]-[23].

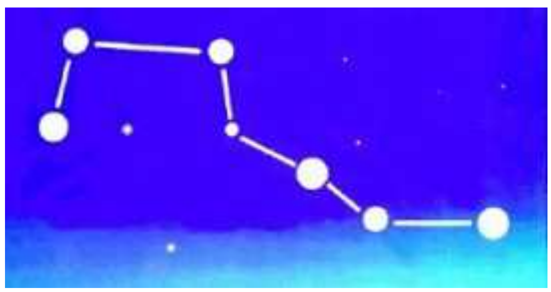

Fig.16

5.1 Global Stability

The stability of systems characterized by differential equations (ES_m) addresses the stability of solutions of (ES_m) and the trajectories of systems with small perturbations on initial values, such as those shown for Big Dipper in Fig.16.

In mathematics, a solution of system (ES_m) of differential equations is called stable or asymptotically stable ([25]) if for all solutions $Y(t)$ of the differential equations (ES_m) with

$$|Y(0) - X(0)| < \delta(\varepsilon),$$

exists for all $t \geq 0$,

$$|Y(t) - X(t)| < \varepsilon$$

for $\forall \varepsilon > 0$ or furthermore,

$$\lim_{t \to 0} |Y(t) - X(t)| = 0.$$

However, by Theorem 2.9 if $\bigcap\limits_{i=1}^{m} S_{T_i} = \emptyset$ there are no solutions of (ES_m). Thus, the classical theory of stability is failed to apply. Then *how can one characterizes the stability of system*

(ES_m)? As we have shown in Subsection 2.4, we always get a labeled graph solution $G^L[ES_m]$ of system (ES_m) whenever it is solvable or not, which can be applied to characterize the stability of system (ES_m).

Without loss of generality, assume $G^L(t)$ be a solution of (ES_m) with initial values $G^L(t_0)$ and let $\omega : V\left(G^L[ES_m]\right) \to \mathbb{R}$ be an index function. It is said to be *ω-stable* if there exists a number $\delta(\varepsilon)$ for any number $\varepsilon > 0$ such that

$$\left\|\omega\left(G^{L_1(t)-L_2(t)}\right)\right\| < \varepsilon,$$

or furthermore, *asymptotically ω-stable* if

$$\lim_{t\to\infty}\left\|\omega\left(G^{L_1(t)-L_2(t)}\right)\right\| = 0$$

if initial values hold with

$$\|L_1(t_0)(v) - L_2(t_0)(v)\| < \delta(\varepsilon)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$. If there is a Liapunov ω-function $L(\omega(t)) : \mathscr{O} \to \mathbb{R}, n \geq 1$ on $\overrightarrow{G}$ with $\mathscr{O} \subset \mathbb{R}^n$ open such that $L(\omega(t)) \geq 0$ with equality hold only if $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ and if $t \geq t_0$, $\dfrac{dL(\omega)}{dt} \leq 0$, for the ω-stability of $\overrightarrow{G}$-flow, we then know a result on ω-stability of (ES_m) following.

Theorem 5.1([22]) *If there is a Liapunov ω-function $L(\omega(t)) : \mathscr{O} \to \mathbf{R}$ on $G^L[ES_m]$ of system (ES_m), then $G^L[ES_m]$ is ω-stable, and furthermore, if $\dot{L}(\omega(t)) < 0$ for $G^L[ES_m] \neq G^{\mathbf{0}}[ES_m]$, then $G^L[ES_m]$ is asymptotically ω-stable.*

For linear differential equations ($LDES_m^1$), we can further introduce the sum-table subgraph following.

Definition 5.2 *Let H^L be a spanning subgraph of $G^L[LDES_m^1]$ of systems ($LDES_m^1$) with initial value $X_v(0), v \in V\left(G[LDES_m^1]\right)$. Then $G^L[LDES_m^1]$ is called sum-stable or asymptotically sum-stable on H^L if for all solutions $Y_v(t)$, $v \in V(H^L)$ of the linear differential equations of ($LDES_m^1$) with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$,*

$$\left|\sum_{v\in V(H^L)} Y_v(t) - \sum_{v\in V(H^L)} X_v(t)\right| < \varepsilon,$$

or furthermore,

$$\lim_{t\to 0}\left|\sum_{v\in V(H^L)} Y_v(t) - \sum_{v\in V(H^L)} X_v(t)\right| = 0.$$

We get a result on the global stability for G-solutions of ($LDES_m^1$) following.

Theorem 5.3([13]) *A labeled graph solution $G^{\mathbf{0}}[LDES_m^1]$ of linear homogenous differential equation systems ($LDES_m^1$) is asymptotically sum-stable on a spanning subgraph H^L of*

$G^L[LDES_m^1]$ *if and only if* $\mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$, $\forall v \in V(H^L)$ *in* $G^L\left[LDES_m^1\right]$.

Example 5.4 Let a labeled graph solution $G^L[LDES_m^1]$ of $(LDES_m^1)$ be shown in Fig.17, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 = \{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the labeled graph solution $G^{\mathbf{0}}[LDES_m^1]$ is sum-stable on the labeled triangle $v_4v_5v_6$ but not on the triangle $v_1v_2v_3$.

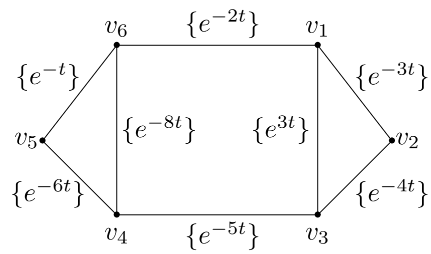

Fig.17

Similarly, let the system $(PDES_m^C)$ of linear partial differential equations be

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ &u|_{t=t_0} = u_0^{[i]}(x_1, x_2, \cdots, x_{n-1}) \end{aligned}\right\} \ 1 \leq i \leq m \qquad (APDES_m^C)$$

A point $X_0^{[i]} = (t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]})$ with $H_i(t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]}) = 0$ for $1 \leq i \leq m$ is called an *equilibrium point* of the ith equation in $(APDES_m^C)$. Then we know the following result, which can be applied to the ecological mathematics for the number of species≥ 3 ([31]).

Theorem 5.5([17]) *Let* $X_0^{[i]}$ *be an equilibrium point of the ith equation in* $(APDES_m^C)$ *for integers* $1 \leq i \leq m$. *If* $\sum\limits_{i=1}^{m} H_i(X) > 0$ *and* $\sum\limits_{i=1}^{m} \frac{\partial H_i}{\partial t} \leq 0$ *for* $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, *then the labeled graph solution* $G^L[APDES_m^C]$ *of system* $(APDES_m^C)$ *is sum-stable. Furthermore, if* $\sum\limits_{i=1}^{m} \frac{\partial H_i}{\partial t} < 0$ *for* $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, *then the labeled graph solution* $G^L[APDES_m^C]$ *of system* $(APDES_m^C)$ *is asymptotically sum-stable.*

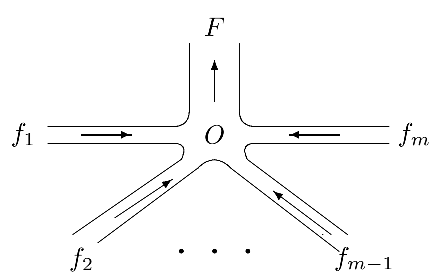

Fig.18

An immediately application of Theorem 5.5 is the control of traffic flows. For example, let O be a node in N incident with m in-flows and 1 out-flow such as those shown in Fig.18. Then, *what conditions will make sure the flow F being stable*? Denote the density of flow F by $\rho^{[F]}$ and f_i by $\rho^{[i]}$ for integers $1 \leq i \leq m$, respectively. Then, by traffic theory,

$$\frac{\partial \rho^{[i]}}{\partial t} + \phi_i(\rho^{[i]})\frac{\partial \rho^{[i]}}{\partial x} = 0, \ 1 \leq i \leq m.$$

We prescribe the initial value of $\rho^{[i]}$ by $\rho^{[i]}(x, t_0)$ at time t_0. Replacing each $\rho^{[i]}$ by ρ in these flow equations of f_i, $1 \leq i \leq m$, we get a non-solvable system $(PDES_m^C)$ of partial differential equations

$$\left.\begin{array}{l} \frac{\partial \rho}{\partial t} + \phi_i(\rho)\frac{\partial \rho}{\partial x} = 0 \\ \rho\mid_{t=t_0} = \rho^{[i]}(x, t_0) \end{array}\right\} \ 1 \leq i \leq m.$$

Denote an equilibrium point of the ith equation by $\rho_0^{[i]}$, i.e., $\phi_i(\rho_0^{[i]})\frac{\partial \rho_0^{[i]}}{\partial x} = 0$. By Theorem 5.5, if

$$\sum_{i=1}^{m} \phi_i(\rho) < 0 \ \text{ and } \ \sum_{i=1}^{m} \phi_(\rho)\left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] \geq 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, then the flow F is stable, and furthermore, if

$$\sum_{i=1}^{m} \phi_(\rho)\left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] < 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, it is asymptotically stable.

5.2 Spacetime

Usually, different spacetime determine different structure of the universe, particularly for the solutions of Einstein's gravitational equations

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu},$$

where $R^{\mu\nu} = R_{\alpha}^{\mu\alpha\nu} = g_{\alpha\beta}R^{\alpha\mu\beta\nu}$, $R = g_{\mu\nu}R^{\mu\nu}$ are the respective Ricci tensor, Ricci scalar curvature, $G = 6.673 \times 10^{-8} cm^3/gs^2$, $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$ ([24]).

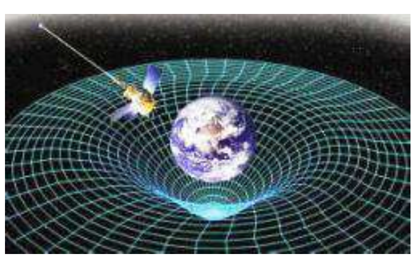

Fig.19

Certainly, Einstein's general relativity is suitable for use only in one spacetime $\mathbb{R}^4$, which implies a curved spacetime shown in Fig.19. But, if the dimension of the universe> 4,

How can we characterize the structure of spacetime for the universe?

Generally, we understanding a thing by observation, i.e., the received information via hearing, sight, smell, taste or touch of our sensory organs and verify results on it in $\mathbb{R}^3 \times \mathbb{R}$. If the dimension of the universe> 4, all these observations are nothing else but a projection of the true faces on our six organs, a partially truth. As a discrete mathematicians, the combinatorial notion should be his world view. A combinatorial spacetime $(\mathscr{C}_G|\bar{t})$ ([7]) is in fact a graph G^L labeled by Euclidean spaces $\mathbb{R}^n, n \geq 3$ evolving on a time vector $\bar{t}$ under smooth conditions in geometry. We can characterize the spacetime of the universe by a complete graph K_m^L labeled by $\mathbb{R}^4$ (See [9]-[11] for details).

For example, if $m = 4$, there are 4 Einstein's gravitational equations for $\forall v \in V\left(K_4^L\right)$. We solve it locally by spherically symmetric solutions in $\mathbb{R}^4$ and construct a graph K_4^L-solution labeled by $S_{f_1}, S_{f_2}, S_{f_3}$ and S_{f_4} of Einstein's gravitational equations, such as those shown in Fig.20,

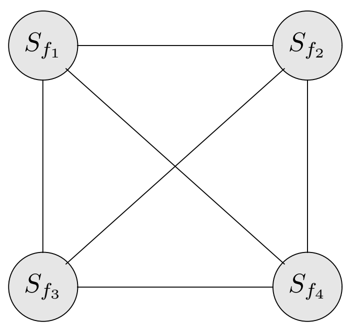

Fig.20

where, each S_{f_i} is a geometrical space determined by Schwarzschild spacetime

$$ds^2 = f(t)\left(1 - \frac{r_s}{r}\right)dt^2 - \frac{1}{1 - \frac{r_s}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for integers $1 \leq i \leq 4$. Certainly, its global behavior depends on the intersections $S_{f_i} \bigcap S_{f_j}, 1 \leq i \neq j \leq 4$.

Notice that $m = 4$ is only an assumption. We do not know the exact value of m at present. Similarly, by Theorem 4.5, we also get a conclusion on spacetime of the Einstein's gravitational equations and we do not know also which labeled graph structure is the real spacetime of the universe.

Theorem 5.6([17]) *There are infinite many $\overrightarrow{G}$-flow solutions on Einstein's gravitational equations*

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -8\pi GT^{\mu\nu}$$

in $\overrightarrow{G}^{\mathbb{C}}$, particularly on those graphs with circuit-decomposition $\overrightarrow{G} = \bigcup\limits_{i=1}^{m} \overrightarrow{C}_i$ with Schwarzschild spacetime on their edges.

For example, let $\overrightarrow{G} = \overrightarrow{C}_4$. We are easily find $\overrightarrow{C}_4$-flow solution of Einstein's gravitational equations such as those shown in Fig.21.

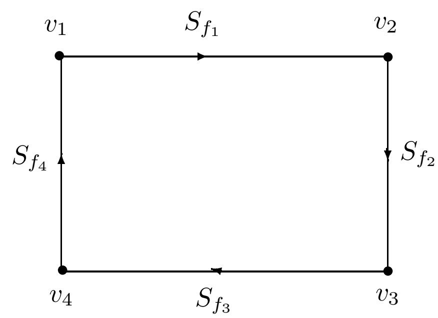

Fig.21

Then, the spacetime of the universe is nothing else but a curved ring such as those shown in Fig.22.

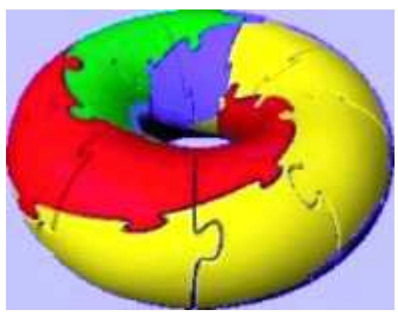

Fig.22

Generally, if $\overrightarrow{G}$ can be decomposed into m orientated circuits $\overrightarrow{C}_i$, $1 \leq i \leq m$, then Theorem 5.6 implies such a spacetime of Einstein's gravitational equations consisting of m curved rings over graph $\overrightarrow{G}$ in space.

§6. Conclusion

What are the elements of mathematics? Certainly, the mathematics consists of elements, include numbers $1, 2, 3, \cdots$, maps, functions $f(\mathbf{x})$, vectors, matrices, points, lines, opened sets $\cdots$, etc. with relations. However, these elements are not enough for understanding the reality of things because they must be a system without contradictions in its subfield of classical mathematics, i.e., a compatible system but contradictions exist everywhere, things are all in full of contradiction in the world. Thus, turn a systems with contradictions to mathematics is an important step for hold on the reality of things in the world. For such an objective, labeled graphs G^L are elements because a non-mathematics in classical is in fact a mathematics over a graph G ([16]), i.e., mathematical combinatorics. Thus, we should pay more attentions to labeled graphs, not only as a labeling technique on graphs but also a really mathematical element.

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Tchebychev and Brahmagupta Polynomials and Golden Ratio: Two New Interconnections

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Abstract: The present paper explores interconnections between sequences related to convergents of generalized golden ratios and four kinds of Tchebychev polynomials. By defining and adding Brahmagupta polynomials of third and fourth kind, the paper also interconnects the four kinds of Brahmagupta polynomials to the four kinds of Tchebychev Polynomials respectively. In this way, the present paper provides two spectacular views of Tchebychev polynomials of all four kinds through golden ratio and Brahmagupta polynomials.

Key Words: Fibonacci and Lucas numbers, Tchebychev polynomials and Brahmagupta polynomials.

AMS(2010): 11B39, 33C47.

§1. Introduction

The algebraic integer $\Phi = \frac{-1+\sqrt{5}}{2}$ obtained as one of the roots of the quadratic equation $t^2+t-1=0$ is well known in the literature as golden ratio. Φ is also given by the beautiful continued fraction expansion

$$\Phi = \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \cdots \, {}_{+} \frac{1}{1} \, {}_{+} \, \cdots = \lim_{n\longrightarrow\infty} \frac{F_n}{F_{n+1}}, \tag{1}$$

where F_n is the well known Fibonacci numbers. Approximating Φ by $\dfrac{F_n}{F_{n+1}}$ for a suitable n, ancient Greek architects have constructed what are called golden triangles, golden rectangles and so on, which have enhanced the beauty of architecture of their buildings. An elegant number theoretic result is that (L_n, F_n), where $L_n = F_{n-1} + F_{n+1}$ is well known as Lucas number, satisfies the quadratic Diophantine equation

$$L_n^2 - 5F_n^2 = 4(-1)^n.$$

For more details please refer [6], [9], [10]. Choosing one of the roots of $xt^2+t-1=0$ and $t^2+xt-1=0$ one gets the following two generalizations of golden ratio with interesting

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continued fraction expansions ([7], [11])

$$\Phi_1(x) = \frac{-1+\sqrt{1+4x}}{2x} = \frac{1}{1} \, {}_{+} \, \frac{x}{1} \, {}_{+} \, \frac{x}{1} \, {}_{+} \, \frac{x}{1} \, {}_{+} \cdots {}_{+}, \qquad x \geq 0, \tag{2}$$

$$\Phi_2(x) = \frac{-x+\sqrt{x^2+4}}{2} = \frac{1}{x} \, {}_{+} \, \frac{1}{x} \, {}_{+} \, \frac{1}{x} \, {}_{+} \, \frac{1}{x} \, {}_{+} \cdots {}_{+}, \qquad x > 0. \tag{3}$$

They have a nontrivial interconnection become

$$\Phi_2(x) = \frac{1}{x} \, \Phi_1\left(\frac{1}{x^2}\right). \tag{4}$$

When $x = 1$, $\Phi_1(1) = \Phi_2(1) = \Phi$.

The four kinds of Tchebychev polynomials well studied in the literature ([1], [3], [7]) are described below when $x = \cos\theta$:

$$\begin{aligned}
T_n(x) &= \cos n\theta \quad ; \; T_0 = 1, \;\; T_1(x) = x, \cdots, \\
U_n(x) &= \frac{\sin(n+1)\theta}{\sin\theta} \;\; ; \;\; U_0 = 1, \; U_1(x) = 2\,x, \cdots, \\
V_n(x) &= \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\,\theta} \;\; ; \; V_0 = 1, \;\; V_1(x) = 2\,x-1, \cdots, \\
W_n(x) &= \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} \;\; ; \;\; W_0 = 1, \; W_1(x) = 2\,x+1, \cdots.
\end{aligned}$$

They satisfy the three term recurrence relations

$$P_{n+1}(x) = 2\,xP_n(x) - P_{n-1}(x)$$

with the above initial condition. Their interrelations are nicely described below in the literature ([1], [3], [7]):

$$\begin{aligned}
U_n(x) &= \frac{T'_{n+1}(x)}{n+1}, \\
V_n(x) &= \frac{T_{n+1}(x)+T_n(x)}{x+1} = U_{n+1}(x) - U_n(x) \\
\text{and} \qquad W_n(x) &= \frac{T_{n+1}(x)-T_n(x)}{x-1} = U_{n+1}(x) + U_n(x) = (-1)^n V_n(-x).
\end{aligned}$$

Their link to trigonometric functions will yield the following worth quoting orthogonality properties ([1], [3], [7]):

$$\int_{-1}^{1} T_m(x) T_n(x) \frac{1}{\sqrt{1-x^2}} dx \;=\; \begin{cases} 0, & m \neq n; \\ \pi, & m = n = 0; \\ \frac{\pi}{2}, & m = n \neq 0, \end{cases}$$

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}dx = \begin{cases} 0, & m \neq n; \\ \frac{\pi}{2}, & m = n, \end{cases}$$

$$\int_{-1}^{1} V_m(x)V_n(x)\sqrt{\frac{1+x}{1-x}}dx = \begin{cases} 0, & m \neq n; \\ \pi, & m = n. \end{cases}$$

$$\int_{-1}^{1} W_m(x)W_n(x)\sqrt{\frac{1-x}{1+x}}dx = \begin{cases} 0, & m \neq n; \\ \pi, & m = n. \end{cases}$$

An amazing result on $\{T_{n+1}, U_n\}$ is that the continued fraction expansion ([11])

$$\sqrt{x^2-1} = x - \frac{1}{2x_-}\frac{1}{2x_-}\frac{1}{2x_-}\frac{1}{2x_-\cdots}, \qquad x > 1, \tag{5}$$

which is constructed using

$$\sqrt{x^2-1} = x - \frac{1}{x+\sqrt{x^2-1}}, \qquad x > 1$$

has the sequence of convergents

$$\left\{\frac{P(x)}{Q(x)}\right\} = \left\{\frac{x}{1}, \frac{2x^2-1}{2x}, \ \ldots \ , \frac{T_{n+1}(x)}{U_n(x)}, \ \ldots\right\}. \tag{6}$$

A related result is that the following continued fraction ([11])

$$\sqrt{\frac{x+1}{x-1}} = 1 + \frac{2}{2x-1_-}\frac{1}{2x_-}\frac{1}{2x_-}\frac{1}{2x_-\cdots}, \qquad x > 1, \tag{7}$$

which can also be written as

$$\sqrt{\frac{x+1}{x-1}} = 1 + \frac{2}{(x-1)+\sqrt{x^2-1}}, \qquad x > 1$$

has the sequence of convergents

$$\left\{\frac{\widetilde{P}(x)}{\widetilde{Q}(x)}\right\} = \left\{\frac{1}{1}, \frac{2x+1}{2x-1}, \ \ldots \ , \frac{W_n(x)}{V_n(x)}, \ \ldots\right\}. \tag{8}$$

A pair of two variable polynomials with a parameter $(x_n(x,y,t), y_n(x,y,t))$ is said to be Brahmagupta polynomials ([5], [6], [9]) if $x_n(x,y,t)$ and $y_n(x,y,t)$ satisfy

$$(x_n \pm y_n\sqrt{t}) = (x \pm y\sqrt{t})^n, \qquad n = 0, 1, 2, \ldots$$

or
$$x_n^2 - ty_n^2 = (x^2 - ty^2)^n$$

or
$$(x_m^2 - ty_m^2)(x_n^2 - ty_n^2) = (x_m x_n + t y_m y_n)^2 - t(x_m y_n + x_n y_m)^2. \tag{9}$$

The last identity (9) is called Brahmagupta identity ([12]), which is a more general form of Diophantine identity

$$(x_m^2+y_m^2)(x_n^2+y_n^2) = (x_mx_n-y_my_n)^2+(x_my_n+x_ny_m)^2.$$

Both x_n and y_n satisfy the following three term recurrence relations:

$$x_{n+1}=2x\ x_n-(x^2-ty^2)x_{n-1},\quad x_0=1,\quad x_1=x,\quad n=1,2,3,\cdots \tag{10}$$

and

$$\frac{y_{n+1}}{y}=2x\ \frac{y_n}{y}-(x^2-ty^2)\ \frac{y_{n-1}}{y},\quad \frac{y_1}{y}=1,\quad \frac{y_2}{y}=2x,\quad n=2,3,4,\cdots. \tag{11}$$

They are related to golden ratio as well as Tchebychev polynomials by the following relations [9]:

(1) For $x=\frac{1}{2}$, $y=\frac{1}{2}$ and $t=5$, one recovers easily

$$\begin{aligned} -x+y\sqrt{t} &= \Phi, \\ 2x_n &= L_n, \\ \frac{y_{n+1}}{y} &= F_{n+1}. \end{aligned}$$

(2) For $x^2-ty^2=1$, one gets directly

$$x_n=T_n(x),\qquad \frac{y_{n+1}}{y}=U_n(x),\ n=0,1,2,\cdots.$$

In the background of the above curious ideas and results the paper intends to do justice to its title. In the next section, the convergents of $\sqrt{\frac{x+1}{x-1}}$ related to $\Phi_1(x)$ are shown to be related to all the four kinds of Tchebychev polynomials in a rigors manner. The convergents of $\Phi_1(x)$ and $\Phi_2(x)$ are shown to be related to $U_n(x)$ and $V_n(x)$ only. In the third and the last section, first two kinds of Brahmagupta polynomials are shown to be related to $T_n(x)$ and $U_n(x)$.

The new things added are Brahmagupta polynomials of third and fourth kind which are defined with the help of Brahmagupta polynomials of second kind. Of course when $x^2-ty^2=1$, all of them will become respective kinds of Tchebychev polynomials.

§2. Generalization of Golden Ratio and Expressions for Their Convergents Interms of Tchebychev Polynomials

First let us consider the generalization of the golden ratio

$$\Phi_1(x)=\frac{-1+\sqrt{1+4x}}{2x}=\sum_{n=0}^{\infty}(-1)^nC_nx^n$$

valid for $|x|<\frac{1}{4}$, where $C_n=\dfrac{1}{n+1}\dbinom{2n}{n}$ is the n^{th} Catalan number. The above series is a

Stieltje's series ([8], [11]) because

$$\begin{aligned}\frac{-1+\sqrt{1+4x}}{2x} &= \frac{1}{4}\int_0^4 \frac{dt}{\sqrt{1+xt}}\\ &= \frac{1}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+}\cdots\ , \qquad x>0\end{aligned}$$

and the sequence of convergents is

$$\left\{\frac{P(x)}{Q(x)}\right\} = \left\{\frac{1}{1}, \frac{1}{x+1}, \frac{1+x}{1+2x}, \frac{1+2x}{1+3x+x^2},\ \cdots\ ,\ \frac{A_n(x)}{A_{n+1}(x)},\ \cdots\right\},$$

where

$$\begin{aligned}A_{n+1}(x) &= A_n(x) + xA_{n-1}(x),\\ A_1(x) &= 1, \quad A_2(x) = 1, \quad n = 2, 3, 4, \cdots .\end{aligned}$$

For $x = 1$, as expected one gets

$$A_n = F_n, \quad n = 1, 2, 3, \cdots .$$

In order to express $A_n(x)$ interms of Tchebychev polynomials, we use

$$\begin{aligned}\frac{1+\sqrt{1+4x}}{2} &= \left[\frac{-1+\sqrt{1+4x}}{2x}\right]^{-1}\\ &= 1 + \frac{x}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+}\cdots{}_{+}\,\frac{x}{1}\,{}_{+}\cdots\ , \qquad x>0\end{aligned} \tag{12}$$

and the sequence of convergents is

$$\left\{\frac{P(x)}{Q(x)}\right\} = \left\{\frac{1}{1}, \frac{1+x}{1}, \frac{1+2x}{1+x},\ \cdots\ ,\ \frac{A_{n+1}(x)}{A_n(x)},\ \cdots\ \right\}.$$

Let us apply the following transformation

$$x = \frac{1}{2(s-1)} \quad \text{or} \quad s-1 = \frac{1}{2x}, \quad x>0,$$

which enables us to wrote

$$\sqrt{1+4x} = \sqrt{\frac{s+1}{s-1}}.$$

Since

$$\begin{aligned}\sqrt{1+4x} &= 1 + 2x\left[\frac{-1+\sqrt{1+4x}}{2x}\right]\\ &= 1 + \frac{2x}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+}\cdots{}_{+}\,\frac{x}{1}\,{}_{+}\cdots\ , \qquad x>0.\end{aligned}$$

Using the above transformation

$$\begin{aligned}\sqrt{\frac{s+1}{s-1}} &= 1+\frac{\frac{1}{s-1}}{1}{}_{+}\frac{\frac{1}{2(s-1)}}{1}{}_{+\cdots}\\ &= 1\ +\ \frac{1}{s-1}{}_{+}\ \frac{1}{2}{}_{+}\ \frac{1}{s-1}{}_{+}\ \frac{1}{2}{}_{+\cdots}\ , \qquad x>0,\end{aligned} \tag{13}$$

which is valid because $s = 1+\frac{1}{2x}$, $x>0$ and the sequence of convergents is

$$\left\{\frac{P(x)}{Q(x)}\right\} = \left\{\frac{1}{1},\ \frac{s}{(s-1)},\ \frac{2s+1}{2s-1},\ \frac{2s^2-1}{2(s-1)2s},\ \cdots\ ,\ \frac{P_{2n-1}(s)}{Q_{2n-1}(s)},\ \frac{P_{2n}(s)}{Q_{2n}(s)},\ \cdots\ \right\}.$$

The numerator and denominator polynomials of the continued fraction (13) satisfy the following relations:

(1) $P_{2n+1}(s) = 2P_{2n}(s) + P_{2n-1}(s)$;

(2) $P_{2n}(s) = (s-1)P_{2n-1}(s) + P_{2n-2}(s)$;

(3) $Q_{2n+1}(s) = 2Q_{2n}(s) + Q_{2n-1}(s)$;

(4) $Q_{2n}(s) = (s-1)Q_{2n-1}(s) + Q_{2n-2}(s)$.

Using the above relation, we get the following three term recurrence relation for the odd and the even convergents of the continued fraction (13):

$$\begin{aligned}P_{2n+1}(s) &= 2\ [(s-1)P_{2n-1}(s) + P_{2n+2}(s)] + P_{2n-1}(s)\\ &= 2s\ P_{2n-1}(s) + [2\ P_{2n-2}(s) - P_{2n-1}(s)],\\ P_{2n+1}(s) &= 2s\ P_{2n-1}(s) - P_{2n-3}(s)\end{aligned}$$

and

$$\begin{aligned}P_{2n}(s) &= (s-1)\ [2\ P_{2n-2}(s) + P_{2n-3}(s)] + P_{2n-2}(s)\\ &= 2s\ P_{2n-2}(s) + [(s-1)\ P_{2n-3}(s) - P_{2n-2}(s)],\\ P_{2n}(s) &= 2s\ P_{2n-2}(s) - P_{2n-4}(s).\end{aligned}$$

Similarly, we obtain the followings:

$$Q_{2n+1}(s) = 2s\ Q_{2n-1}(s) - Q_{2n-3}(s)$$

and

$$Q_{2n}(s) = 2s\ Q_{2n-2}(s) - Q_{2n-4}(s).$$

Since

$$\begin{aligned}P_1(s) &= 1, P_3(s) = 2s-1,\\ P_{2n-1}(s) &= V_{n-1}(s); \qquad n = 1,2,3,\cdots\ ,\end{aligned}$$

$$\begin{aligned} Q_1(s) &= 1, Q_3(s) = 2s+1, \\ Q_{2n-1}(s) &= W_{n-1}(s); \quad n = 1, 2, 3, \cdots , \end{aligned}$$

$$\begin{aligned} P_2(s) &= s, P_4(s) = 2s^2 - 1, \\ P_{2n}(s) &= T_n(s); \quad n = 1, 2, 3, \cdots \end{aligned}$$

and

$$\begin{aligned} Q_2(s) &= (s-1), Q_4(s) = (s-1)2s, \\ Q_{2n}(s) &= (s-1)U_{n-1}(s); \quad n = 1, 2, 3, \cdots . \end{aligned}$$

The odd and even convergents of the continued fraction (12) are:

$$\begin{aligned} \frac{A_{2n}(x)}{A_{2n-1}(x)} = \frac{1}{2}\left[1 + \frac{W_{n-1}(s)}{V_{n-1}(s)}\right] &= \frac{U_{n-1}(s)}{V_{n-1}(s)} \\ &= \frac{x^{n-1}U_{n-1}\left(1+\frac{1}{2x}\right)}{x^{n-1}V_{n-1}\left(1+\frac{1}{2x}\right)} \end{aligned}$$

and

$$\begin{aligned} \frac{A_{2n+1}(x)}{A_{2n}(x)} = \frac{1}{2}\left[1 + \frac{T_n(s)}{(s-1)U_{n-1}(s)}\right] &= \frac{1}{2(s-1)}\left[\frac{(s-1)U_{n-1}(s) + T_n(s)}{U_{n-1}(s)}\right] \\ &= \frac{1}{2(s-1)}\,\frac{V_n(s)}{U_{n-1}(s)} \\ &= \frac{x^n V_n\left(1+\frac{1}{2x}\right)}{x^{n-1}U_{n-1}\left(1+\frac{1}{2x}\right)}. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} V_n\left(1+\frac{x}{2}\right) &= x^n\ A_{2n+1}\left(\frac{1}{x}\right), \\ U_n\left(1+\frac{x}{2}\right) &= x^n\ A_{2n+2}\left(\frac{1}{x}\right). \end{aligned}$$

Now, we obtain the odd and even convergents of the continued fraction (12) interms of second and third kind of Tchebychev polynomials

$$\frac{A_{2n-1}(x)}{A_{2n}(x)} = \frac{x^{n-1}V_{n-1}\left(1+\frac{1}{2x}\right)}{x^{n-1}U_{n-1}\left(1+\frac{1}{2x}\right)}$$

and

$$\frac{A_{2n}(x)}{A_{2n+1}(x)} = \frac{x^{n-1}U_{n-1}\left(1+\frac{1}{2x}\right)}{x^{n}V_{n}\left(1+\frac{1}{2x}\right)}.$$

(Similar results are derived in [7].)

Similarly the following continued fraction

$$\Phi_2(x) = \frac{-x+\sqrt{x^2+4}}{2} = \frac{1}{x}\left[\frac{-1+\sqrt{1+\frac{4}{x^2}}}{\frac{2}{x^2}}\right] = \frac{1}{x+}\,\frac{1}{x+}\cdots+\frac{1}{x+}\cdots, \qquad x>0 \tag{14}$$

has the following odd and even convergents:

$$\frac{B_{2n-1}(x)}{B_{2n}(x)} = \frac{1}{x}\,\frac{A_{2n-1}\left(\frac{1}{x^2}\right)}{A_{2n}\left(\frac{1}{x^2}\right)} = \frac{1}{x}\,\frac{(x^2)^{n-1}A_{2n-1}\left(\frac{1}{x^2}\right)}{(x^2)^{n-1}A_{2n}\left(\frac{1}{x^2}\right)} = \frac{1}{x}\,\frac{V_{n-1}\left(1+\frac{x^2}{2}\right)}{U_{n-1}\left(1+\frac{x^2}{2}\right)}$$

and

$$\frac{B_{2n}(x)}{B_{2n+1}(x)} = \frac{1}{x}\,\frac{A_{2n}\left(\frac{1}{x^2}\right)}{A_{2n+1}\left(\frac{1}{x^2}\right)} = \frac{1}{x}\,\frac{x^2(x^2)^{n-1}A_{2n}\left(\frac{1}{x^2}\right)}{(x^2)^{n}A_{2n+1}\left(\frac{1}{x^2}\right)} = x\,\frac{U_{n-1}\left(1+\frac{x^2}{2}\right)}{V_n\left(1+\frac{x^2}{2}\right)}.$$

Hence

$$A_{2n+1}(x) = x^nV_n\left(1+\frac{1}{2x}\right), \qquad A_{2n+2}(x) = x^nU_n\left(1+\frac{1}{2x}\right),$$
$$B_{2n+1}(x) = V_n\left(1+\frac{x^2}{2}\right), \qquad B_{2n+2}(x) = x\,U_n\left(1+\frac{x^2}{2}\right).$$

For $x=1$, we obtain

$$F_{2n+1} = A_{2n+1}(1) = B_{2n+1}(1) = V_n\left(\frac{3}{2}\right),$$
$$F_{2n+2} = A_{2n+2}(1) = B_{2n+2}(1) = U_n\left(\frac{3}{2}\right), \qquad n=0,1,2,3,\cdots.$$

§3. Connections Between Tchebychev Polynomials and Brahmagupta Polynomials of All Four Kinds

Brahmagupta polynomials have the following binet forms ([9]):

$$x_n(x,y;t) = \frac{1}{2}[(x+y\sqrt{t})^n + (x-y\sqrt{t})^n], \qquad n=0,1,2,3,\cdots$$

and

$$\frac{y_{n+1}(x,y;t)}{y} = \frac{1}{2y\sqrt{t}}[(x+y\sqrt{t})^{n+1} - (x-y\sqrt{t})^{n+1}], \qquad n=0,1,2,3,\cdots.$$

Put $\beta = x^2 - ty^2$ or $y\sqrt{t} = \sqrt{x^2-\beta}$, then we obtain

$$\begin{aligned}
x_n(x,y;t) &= \frac{1}{2}\,[(x+\sqrt{x^2-\beta})^n + (x-\sqrt{x^2-\beta})^n] \\
&= \frac{\beta^{\frac{n}{2}}}{2}\left[\left(\frac{x}{\sqrt{\beta}} + \sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2 - 1}\right)^n + \left(\frac{x}{\sqrt{\beta}} - \sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2 - 1}\right)^n\right] \\
&= \frac{\beta^{\frac{n}{2}}}{2} T_n\left(\frac{x}{\sqrt{\beta}}\right)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{y_{n+1}(x,y;t)}{y} &= \frac{1}{2\sqrt{x^2-\beta}}[(x+\sqrt{x^2-\beta})^{n+1} - (x-\sqrt{x^2-\beta})^{n+1}] \\
&= \frac{\beta^{\frac{n}{2}}}{2\sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2 - 1}}\left[\left(\frac{x}{\sqrt{\beta}} + \sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2 - 1}\right)^{n+1}\right. \\
&\qquad \left. - \left(\frac{x}{\sqrt{\beta}} - \sqrt{\left(\frac{x}{\sqrt{\beta}}\right)^2 - 1}\right)^{n+1}\right] \\
&= \beta^{\frac{n}{2}}\, U_n\left(\frac{x}{\sqrt{\beta}}\right).
\end{aligned}$$

Motivated by Tchebychev polynomials of third and forth kind, we can define Brahmagupta polynomials of third and forth kind respectively as follows:

$$\begin{aligned}
v_n(x,y;t) &= \frac{y_{n+1}(x,y;t)}{y} - \beta\,\frac{y_n(x,y;t)}{y}, \\
w_n(x,y;t) &= \frac{y_{n+1}(x,y;t)}{y} + \beta\,\frac{y_n(x,y;t)}{y}.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
v_0 &= w_0 = \frac{y_1}{y} = 1, \\
v_1 &= 2x - \beta, \quad w_1 = 2x + \beta, \\
v_{n+1}(x,y;t) &= 2x\; v_n(x,y;t) - \beta\; v_{n-1}(x,y;t), \\
w_{n+1}(x,y;t) &= 2x\; w_n(x,y;t) - \beta\; w_{n-1}(x,y;t).
\end{aligned}$$

Hence

$$\begin{aligned}
v_{n+1}(x,y;t) &= \beta^{\frac{n}{2}}\left[U_n\left(\frac{x}{\sqrt{\beta}}\right) - \sqrt{\beta}\,U_{n-1}\left(\frac{x}{\sqrt{\beta}}\right)\right], \\
w_{n+1}(x,y;t) &= \beta^{\frac{n}{2}}\left[U_n\left(\frac{x}{\sqrt{\beta}}\right) + \sqrt{\beta}\,U_{n-1}\left(\frac{x}{\sqrt{\beta}}\right)\right].
\end{aligned}$$

If $\beta = 1$, we get back

$$\begin{aligned} v_n\left(x, y;\ \frac{x^2-1}{y^2}\right) &= U_n(x) - U_{n-1}(x) = V_n(x), \\ w_n\left(x, y;\ \frac{x^2-1}{y^2}\right) &= U_n(x) + U_{n-1}(x) = W_n(x), \end{aligned}$$

which are the Tchebychev polynomials of third and forth kind respectively.

The following are generating functions of $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ ([1], [2], [4]):

(1) $T(s) = 2 \displaystyle\sum_{n=1}^{\infty} \frac{T_n(x)}{n} s^n;$

(2) $U(s) = \displaystyle\sum_{n=1}^{\infty} U_n(x) s^n = \frac{1}{1-2xs+s^2};$

(3) $V(s) = \displaystyle\sum_{n=0}^{\infty} V_n(x) s^n = (1-s)\ U(s);$

(4) $W(s) = \displaystyle\sum_{n=0}^{\infty} W_n(x) s^n = (1+s)\ U(s).$

It is shown that $U(s) = e^{T(s)}$ ([2]). One can extend the above results to $x_n(x, y; t)$, $\frac{y_{n+1}(x,y;t)}{y}$, $v_n(x, y; t)$ and $w_n(x, y; t)$ including the results in [9]:

(1) $X(s) = 2 \displaystyle\sum_{n=1}^{\infty} \frac{x_n(x, y; t)}{n} s^n;$

(2) $Y(s) = \displaystyle\sum_{n=1}^{\infty} \frac{y_{n+1}(x, y; t)}{y} s^n = \frac{1}{1-2xs+\beta\ s^2};$

(3) $\widetilde{V}(s) = \displaystyle\sum_{n=0}^{\infty} v_n(x, y; t) s^n = (1-\beta\ s)\ U(s);$

(4) $\widetilde{W}(s) = \displaystyle\sum_{n=0}^{\infty} w_n(x, y; t) s^n = (1+\beta\ s)\ U(s);$

(5) $Y(s) = e^{X(s)}.$

In this way, the present paper provides two spectacular views of Tchebychev polynomials of all four kinds through golden ratio and Brahmagupta polynomials.

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On the Quaternionic Normal Curves in the Semi-Euclidean Space E_2^4

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Abstract: In this paper, we define the semi-real quaternionic normal curves in four dimensional semi-Euclidean space E_2^4. We obtain some characterizations of semi-real quaternionic normal curves in terms of their curvature functions. Moreover, we give necessary and sufficient condition for a semi-real quaternionic curve to be a semi-real quaternionic normal curves in E_2^4.

Key Words: Normal curves, semi-real quaternion, semi-quaternionic curve, position vector.

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§1. Introduction

In mathematics, the quaternion were discovered by Irish mathematician S. W.R. Hamilton, in 1843, which are more general form of complex number [5]. He defined a quaternion as the quotient of two directed lines in a three-dimensional space. Also, quaternions can be written as sum of a scalar and a vector. A special feature of quaternions is that the product of two quaternions is noncommutative. Quaternions have an important role in diverse areas such as kinematics and mechanics. They provide us opportunity representation for describing finite rotation in space.

In [1], Serret–Frenet formulae for a quaternionic curves in E^3 and E^4 are given by Baharathi and Nagaraj. After them Coken and Tuna defined Serret–Frenet formulae for a quaternionic curves in semi-Euclidean space E_2^4 ([3]).

In analogy with the Euclidean case, Serret–Frenet formulae for a quaternionic curves in semi-Euclidean space E_2^4 is defined in [11]. Moreover, characterization of quaternionic B_2-slant helices in Euclidean space E^4 given in [3] and quaternionic mannheim curves are studied in semi Eucliden space E^4 in [9].

In the Euclidean Space E^3, normal curves defined as the curves whose position vector always lying in their normal plane [2]. Analogously, normal curves in other space are defined as the curves whose normal planes always contain a fixed point. As well, normal curves have same characterization with spherical curves which case has interesting corollaries for curve theory.

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Recently, İlarslan [6], has been studied some characterizations of spacelike normal curves in the Minkowski 3-space E_1^3. Also, Ilarslan and Nesovic [8] have been investigated spacelike and timelike normal curves in Minkowski space-time.

In this paper, we define the semi-real quaternionic normal curves in four dimensional semi-Euclidean E_2^4. We obtain some characterizations of semi-real quaternionic normal curves in terms of their curvature functions. Moreover, we give necessary and sufficient condition for a semi-real quaternionic curve to be a semi-real quaternionic normal curves in E_2^4.

§2. **Preliminary**

A brief summary of the theory of semi-real quaternions in the semi-Euclidean space and normal curves are presented in this section.

A pseudo-Riemannian manifold is a differentiable manifold equipped with pseudo-Riemannian metric which is nondegenerate, smooth, symmetric metric tensor. This metric tensor need not be positive definite. We denote the pseudo (semi)-Euclidean $(n+1)$-space by E_ν^{n+1}. If $\nu = 0$, E_ν^{n+1} semi-Euclidean spaces reduce to E^{n+1} Euclidean space, that is, semi-Euclidean space is a generalization of Euclidean space. For $\nu = 1$ and $n \geqslant 1$; E_1^{n+1} is called Lorentz–Minkowski $(n+1)$ space. The Lorentz manifold form the most important subclass of semi-Riemannian manifolds because of their physical application to the theory of relativity. Due to semi-Riemannian metric there are three different kind of curves, namely spacelike, timelike, lightlike (null) depending on the casual character of their tangent vectors, that is, the curve α is called a spacelike (resp. timelike and lightlike) if $\alpha'(t)$ is spacelike (resp. timelike and lightlike) for any $t \in I$.

A semi-real quaternion q is an expression of the form

$$q = ae_1 + be_2 + ce_3 + d \tag{1}$$

such that

$$\begin{cases} e_i \times e_i = -\varepsilon_{e_i}, \quad 1 \leq i \leq 3, \\ e_i \times e_j = \varepsilon_{e_i}\varepsilon_{e_j} e_k, \quad \text{in } E_1^3, \\ e_i \times e_j = -\varepsilon_{e_i}\varepsilon_{e_j} e_k, \quad \text{in } E_2^4, \end{cases} \tag{2}$$

where (ijk) is an even permutation of (123) and $a, b, c, d \in R$.

We can write quaternion as $q = S_q + V_q$ where $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$ denote scalar and vector part of q, respectively. For every $p, q \in Q_\nu$, the multiplication of two semi-real quaternions p and q is defined as

$$p \times q = S_p S_q + < V_p, V_q > + S_p V_q + S_q V_p + V_p \wedge V_q, \text{ for every } p, q \in Q_\nu, \tag{3}$$

where $<,>$ and $\wedge$ are scalar and cross product in E_1^3, respectively. The conjugate of the semi-real quaternion q is denoted by γq and defined $\gamma q = S_q - V_q = d - ae_1 - be_2 - ce_3$. This helps to define the symmetric, non-degenerate, bilinear form h as follows.

$$h : Q_\nu \times Q_\nu \to R,$$

$$h(p,q)=\frac{1}{2}\left[\varepsilon_p\varepsilon_{\gamma q}\left(p\times\gamma q\right)+\varepsilon_q\varepsilon_{\gamma p}\left(q\times\gamma p\right)\right]\quad\text{for } E_1^3 \tag{4}$$

$$h(p,q)=\frac{1}{2}\left[\varepsilon_p\varepsilon_{\gamma q}\left(p\times\gamma q\right)+\varepsilon_q\varepsilon_{\gamma p}\left(q\times\gamma p\right)\right]\quad\text{for } E_2^4,$$

the norm of semi-real quaternion $q\in Q_\nu$ is

$$\|q\|^2=-a^2-b^2+c^2+d^2$$

q is called a semi-real spatial quaternion whenever $q\times\gamma q=0$. For $p,q\in Q_\nu$ where if $h(p,q)=0$ then p and q are called h-orthogonal [11]. If $\|q\|^2=1$, the q is called a semi real unit quaternion.

Recall that the pseudosphere, the pseudohyperbolic space and the lightcone are hyperquadrics in E_2^4, respectively defined by

$$\begin{aligned} S_1^3(m,r) &= \left\{x\in E_2^4 : h(x-m,x-m)=r^2\right\}\\ H_0^3(m,r) &= \left\{x\in E_2^4 : h(x-m,x-m)=-r^2\right\}\\ C_3^3(m,r) &= \left\{x\in E_2^4 : h(x-m,x-m)=0\right\}\end{aligned}$$

where $r>0$ is the radius and $m\in E_2^4$ is the center of hyperquadric.

In the Euclidean space E^3, it is well-known that to each unit speed curve $\alpha: I\subset R\to E^3$ with at least four continuous derivatives has Frenet frame $\{t,n,b\}$. At each point of the curve which is spanned by $\{t,n\}$, $\{t,b\}$ and $\{n,b\}$ are known as the osculating plane, the rectifying plane and the normal plane, respectively. Rectifying curve is introduced by B.Y.Chen, whose position vector always lies its rectifying plane $\{t,b\}$ ([2]). Similarly, a curve is called a osculating curve if its position vector always lies its osculating plane $\{t,n\}$. İlarslan and Nesovic defined normal curve as

$$\alpha(s)=\lambda(s)n(s)+\mu(s)b(s),$$

where λ and μ are arbitrary differentiable functions in terms of the arc length parameter s ([7]). This means that normal curve's position vector always lies its normal plane $\{n,b\}$.

Analogously, in E^4 the normal curve defined by İlarslan whose position vector always lies in orthogonal complement $T^\perp$ of its tangent vector field of the curve. The position vector of a normal curve α in E^4, satisfies the equation

$$\alpha(s)=\lambda(s)N(s)+\mu(s)B_1(s)+\nu(s)B_2(s),$$

where λ, μ and ν are arbitrary differentiable functions in terms of the arc length parameter s, respectively ([8]).

§3. Some Characterization of Quaternionic Normal Curves in Semi Euclidean Space

In this section, the four-dimensional Euclidean space E_2^4 is identified with the space of unit semi-real quaternion. Let

$$\beta : I \subset R \longrightarrow Q, \quad \beta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1 \tag{5}$$

be a smooth curve β in E_2^4 defined over the interval I. Let the parameter s be chosen such that the tangent $T = \beta'(s) = \sum_{i=1}^{4} \gamma_i'(s) e_i$ has unit magnitude. Let $\{T, N, B_1, B_2\}$ be the Frenet apparatus of the differentiable Euclidean space curve in the Euclidean space E_2^4. Then the Frenet equations are

$$\begin{cases} T'(s) = \varepsilon_N K N(s) \\ N'(s) = -\varepsilon_t \varepsilon_N K T(s) + \varepsilon_n k B_1(s) \\ B_1'(s) = -\varepsilon_t k N(s) + \varepsilon_n (r - \varepsilon_t \varepsilon_T \varepsilon_N K) B_2(s) \\ B_2'(s) = -\varepsilon_b (r - \varepsilon_t \varepsilon_T \varepsilon_N K) B_1, \end{cases} \tag{6}$$

where $T(s)$ is the tangent vector of the curve β and $K = \varepsilon_N \|T'(s)\|$ ([3]).

It is obtained the Frenet formulae in [1] and the apparatus for the curve β by making use of the Frenet formulae for a curve γ in R^3. Moreover, there are relationships between curvatures of the curves β and γ. These relations can be explained that the torsion of β is the principal curvature of the curve γ. Also, the bitorsion of β is $(r - \varepsilon_t \varepsilon_T \varepsilon_N K)$, where r is the torsion of γ and K is the principal curvature of β. These relations are only determined for quaternions, [1].

In this section, we characterize the semi-real quaternionic normal curves with the third curvature $(r - \varepsilon_t \varepsilon_T \varepsilon_N K) \neq 0$ for each s.

Let $\beta = \beta(s)$ be a unit speed semi-real quaternionic normal curve, lying fully in Q_ν. Then its position vector satisfies

$$\beta(s) = \lambda(s) N(s) + \mu(s) B_1(s) + \nu(s) B_2(s) \tag{7}$$

By taking the derivative of (7) with respect to s and using the Frenet equations (6), we obtain

$$T = -\varepsilon_t \varepsilon_N K \lambda T + (\lambda' - \varepsilon_t k \mu) N + (\varepsilon_n k \lambda + \mu' - \varepsilon_b (r - \varepsilon_t \varepsilon_T \varepsilon_N K) \upsilon) B_1 + (\varepsilon_n (r - \varepsilon_t \varepsilon_T \varepsilon_N K) \mu + \upsilon') B_2$$

and therefore

$$\begin{cases} -\varepsilon_t \varepsilon_N K \lambda = 1\ , = 1, \\ \lambda' - \varepsilon_t k \mu = 0, \\ \varepsilon_n k \lambda + \mu' - \varepsilon_b (r - \varepsilon_t \varepsilon_T \varepsilon_N K) \upsilon = 0, \\ \varepsilon_n (r - \varepsilon_t \varepsilon_T \varepsilon_N K) \mu + \upsilon' = 0. \end{cases} \tag{8}$$

From the first three equations we find

$$\begin{cases} \lambda(s) = -\dfrac{\varepsilon_t\varepsilon_N}{K(s)}\ , \quad \mu(s) = -\dfrac{\varepsilon_N}{k(s)}\left(\dfrac{1}{K(s)}\right)', \\ \upsilon(s) = -\dfrac{\varepsilon_b\varepsilon_N}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\varepsilon_n\dfrac{k(s)}{K(s)} + \left(\dfrac{1}{k(s)}\left(\dfrac{1}{K(s)}\right)'\right)'\right] \end{cases} \tag{9}$$

Substituting relation (9) into (7), we get that the position vector of the semi-real quaternionic normal curve β is given by

$$\begin{aligned} \beta(s) &= -\frac{\varepsilon_t\varepsilon_N}{K(s)}N - \frac{\varepsilon_N}{k(s)}\left(\frac{1}{K(s)}\right)' B_1 \\ &\quad -\frac{\varepsilon_b\varepsilon_N}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\varepsilon_n\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right] B_2 \end{aligned} \tag{10}$$

Then we have the following theorem.

Theorem 3.1 *Let $\beta(s)$ be a unit speed semi-real quaternionic curve, lying fully in Q_ν. Then $\beta(s)$ is a semi-real quaternionic normal curve if and only if*

$$-\frac{(r-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}{k(s)}\left(\frac{1}{K(s)}\right)' = \left[\frac{\varepsilon_n\varepsilon_b}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\begin{array}{c} \varepsilon_t\varepsilon_n\frac{k(s)}{K(s)} + \\ \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)' \end{array}\right]\right]'. \tag{11}$$

Proof Let us first assume that $\beta(s)$ is a semi-real quaternionic normal curve. Then relations (8) and (9) imply that (11) holds.

Conversely, assume that relation (11) holds. Let us consider the vector $m \in Q_\nu$ given by

$$\begin{aligned} m(s) &= \beta(s) + \frac{\varepsilon_t\varepsilon_N}{K(s)}N + \frac{\varepsilon_N}{k(s)}\left(\frac{1}{K(s)}\right)' B_1 \\ &\quad +\frac{\varepsilon_b\varepsilon_N}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\varepsilon_n\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right] B_2. \end{aligned} \tag{12}$$

Differentiating (12) with respect to s and by applying (6), we get

$$\begin{aligned} m'(s) &= \frac{\varepsilon_n\varepsilon_N(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}{k(s)}\left(\frac{1}{K(s)}\right)' B_2 \\ &\quad + \left(\frac{\varepsilon_b\varepsilon_N}{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\varepsilon_n\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right]\right)' B_2. \end{aligned}$$

From relation (11) it follows that m is a constant vector, which means that β is congruent to a semi-real quaternionic normal curve. □

Theorem 3.2 *Let $\beta(s)$ be a unit speed semi-real quaternionic curve, lying fully in Q_ν. If β is a semi-real quaternionic normal curve, then the following statements hold:*

(i) the principal normal and the first binormal component of the position vector β are respectively given by

$$\begin{cases} h(\beta, N) = -\dfrac{\varepsilon_t}{K(s)}, \\ h(\beta, B_1) = -\dfrac{\varepsilon_n \varepsilon_T \varepsilon_N}{k(s)} \left(\dfrac{1}{K(s)}\right)'. \end{cases} \tag{13}$$

(ii) the first binormal and the second binormal component of the position vector β are respectively given by

$$\begin{cases} h(\beta, B_1) = -\dfrac{\varepsilon_n \varepsilon_T \varepsilon_N}{k(s)} \left(\dfrac{1}{K(s)}\right)', \\ h(\beta, B_2) = -\dfrac{\varepsilon_T \varepsilon_N}{(r(s) - \varepsilon_t \varepsilon_T \varepsilon_N K(s))} \left[\varepsilon_t \varepsilon_n \dfrac{k(s)}{K(s)} + \left(\dfrac{1}{k(s)} \left(\dfrac{1}{K(s)}\right)'\right)'\right]. \end{cases} \tag{14}$$

Conversely, if $\beta(s)$ is a unit speed semi-real quaternionic curve, lying fully in Q_ν, and one of statements (i) or (ii) holds, then β is a normal curve.

Proof If $\beta(s)$ is a semi-real quaternionic normal curve, it is easy to check that relation (10) implies statements (i) and (ii).

Conversely, if statement (i) holds, differentiating equation $h(\beta, N) = -\frac{\varepsilon_t}{K(s)}$ with respect to s and by applying (6), we find $h(\beta, T) = 0$ which means that β is a semi-real quaternionic normal curve. If statement (ii) holds, in a similar way we conclude that β is a semi-real quaternionic normal curve. $\square$

In the next theorem, we obtain interesting geometric characterization of semi-real quaternionic normal curves.

Theorem 3.3 *Let $\beta(s)$ be a unit speed semi-real quaternionic curve, lying fully in Q_ν. Then β is a semi-real quaternionic normal curve if and only if β lies in some hyperquadrics in Q_ν.*

Proof First assume that β is a semi-real quaternionic normal curve. It follows, by straightforward calculations using Theorem 3.1, we get

$$\begin{aligned} &2\frac{\varepsilon_N}{K}\left(\frac{1}{K}\right)' + 2\frac{\varepsilon_n \varepsilon_T}{k}\left(\frac{1}{K}\right)'\left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)' \\ &+2\frac{\varepsilon_b \varepsilon_T}{(r - \varepsilon_t \varepsilon_T \varepsilon_N K)}\left[\varepsilon_t \varepsilon_n \frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\left(\frac{1}{(r - \varepsilon_t \varepsilon_T \varepsilon_N K)}\left[\varepsilon_t \varepsilon_n \frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\right)' = 0. \end{aligned} \tag{15}$$

On the other hand, the previous equation is differential of the equation

$$\varepsilon_N \left(\frac{1}{K}\right)^2 + \varepsilon_n \varepsilon_T \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)^2 + \left(\frac{1}{(r - \varepsilon_t \varepsilon_T \varepsilon_N K)}\left[\varepsilon_t \varepsilon_n \frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\right)^2 = r, \quad r \in R. \tag{16}$$

By using (12), it is easy to check that

$$h(\beta - m, \beta - m) = \left(\frac{1}{K}\right)^2 + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)^2 + \left(\frac{1}{(r - \varepsilon_t \varepsilon_T \varepsilon_N K)}\left[\varepsilon_t \varepsilon_n \frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\right)^2, \tag{17}$$

which together with (16) gives $h(\beta-m,\beta-m)=r$. Consequently, β lies in some hypersphere in Q_ν.

Conversely, if β lies in some hyperquadric in Q_ν, then

$$h(\beta-m,\beta-m)=r,\ r\in R, \tag{18}$$

where $m\in Q_\nu$ is a constant vector. By taking the derivative of the previous equation with respect to s, we easily obtain $h(\beta-m,T)=0$ which proves the theorem. □

Recall that arbitrary curve β in Q_ν is called a W-curve (or a helix), if it has constant curvature functions ([10]). The following theorem gives the characterization of semi-real quaternionic W-curve in Q_ν, in terms of semi-real quaternionic normal curves.

Theorem 3.4 *Every unit speed semi-real quaternionic W-curve, lying fully in Q_ν, is to a semi-real quaternionic normal curve.*

Proof By assumption we have $K(s)=c_1$, $k(s)=c_2$, $(r-\varepsilon_t\varepsilon_T\varepsilon_N K)(s)=c_3$, where $c_1,c_2,c_3\in R-\{0\}$. Since the curvature functions obviously satisfy relation (11), according to Theorem 3.1, β is a semi-real quaternionic normal curve. □

Lemma 3.1 *A unit speed semi-real quaternionic $\beta(s)$, lying fully in Q_ν, is a semi-real quaternionic normal curve if and only if there exists a differentiable function $f(s)$ such that*

$$\begin{cases} f(s)(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s)) = \varepsilon_t\varepsilon_n \dfrac{k(s)}{K(s)} + \left(\dfrac{1}{k(s)}\left(\dfrac{1}{K(s)}\right)'\right)', \\ f'(s) = -\varepsilon_n\varepsilon_b \dfrac{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}{k(s)}\left(\dfrac{1}{K(s)}\right)'. \end{cases} \tag{19}$$

By using the similar methods as in [8], as well as Lemma 3.1, we obtain the following theorem which give the necessary and the sufficient conditions for semi-real quaternionic curves in Q_ν to be the semi-real quaternionic normal curves.

Theorem 3.5 *Let $\beta(s)$ be a unit speed semi-real quaternionic curve in Q_ν whose Frenet formulas obtained from spacelike semi-real spatial quaternionic curve with spacelike principal normal n. Then β is a semi-real quaternionic normal curve if and only if there exist constants a_0, $b_0\in R$ such that*

$$\begin{aligned} \frac{1}{k(s)}\left(\frac{1}{K(s)}\right)' &= \left(a_0+\varepsilon_t\int\frac{k(s)}{K(s)}\cos\theta(s)ds\right)\cos\theta(s) \\ &\quad+\left(b_0+\varepsilon_t\int\frac{k(s)}{K(s)}\sin\theta(s)ds\right)\sin\theta(s), \end{aligned} \tag{20}$$

where $\theta(s)=\int\limits_0^s(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))\ ds$.

Proof If $\beta(s)$ is a semi-real quaternionic normal curve, according to Lemma 3.1 there exists a differentiable function $f(s)$ such that relation (19) holds, whereby $\varepsilon_b = -1$. Let us define differentiable functions $\theta(s)$, $a(s)$ and $b(s)$ by

$$\begin{cases} \theta(s) = \int\limits_0^s (r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s))ds \\ a(s) = -\dfrac{1}{k(s)}\left(\dfrac{1}{K(s)}\right)' \cosh\theta(s) + f(s)\sinh\theta(s) - \varepsilon_t \displaystyle\int \dfrac{k(s)}{K(s)}\cosh\theta(s)ds \\ b(s) = -\dfrac{1}{k(s)}\left(\dfrac{1}{K(s)}\right)' \sinh\theta(s) - f(s)\cosh\theta(s) - \varepsilon_t \displaystyle\int \dfrac{k(s)}{K(s)}\sinh\theta(s)ds = \end{cases} \tag{21}$$

By using (19), we easily find $\theta'(s) = (r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s))$, $a'(s) = 0$, $b'(s) = 0$ and thus

$$a(s) = a_0\ ,\ \ b(s) = b_0\ ,\ \ a_0, b_0 \in R. \tag{22}$$

Multiplying the second and the third equations in (21), respectively with $\cosh\theta(s)$ and $-\sinh\theta(s)$, adding the obtained equations and using (22), we conclude that relation (20) holds.

Conversely, assume that there exist constants a_0, $b_0 \in R$ such that relation (20) holds. By taking the derivative of (20) with respect to s, we find

$$-\varepsilon_t\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)' = (r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s))\begin{bmatrix} \left(a_0 + \varepsilon_t\int \frac{k(s)}{K(s)}\cosh\theta(s)ds\right)\sinh\theta(s) \\ -\left(b_0 + \varepsilon_t\int \frac{k(s)}{K(s)}\sinh\theta(s)ds\right)\cosh\theta(s) \end{bmatrix}. \tag{23}$$

Let us define the differentiable function $f(s)$ by

$$f(s) = \frac{1}{(r(s) - \varepsilon_t\varepsilon_T\varepsilon_N K(s))}\left[\varepsilon_t\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right] \tag{24}$$

Next, relations (23) and (24) imply

$$f(s) = \left(a_0 + \varepsilon_t\int \frac{k(s)}{K(s)}\cosh\theta(s)ds\right)\sinh\theta(s) - \left(b_0 + \varepsilon_t\int \frac{k(s)}{K(s)}\sinh\theta(s)ds\right)\cosh\theta(s)$$

By using this and (20), we obtain $f'(s) = \frac{(r(s)-\varepsilon_t\varepsilon_T\varepsilon_N K(s))}{k(s)}\left(\frac{1}{K(s)}\right)'$. Finally, Lemma 3.1 implies that β is congruent to a semi-real quaternionic normal curve. □

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Global Equitable Domination Number of Some Wheel Related Graphs

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Abstract: A dominating set is called a global dominating set if it is a dominating set of a graph G and its complement $\overline{G}$. A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V(G) - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$. An equitable dominating set D of a graph G is a global equitable dominating set if it is also an equitable dominating set of the complement of G. The global equitable domination number $\gamma_g^e(G)$ of G is the minimum cardinality of a global equitable dominating set of G. In this paper, we investigate the global equitable domination number of some wheel related graphs.

Key Words: Global dominating set, equitable dominating set, Smarandachely equitable dominating set, global equitable dominating set, global equitable domination number.

AMS(2010): 05C69, 05C76.

§1. Introduction

The study of domination in graphs is one of the fastest growing areas within graph theory. An excellent survey on the concept of domination and its related parameters can be found in the book by Haynes *et al.* [4] while some advanced topics on domination are explored in Haynes *et al.* [5]. The concept of domination has interesting applications in the study of social networks which motivated Prof. E. Sampathkumar to introduce the concept of equitable domination in graphs.

Secondly, let G be a graph of road network linking various locations. It is desirable to maintain the supply to these locations uninterruptedly by using the alternative links even if the original links get disturbed. Then the problem of finding the minimum number of supplying stations needed to accomplish this task is equivalent to find the global domination number. The concept of global domination was introduced by Sampathkumar [9].

Many domination models are introduced by combining two different domination parameters. Independent domination, global domination, equitable domination, connected domination are among worth to mention. Motivated through the concepts of global domination and equitable domination, a new concept of global equitable domination was conceived by Basavanagoud

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and Teli [2] and formalized by Vaidya and Pandit [14]. In the present paper, we obtain the global equitable domination number of some wheel related graphs.

Throughout the paper, a graph $G = (V(G), E(G))$ we mean a finite and undirected graph without loops and multiple edges. The set $D \subseteq V(G)$ of vertices in a graph G is called a dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. The minimum cardinality of a dominating set of G is called the domination number of G which is denoted by $\gamma(G)$.

The complement $\overline{G}$ of G is the graph with vertex set $V(G)$ in which two vertices are adjacent in $\overline{G}$ if they are not adjacent in G.

For a vertex $v \in V(G)$, the open neighborhood of v, denoted by $N(v)$, is $\{u \in V(G) : uv \in E(G)\}$. We denote the degree of a vertex v in G by $d_G(v)$. A vertex of degree one is called a pendant vertex and a vertex which is not the end of any edge is called an isolated vertex. An edge e of a graph G is said to be incident with the vertex v if v is an end vertex of e. An edge incident with a pendant vertex is called a pendant edge.

A set $D \subseteq V(G)$ is called a global dominating set of G if D is a dominating set of both G and $\overline{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set in G. Many researchers have explored this concept. For example, Gangadharappa and Desai [3] have discussed the global domination in graphs of small diameters. Vaidya and Pandit [12, 13] have investigated the global domination number of the larger graphs obtained by some graph operations on a given graph while Kulli and Janakiram [6] have introduced the concept of total global dominating sets.

A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V(G) - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$, otherwise, a Smarandachely equitable dominating set, i.e., $|d_G(u) - d_G(v)| \geq 2$ for each edge $uv \in E(G)$ with $u \in D$ and $v \in V(G) - D$. The minimum cardinality of such a dominating set is called the equitable domination number of G which is denoted by $\gamma^e(G)$. Swaminathan and Dharmalingam [11] have studied the equitable domination in graphs and characterized the minimal equitable dominating sets. Sivakumar *et al.* [10] have discussed the connected equitable domination in graphs while Murugan and Emmanuel [7] have identified the inter relationship among domination, equitable domination and independent domination in graphs. Revathi and Harinarayanan [8] have studied the equitable domination in fuzzy graphs while Basavanagoud *et al.* [1] have studied the equitable total domination in graphs.

A vertex $v \in V(G)$ is equitably adjacent with a vertex $u \in V(G)$ if $|d_G(u) - d_G(v)| \leq 1$ and $uv \in E(G)$. A vertex $u \in V(G)$ is called an equitable isolate if $|d_G(u) - d_G(v)| \geq 2$ for all $v \in N(u)$. Analogous to the characteristic of an isolated vertex in a dominating set, an equitable isolate must belong to any equitable dominating set of G. Clearly, the isolated vertices are the equitable isolates. Hence, $I_s \subseteq I_e \subseteq D$ for every equitable dominating set D where I_s and I_e denote the sets of all isolated vertices and all equitable isolates of G respectively.

A subset D of $V(G)$ is called a global equitable dominating set of G if D is an equitable dominating set of both G and $\overline{G}$. The minimum cardinality of a global equitable dominating set of G is called the global equitable domination number of G and it is denoted by $\gamma_g^e(G)$.

Since at least two vertices are required to equitably dominate both G and $\overline{G}$, we have

$2 \leq \gamma_g^e(G) \leq n$ for every graph of order $n > 1$. Both of these bounds are sharp. In particular, the equality of the lower bound is attained by P_n $(2 \leq n \leq 6)$ and $K_{r,s}$ $(\,|r-s| \leq 1\,)$ while the upper bound is achieved by K_n, $K_{1,p}$ and $K_{r,s}$ $(\,|r-s| \geq 2\,)$.

The wheel W_n is defined to be the join $C_{n-1} + K_1$ where $n \geq 4$. The vertex corresponding to K_1 is known as the apex vertex and the vertices corresponding to cycle C_{n-1} are known as the rim vertices. For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n.

For notations and graph theoretic terminology not defined herein, we refer the readers to West [15] while the terms related to the concept of domination are used in the sense of Haynes *et al.* [4].

§2. Main Results

Definition 2.1 *The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each of its rim vertices.*

Proposition 2.2 ([2])

(*i*) *For the path P_n $(n \geq 4)$, $\gamma_g^e(P_n) = \lceil \frac{n}{3} \rceil$;*

(*ii*) *For the cycle C_n,* $\gamma_g^e(C_n) = \begin{cases} 3 & \text{if } n = 3, 5 \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$

Theorem 2.3 *For the helm,* $\gamma_g^e(H_n) = \begin{cases} 7 & \text{if } n = 4 \\ n+2 & \text{if } n = 5, 6 \\ \lceil \frac{4n-1}{3} \rceil & \text{otherwise.} \end{cases}$

Proof Let $v_1, v_2, \cdots, v_{n-1}$ be the rim vertices of wheel W_n and let c denotes the apex vertex of the helm H_n. Let $u_1, u_2, \ldots, u_{n-1}$ be the pendant vertices of H_n. Then, $|V(H_n)| = 2n-1$ and $|E(H_n)| = 3(n-1)$.

Case 1. $n = 4$

For $n = 4$, the pendant vertices of H_n are equitable isolates in H_n while the remaining vertices of H_n are equitable isolates in $\overline{H_n}$. Hence, the vertex set of H_n is the only global equitable dominating set of H_n implying that $\gamma_g^e(H_n) = |V(H_n)| = 7$.

Case 2. $n = 5, 6$

Since the $n-1$ pendant vertices of H_n are equitable isolates in H_n and the apex vertex c is an equitable isolate in $\overline{H_n}$, it follows that every global equitable dominating set of H_n must contain these vertices. Now, these vertices equitably dominate all the vertices of H_n but do not equitably dominate all the vertices of $\overline{H_n}$. Moreover, any two adjacent rim vertices of W_n can equitably dominate the remaining vertices of $\overline{H_n}$. Hence, every global equitable dominating set of H_n must contain at least $n+2$ vertices of H_n. Therefore, $\gamma_g^e(H_n) = n+2$.

Case 3. $n \geq 7$

In this case, the apex vertex is an equitable isolate in H_n as well as in $\overline{H_n}$ while the $n-1$ pendant vertices of H_n are equitable isolates in H_n only. Therefore, these vertices must belong to every global equitable dominating set of H_n. Now, the remaining vertices induce a cycle C_{n-1} and by Proposition 2.2, $\gamma_g^e(C_{n-1}) = \lceil \frac{n-1}{3} \rceil$. Therefore, $\gamma_g^e(H_n) = n-1+1+\lceil \frac{n-1}{3} \rceil = \lceil \frac{4n-1}{3} \rceil$.

Thus, we have proved that

$$\gamma_g^e(H_n) = \begin{cases} 7 & \text{if } n=4 \\ n+2 & \text{if } n=5,6 \\ \lceil \frac{4n-1}{3} \rceil & \text{otherwise.} \end{cases}$$

□

Definition 2.4 *The flower graph Fl_n is the graph obtained from the helm H_n by joining each pendant vertex to the apex vertex of the helm H_n.*

Theorem 2.5 *For the flower graph,*

$$\gamma_g^e(Fl_n) = \begin{cases} n+3 & \text{if } n=4,6 \\ \lceil \frac{4n-1}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof Let $v_1, v_2, \cdots, v_{n-1}$ be the rim vertices of wheel W_n and let $u_1, u_2, \cdots, u_{n-1}$ be the pendant vertices of the helm H_n. Let c denotes the apex vertex of Fl_n. Then $|V(Fl_n)| = 2n-1$. Here, $d_G(v_i) = 4$, $d_G(u_i) = 2$ for $1 \leq i \leq n-1$ and $d_G(c) = 2(n-1)$ where $G = Fl_n$.

Now, the vertex c is an equitable isolate in G as well as in $\overline{G}$. Therefore, every global equitable dominating set of G must contain c. Moreover, the vertices $u_1, u_2, \ldots, u_{n-1}$ being the equitable isolates in G, must belong to every global equitable dominating set of G. Now, the remaining vertices $v_1, v_2, \ldots, v_{n-1}$ in G induce a cycle C_{n-1} and by Proposition 2.2,

$$\gamma_g^e(C_n) = \begin{cases} 3 & \text{if } n=3,5 \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \gamma_g^e(Fl_n) &= \gamma_g^e(C_{n-1}) + (n-1) + 1 \\ &= \gamma_g^e(C_{n-1}) + n \\ &= \begin{cases} n+3 & \text{if } n=4,6 \\ \lceil \frac{n-1}{3} \rceil + n & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$\gamma_g^e(Fl_n) = \begin{cases} n+3 & \text{if } n=4,6 \\ \lceil \frac{4n-1}{3} \rceil & \text{otherwise.} \end{cases}$$

□

Definition 2.6 *The sunflower graph Sf_n is the resultant graph obtained from the flower graph by attaching $(n-1)$ pendant edges to the apex vertex of wheel W_n.*

Theorem 2.7 *For the sunflower graph, $\gamma_g^e(Sf_n) = 3n-2$.*

Proof Let c denotes the apex vertex of wheel W_n and let $v_1, v_2, \cdots, v_{n-1}$ be the rim vertices of W_n. Let $u_1, u_2, \cdots, u_{n-1}$ be the vertices of degree 2 in Sf_n and let $x_1, x_2, \ldots, x_{n-1}$ be the pendant vertices of Sf_n. Then, $|V(Sf_n)| = 3n-2$.

Now, c is the equitable isolate in both Sf_n and $\overline{Sf_n}$. Moreover, the vertices $u_1, u_2, \cdots, u_{n-1}$, $x_1, x_2, \cdots, x_{n-1}$ are equitable isolates in Sf_n while the vertices $v_1, v_2, \cdots, v_{n-1}$ are equitable isolates in $\overline{Sf_n}$. Since an equitable isolate must belong to every equitable dominating set, it follows that the vertex set $V(Sf_n)$ is the only global equitable dominating set of Sf_n. Therefore, $\gamma_g^e(Sf_n) = |V(Sf_n)| = 3n-2$. □

Definition 2.8 *The closed helm CH_n is the graph obtained from a helm by joining each pendant vertex to form a cycle.*

Theorem 2.9 *For the closed helm CH_n $(n > 5)$,*

$$\gamma_g^e(CH_n) = \begin{cases} \left\lfloor \frac{n+2}{2} \right\rfloor & \text{if n} \equiv 1 \,(\text{mod}\, 4) \\ \left\lceil \frac{n+2}{2} \right\rceil & \text{if n} \equiv 0,\, 2 \text{ } or \text{ } 3 \text{ } (\text{mod}\, 4). \end{cases}$$

Proof Let $v_1, v_2, \cdots, v_{n-1}$ be the vertices of degree 4 and let $u_1, u_2, \cdots, u_{n-1}$ be the vertices of degree 3 in $G = CH_n$. Let c denotes the apex vertex of CH_n. Then the closed helm CH_n has $2n-1$ vertices. The vertex c is an equitable isolate in CH_n and the remaining vertices which are adjacent in CH_n, are also equitably adjacent in CH_n. The vertex c being an equitable isolate, must belong to every global equitable dominating set of CH_n. Hence, we construct a vertex set $D \subset V(CH_n)$ as follows:

$$D = \{c, v_{4i+1}, u_{4j+3}\},$$

where $0 \le i \le \left\lfloor \frac{n-2}{4} \right\rfloor$ and $0 \le j \le \left\lfloor \frac{n}{4} \right\rfloor$ with

$$|D| = \begin{cases} \left\lfloor \frac{n+2}{2} \right\rfloor & \text{if n} \equiv 1 \,(\text{mod}\, 4) \\ \left\lceil \frac{n+2}{2} \right\rceil & \text{if n} \equiv 0,\, 2 \text{ } or \text{ } 3 \text{ } (\text{mod}\, 4). \end{cases}$$

Now, $d_G(v_{4i+1}) = 4$, $d_G(u_{4j+3}) = 3$ and $d_{\overline{G}}(u_{4j+3}) - d_{\overline{G}}(v_{4j+1}) = 1$. Then for every $v \in V(G) - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \le 1$. Moreover, for every $v' \in V(G) - D$ there exists a vertex $u' \in D$ such that $u'v' \in E(\overline{G})$ and $|d_{\overline{G}}(u') - d_{\overline{G}}(v')| \le 1$. Hence, the set D is an equitable dominating set of G as well as of $\overline{G}$. Therefore, D is a global equitable dominating set of G. Moreover, from the adjacency nature of the vertices of G, one can observe that the set D is of minimum cardinality.

Thus, the set D is a global equitable dominating set of $G = CH_n$ $(n > 5)$ with minimum cardinality implying that

$$\gamma_g^e(CH_n) = \begin{cases} \lfloor \frac{n+2}{2} \rfloor & \text{if n} \equiv 1 \,(\text{mod}\, 4) \\ \lceil \frac{n+2}{2} \rceil & \text{if n} \equiv 0,\, 2 \; or \, 3 \; (\text{mod}\, 4). \end{cases}$$

□

Remark 2.10 For $n = 4, 5$, at least two vertices are required to equitably dominate all the vertices of CH_n as well as of $\overline{CH_n}$. Therefore, $\gamma_g^e(CH_n) = 2$ for $n = 4, 5$.

Definition 2.11 *A web graph is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. We denote the web graph by Wb_n.*

Theorem 2.12 *For the web graph Wb_n $(n > 6)$,*

$$\gamma_g^e(Wb_n) = \begin{cases} \lfloor \frac{3n}{2} \rfloor & \text{if n} \equiv 1 \,(\text{mod}\, 4) \\ \lceil \frac{3n}{2} \rceil & \text{if n} \equiv 0,\, 2 \; or \, 3 \; (\text{mod}\, 4). \end{cases}$$

Proof Let c denotes the apex vertex of web graph Wb_n. Let $v_1, v_2, \cdots, v_{n-1}$ and $u_1, u_2, \cdots, u_{n-1}$ be the vertices of inner cycle and outer cycle of Wb_n respectively. Let $x_1, x_2, \cdots, x_{n-1}$ denote the pendant vertices of Wb_n.

Since the apex vertex c and the $n-1$ pendant vertices are the equitable isolates in Wb_n as well as in $\overline{Wb_n}$ for $n > 6$, it follows that these vertices must belong to every global equitable dominating set of Wb_n. Moreover, the vertices except the pendant vertices induce the closed helm CH_n and by Theorem 2.9, we have

$$\gamma_g^e(CH_n) = \begin{cases} \lfloor \frac{n+2}{2} \rfloor & \text{if n} \equiv 1 \,(\text{mod}\, 4) \\ \lceil \frac{n+2}{2} \rceil & \text{if n} \equiv 0,\, 2 \; or \, 3 \; (\text{mod}\, 4). \end{cases}$$

Hence,

$$\gamma_g^e(Wb_n) = \gamma_g^e(CH_n) + (n-1)$$

$$= \begin{cases} \lfloor \frac{n+2}{2} \rfloor + (n-1) & \text{if n} \equiv 1 \,(\text{mod}\, 4) \\ \lceil \frac{n+2}{2} \rceil + (n-1) & \text{if n} \equiv 0,\, 2 \; or \, 3 \; (\text{mod}\, 4). \end{cases}$$

Thus, for $n > 6$,

$$\gamma_g^e(Wb_n) = \begin{cases} \lfloor \frac{3n}{2} \rfloor & \text{if n} \equiv 1 \,(\text{mod}\, 4) \\ \lceil \frac{3n}{2} \rceil & \text{if n} \equiv 0,\, 2 \; or \, 3 \; (\text{mod}\, 4). \end{cases}$$

□

Remark 2.13 (i) For $n = 4, 5$, the $n-1$ pendant vertices are the equitable isolates in Wb_n

as well as in $\overline{Wb_n}$ and the remaining vertices induce the closed helm CH_n. Thus, $\gamma_g^e(Wb_n) = \gamma_g^e(CH_n) + (n-1)$ implying that $\gamma_g^e(Wb_4) = 5$ and $\gamma_g^e(Wb_5) = 6$.

(ii) For $n = 6$, the apex vertex is not an equitable isolate in Wb_6 as well as in $\overline{Wb_6}$ while the pendant vertices are the equitable isolates in both Wb_6 and $\overline{Wb_6}$. Hence, $\gamma_g^e(Wb_6) = 8$.

Definition 2.14 *A gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the $(n-1)$ - cycle of W_n.*

Theorem 2.15 *For the gear graph,*

$$\gamma_g^e(G_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n = 4, 5 \\ \lceil \frac{2n+1}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof Let c denotes the apex vertex of wheel W_n and let $v_1, v_2, \cdots, v_{n-1}$ be the rim vertices of W_n. To obtain the gear graph G_n, subdivide each rim edge of wheel by the vertices $u_1, u_2, \cdots, u_{n-1}$ where each u_i is added between v_i and v_{i+1} for $i = 1, 2, \cdots, n-2$ and u_{n-1} is added between v_1 and v_{n-1}. Then $|V(G_n)| = 2n-1$ and $|E(G_n)| = 3(n-1)$. The graph G_n contains the outer cycle $C_{2(n-1)}$.

For $n = 4, 5$, the sets $D = \{v_1, u_3\}$ and $D = \{c, v_1, v_3\}$ are clearly the global equitable dominating sets of G_4 and G_5 respectively with minimum cardinality. Therefore, $\gamma_g^e(G_n) = \lceil \frac{n}{2} \rceil$ for $n = 4, 5$.

For $n > 5$, since the vertex c is the equitable isolate in G_n as well as in $\overline{G_n}$, it must belong to every global equitable dominating set of G_n. Moreover, the vertices other than c induce a cycle $C_{2(n-1)}$. Furthermore, $V(G_n) = V(C_{2(n-1)}) \bigcup \{c\}$ and by Proposition 2.2, $\gamma_g^e(C_n) = \lceil \frac{n}{3} \rceil$ for $n > 5$. This implies that $\gamma_g^e(G_n) = \gamma_g^e(C_{2(n-1)}) + 1 = \left\lceil \frac{2(n-1)}{3} \right\rceil + 1 = \lceil \frac{2n+1}{3} \rceil$. Hence, we have proved that

$$\gamma_g^e(G_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n = 4, 6 \\ \lceil \frac{2n+1}{3} \rceil & \text{otherwise.} \end{cases}$$

□

Definition 2.16 *The splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.*

Theorem 2.17 *For the splitting graph of wheel W_n $(n > 7)$,*

$$\gamma_g^e(S'(W_n)) = \begin{cases} \lceil \frac{4n+3}{3} \rceil & \text{if n} \equiv 0 \text{ } or \text{ } 2 \pmod 3 \\ \lfloor \frac{4n+3}{3} \rfloor & \text{if n} \equiv 1 \pmod 3. \end{cases}$$

Proof Let $v_1, v_2, \cdots, v_{n-1}$ be the rim vertices of wheel W_n and let c denotes the apex vertex of W_n. Let $c', v_1', v_2', \cdots, v_{n-1}'$ be the added vertices corresponding to the vertices $c, v_1, v_2, \cdots, v_{n-1}$ of W_n to obtain $G = S'(W_n)$. Then $|V(G)| = 2n$.

For $n = 8$, the vertices c and c' are equitable isolates in $\overline{G}$ and $c, v_1', v_2', \cdots, v_7'$ are equitable isolates in G. For $n > 8$, the vertices c and c' are equitable isolates in both G and $\overline{G}$ while the vertices $v'_1, v'_2, \cdots, v'_{n-1}$ are equitable isolates in G. Since an equitable isolate must belong to every equitable dominating set of G, the vertices $c, c', v'_1, v'_2, \cdots, v'_{n-1}$ being equitable isolates, must belong to every global equitable dominating set of G. Now, the remaining vertices $v_1, v_2, \cdots, v_{n-1}$ of G induce a cycle C_{n-1} and by Proposition 2.2, $\gamma_g^e(C_n) = \lceil \frac{n}{3} \rceil$ for $n > 5$. This implies that $\gamma_g^e(G) = \gamma_g^e(C_{n-1}) + n + 1 = \lceil \frac{n-1}{3} \rceil + n + 1$. Thus, for $n > 7$,

$$\gamma_g^e(S'(W_n)) = \begin{cases} \lceil \frac{4n+3}{3} \rceil & \text{if n} \equiv 0 \; or \; 2 \pmod 3) \\ \lfloor \frac{4n+3}{3} \rfloor & \text{if n} \equiv 1 \pmod 3). \end{cases}$$

□

Remark 2.18 For $4 \leq n \leq 7$, the apex vertex and all the duplicated vertices are the equitable isolates either in $S'(W_n)$ or in $\overline{S'(W_n)}$ and by Proposition 2.2,

$$\gamma_g^e(C_n) = \begin{cases} 3 & \text{if } n = 3, 5 \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Hence,

$$\gamma_g^e(S'(W_n)) = \begin{cases} 8 & \text{if } n = 4, 5 \\ 10 & \text{if } n = 6, 7. \end{cases}$$

§3. Concluding Remarks

The concept of global equitable domination is a variant of global domination and equitable domination. We obtain the exact values of global equitable domination number of the helm H_n, the flower graph Fl_n, the sunflower graph Sf_n, the closed helm CH_n, the web graph Wb_n, the gear graph G_n and the splitting graph of wheel $S'(W_n)$.

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The Pebbling Number of Jahangir Graph $J_{2,m}$

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Abstract: The t-pebbling number, $f_t(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, t pebbles can be moved to any specified target vertex by a sequence of pebbling moves, each move taking two pebbles off a vertex and placing one on an adjacent vertex. When $t = 1$, we call it as the pebbling number of G, and we denote it by $f(G)$. In this paper, we are going to give an alternate proof for the pebbling number of the graph $J_{2,m}$ $(m \geq 3)$.

Key Words: Graph pebbling, pebbling move, Jahangir graph.

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§1. Introduction

An n-dimensional cube Q_n, or n-cube for short, consists of 2^n vertices labelled by $(0, 1)$-tuples of length n. Two vertices are adjacent if their labels are different in exactly one entry. Saks and Lagarias (see [1]) propose the following question: suppose 2^n pebbles are arbitrarily placed on the vertices of an n-cube. Does there exist a method that allows us to make a sequence of moves, each move taking two pebbles off one vertex and placing one pebble on an adjacent vertex, in such a way that we can end up with a pebble on any desired vertex? This question is answered in the affirmative in [1].

We begin by introducing relevant terminology and background on the subject. Here, the term graph refers to a simple graph without loops or multiple edges. A configuration C of pebbles on a graph $G = (V, E)$ can be thought of as a function $C : V(G) \to N \cup \{0\}$. The value $C(v)$ equals the number of pebbles placed at vertex v, and the size of the configuration is the number $|C| = \sum_{v \in V(G)} C(v)$ of pebbles placed in total on G. Suppose C is a configuration of pebbles on a graph G. A pebbling move (step) consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex v, the target vertex, if we can apply pebbling moves repeatedly (if necessary) so that in the resulting configuration the vertex v has at least one pebble.

Definition 1.1([2]) *The t-pebbling number of a vertex v in a graph G, $f_t(v, G)$, is the smallest positive integer n such that however n pebbles are placed on the vertices of the graph, t pebbles can be moved to v in finite number of pebbling moves, each move taking two pebbles off one*

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vertex and placing one on an adjacent vertex. The t*-pebbling number of* G*,* $f_t(G)$*, is defined to be the maximum of the pebbling numbers of its vertices.*

Thus the t-pebbling number of a graph G, $f_t(G)$, is the least n such that, for any configuration of n pebbles to the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the pebbling number of G.

Fact 1.2 ([12], [13]) For any vertex v of a graph G, $f(v, G) \geq n$ where $n = |V(G)|$.

Fact 1.3 ([12]) The pebbling number of a graph G satisfies

$$f(G) \geq max\{2^{diam(G)}, |V(G)|\}.$$

Saks and Lagarias question then reduces to asking whether $f(Q_n) \leq n$, where Q_n is the n-cube. Chung [1] answered this question in the affirmative, by proving a stronger result.

Theorem 1.4 ([1]) *In an* n*-cube with a specified vertex* v*, the following are true:*

(1) *If* 2^n *pebbles are assigned to vertices of the* n*-cube, one pebble can be moved to* v*;*

(2) *Let* q *be the number of vertices that are assigned an odd number of pebbles. If there are all together more than* $2^{n+1} - q$ *pebbles, then two pebbles can be moved to* v*.*

With regard to t-pebbling number of graphs, we find the following theorems.

Theorem 1.5 ([9]) *Let* K_n *be the complete graph on* n *vertices where* $n \geq 2$*. Then* $f_t(K_n) = 2t + n - 2$*.*

Theorem 1.6 ([3]) *Let* $K_1 = \{v\}$*. Let* $C_{n-1} = (u_1, u_2, \cdots, u_{n-1})$ *be a cycle of length* $n - 1$*. Then the* t*-pebbling number of the wheel graph* W_n *is* $f_t(W_n) = 4t + n - 4$ *for* $n \geq 5$*.*

Theorem 1.7 ([5]) *For* $G = K^*_{s_1, s_2, \cdots, s_r}$*,*

$$f_t(G) = \begin{cases} 2t + n - 2, & \text{if } 2t \leq n - s_1 \\ 4t + s_1 - 2, & \text{if } 2t \geq n - s_1 \end{cases}.$$

Theorem 1.8 ([9]) *Let* $K_{1,n}$ *be an* n*-star where* $n > 1$*. Then* $f_t(K_{1,n}) = 4t + n - 2$*.*

Theorem 1.9 ([9]) *Let* C_n *denote a simple cycle with* n *vertices, where* $n \geq 3$*. Then* $f_t(C_{2k}) = t2^k$ *and* $f_t(C_{2k+1}) = \frac{2^{k+1} - (-1)^{k+2}}{3} + (t-1)2^k$*.*

Theorem 1.10 ([9]) *Let* P_n *be a path on* n *vertices. Then* $f_t(P_n) = t(2^{n-1})$*.*

Theorem 1.11 ([9]) *Let* Q_n *be the* n*-cube. Then* $f_t(Q_n) = t(2^n)$*.*

Now, we state the known pebbling results of the Jahangir graph $J_{2,m}$ and then we give an alternate proof for those results in Section 2.

Definition 1.12 ([11]) *Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm+1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.*

A labeling for $J_{2,m}$ for $m \geq 3$ is defined as follows:

Let v_{2m+1} be the label of the center vertex and $v_1, v_2, \cdots, v_{2m}$ be the label of the vertices that are incident clockwise on cycle C_{2m} so that $deg(v_1) = 3$.

The pebbling number of Jahangir graph $J_{2,m}$ ($m \geq 3$) is determined as follows:

Theorem 1.13 ([6]) *For the Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$.*

Theorem 1.14 ([6]) *For the Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$.*

Theorem 1.15 ([6]) *For the Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.*

Theorem 1.16 ([6]) *For the Jahangir graph $J_{2,6}$, $f(J_{2,6}) = 21$.*

Theorem 1.17 ([6]) *For the Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.*

Theorem 1.18 ([7]) *For the Jahangir graph $J_{2,m}$ where $m \geq 8$, $f(J_{2,m}) = 2m + 10$.*

The t-pebbling number of Jahangir graph $J_{2,m}$ ($m \geq 3$)is as follows:

Theorem 1.19 ([8]) *For the Jahangir graph $J_{2,3}$, $f_t(J_{2,3}) = 8t$.*

Theorem 1.20 ([8]) *For the Jahangir graph $J_{2,4}$, $f_t(J_{2,4}) = 16t$.*

Theorem 1.21 ([8]) *For the Jahangir graph $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.*

Theorem 1.22 ([8]) *For the Jahangir graph $J_{2,m}$, $f_t(J_{2,m}) = 16(t-1) + f(J_{2,m})$ where $m \geq 6$.*

Notation 1.23 Let $p(v)$ denote the number of pebbles on the vertex v and $p(A)$ denote the number of pebbles on the vertices of the set $A \subseteq V(G)$. We define the sets $S_1 = \{v_1, v_3, \cdots, v_{2m-1}\}$ and $S_2 = \{v_2, v_4, \cdots, v_{2m}\}$ from the labelling of $J_{2,m}$.

Remark 1.24 Consider a graph G with n vertices and $f(G)$ pebbles on it and we choose a target vertex v from G. If $p(v) = 1$ or $p(u) \geq 2$ where $uv \in E(G)$, then we can move one pebble to v easily. So, we always assume that $p(v) = 0$ and $p(u) \leq 1$ for all $uv \in E(G)$ when v is the target vertex.

§2. Alternate Proof for the Pebbling Number of $J_{2,m}$

Theorem 2.1 *For the Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$.*

Proof Put seven pebbles at v_4. Clearly we cannot move a pebble to v_1, since $d(v_4, v_1) = 3$. Thus $f(J_{2,3}) \geq 8$.

We have three cases to prove $f(J_{2,3}) \leq 8$.

Case 1. Let v_7 be the target vertex.

Clearly, $p(v_7) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq 5$, there exists a vertex, say v_2, such that $p(v_2) \geq 2$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_7 easily. Also, we can move one pebble to v_7, if $p(v_2) \geq 4$. Assume that $p(v_1) = 0$, $p(v_3) = 0$ and $p(v_2) = 2$ or 3. Thus either $p(v_4) \geq 2$ or $p(v_6) \geq 2$ and hence we can move one pebble to v_7 through v_3 or v_1.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$, $p(v_2) \leq 1$, $p(v_6) \leq 1$ and $p(v_7) \leq 1$, by Remark **??**. If $p(v_3) \geq 4$ or $p(v_5) \geq 4$ or $p(v_3) \geq 2$ and $p(v_5) \geq 2$ then we can move one pebble to v_1 through v_7. Without loss of generality, let $p(v_3) \geq 2$ and so $p(v_5) \leq 1$. If $p(v_2) = 1$ or $p(v_7) = 1$ then also we can move one pebble to v_1. So, we assume $p(v_2) = p(v_7) = 0$. Clearly, $p(v_4) \geq 3$. If $p(v_3) = 3$ then we move one pebble to v_3 from v_4 and hence we are done. Let $p(v_3) = 2$ and thus we move two pebbles to v_3 from v_4 and hence we are done. Assume $p(v_3) \leq 1$. In a similar way, we may assume that $p(v_5) \leq 1$ and hence $p(v_4) \geq 3$. Let $p(v_2) = 1$. If $p(v_3) = 1$ then clearly we can move one pebble to v_1. If $p(v_3) = 0$ then $p(v_4) \geq 4$ and hence we can move one pebble to v_2 and so one pebble is moved to v_1. Assume $p(v_2) = 0$. In a similar way, we may assume that $p(v_6) = 0$ and hence $p(v_4) \geq 5$. If $p(v_7) = 1$ then we are done easily. Let $p(v_7) = 0$. If $p(v_3) = 1$ or $p(v_5) = 1$ then we move three pebbles to v_3 or v_5, respectively. Thus we can move one pebble to v_1. Assume $p(v_3) = p(v_5) = 0$. Then $p(v_4) = 8$ and hence we can move one pebble to v_1 easily.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$, by Remark 1.24. Let $p(v_4) \geq 2$. If $p(v_4) \geq 4$ then clearly we are done.

Assume $p(v_4) = 2$ or 3 then clearly $p(v_3) = 0$ and $p(v_7) \leq 1$ (otherwise, we can move one pebble to v_2). Since, $p(v_5) + p(v_6) \geq 3$, first we let $p(v_6) \geq 2$. Clearly we are done if $p(v_1) = 0$ and $p(v_6) \geq 4$. Assume $p(v_1) = 0$ and $p(v_6) = 2$ or 3. If $p(v_7) = 1$ then we move one pebble to v_7 from v_4 since $p(v_4) \geq 2$ and $p(v_5) = 1$ and thus we move one pebble to v_1. Then we move one more pebble to v_1 from v_6 and hence one pebble can be moved to v_2. Assume $p(v_7) = 0$ and so $p(v_5) \geq 2$. If $p(v_4) = 3$ or $p(v_6) = 3$ then clearly we can move one pebble to v_2 by moving one pebble to v_3 or v_6. Thus we assume $p(v_4) = 2$ and $p(v_6) = 2$ and so $p(v_5) = 4$ and hence we are done. Assume $p(v_6) \leq 1$ and so $p(v_4) = 2$. Clearly, we are done if $p(v_5) \geq 4$. Assume $p(v_5) = 3$ and hence we move one pebble to v_2 since $p(v_7) = p(v_1) = 1$.

Assume $p(v_4) \leq 1$. In a similar way, we may assume that $p(v_6) \leq 1$ and so $p(v_7) \leq 1$. Let $p(v_1) = 1$. Clearly we are done if $p(v_7) = 1$ or $p(v_6) = 1$. Assume $p(v_6) = p(v_7) = 0$ and so $p(v_5) \geq 4$. Thus we move one pebble to v_1 and hence we are done. Assume $p(v_1) = 0$. In a similar way, we assume that $p(v_3) = 0$. We have $p(v_5) \geq 5$. Let $p(v_5) = 5$. Clearly, $p(v_6) = p(v_7) = 1$ and hence we can move one pebble to v_2 through v_1. Let $p(v_5) \geq 6$. If $p(v_4) = 1$ or $p(v_6) = 1$ or $p(v_7) = 1$ then we move three pebbles to v_4 or v_6 or v_7 and hence we are done. Assume $p(v_4) = p(v_6) = p(v_7) = 0$ and so $p(v_5) = 8$. Thus we can move one pebble to v_2 easily. □

Theorem 2.2 *For the Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$.*

Proof Put fifteen pebbles at v_8. Clearly we cannot move a pebble to v_4, since $d(v_8, v_4) = 4$. Thus $f(J_{2,4}) \geq 16$.

We have three cases to prove $f(J_{2,4}) \leq 16$.

Case 1. Let v_9 be the target vertex.

Clearly, $p(v_9) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark **??**. Since, $p(S_2) \geq 12$, there exists a vertex, say v_2, such that $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_9 easily. Assume $p(v_1) = 0$ and $p(v_3) = 0$. So, we can move one pebble to v_9 easily, since $p(v_2) \geq 4$.

Case 2: Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$, $p(v_2) \leq 1$, $p(v_8) \leq 1$ and $p(v_9) \leq 1$, by Remark **??**. If $p(v_3) \geq 4$ or $p(v_5) \geq 4$ or $p(v_7) \geq 4$ then we can move one pebble to v_1 through v_9. Assume $p(v_i) \leq 3$ for all $i \in \{3, 5, 7\}$. Let $p(v_3) \geq 2$ and if $p(v_9) = 1$ or $p(v_2) = 1$ or $p(v_5) \geq 2$ or $p(v_7) \geq 2$ then we can move one pebble to v_1 through v_9 easily. Assume $p(v_2) = 0$, $p(v_9) = 0$, $p(v_5) \leq 1$ and $p(v_7) \leq 1$. Clearly, either $p(v_4) \geq 4$ or $p(v_6) \geq 4$ and hence we can move one pebble to v_1 through v_9.

Assume $p(v_3) \leq 1$. In a similar way, we may assume that $p(v_5) \leq 1$ and $p(v_7) \leq 1$ and hence either $p(v_4) \geq 5$ or $p(v_6) \geq 5$. Without loss of generality, let $p(v_4) \geq 5$. If $p(v_2) = 1$ or $p(v_9) = 1$ then we move one pebble to v_2 or v_9 from v_4 and hence we can move one pebble to v_1. Assume $p(v_2) = 0$ and $p(v_9) = 0$ then clearly $p(v_4) \geq 6$. If $p(v_3) = 1$ or $p(v_5) = 1$ or $p(v_6) \geq 2$ then we can move one pebble to v_1 easily by moving three pebbles to v_3 or v_5 from v_4. Let $p(v_3) = 0$, $p(v_5) = 0$ and $p(v_6) \leq 1$ and hence $p(v_4) \geq 13$. Thus we can move one pebble to v_1 easily.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$, by Remark 1.24. Let $p(v_4) \geq 2$. If $p(v_4) \geq 4$ then clearly we are done.

Assume $p(v_4) = 2$ or 3 then clearly $p(v_3) = 0$ and $p(v_9) \leq 1$ (otherwise, we can move one pebble to v_2). If $p(v_5) \geq 4$ or $p(v_7) \geq 4$ or $p(v_5) \geq 2$ and $p(v_7) \geq 2$ then we can move one pebble to v_3 and then we move one pebble to v_3 from v_4 and hence one pebble can be moved to v_2 from v_3. Asssume $p(v_5) \leq 3$ and $p(v_7) \leq 4$ such that we cannot move one pebble to v_9. So, $p(v_5) + p(v_7) \leq 4$. Clearly, $p(v_8) + p(v_1) \leq 3$ and hence $p(v_6) \geq 6$. If $p(v_5) = 1$ or $p(v_7) = 1$ then we move three pebbles to v_5 or v_7 and then we can move two pebbles to v_3 from v_5 and v_4 and hence we are done. Assume $p(v_5) = 0$ and $p(v_7) = 0$. So, $p(v_6) \geq 8$. We move two pebbles to v_4 from v_6 and hence we can move one pebble to v_2 from v_4 easily.

Assume $p(v_4) \leq 1$. In a similar way, we may assume that $p(v_8) \leq 1$ and so $p(v_9) \leq 1$. Clearly, $p(v_5) + p(v_6) + p(v_7) \geq 11$ and so we can move two pebbles to v_9. If $p(v_1) = 1$ or $p(v_3) = 1$ then we move one more pebble to v_1 or v_3 from v_9 and hence we are done. Assume $p(v_1) = 0$ and $p(v_3) = 0$ then we have $p(v_5) + p(v_6) + p(v_7) \geq 13$. Let $p(v_5) \geq 4$. Clearly, we are done if $p(v_6) + p(v_7) \geq 8$. Assume $p(v_6) + p(v_7) \leq 7$ and so $p(v_5) \geq 6$. If $p(v_4) = 1$ or $p(v_9) = 1$ then we move three pebbles to v_4 or v_9 from v_6 and hence we are done. Let

$p(v_4) = 0$ and $p(v_9) = 0$ and hence we can move one pebble to v_2 since $p(v_5) \geq 8$. Assume $p(v_5) = 2$ or 3 and so $p(v_6) + p(v_7) \geq 8$. Thus we can move one pebble to v_1 or v_3 from the vertices v_6 and v_7. Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$. Otherwise, we can move one pebble to v_1 or v_3 if $p(v_4) = 1$ or $p(v_9) = 1$. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) = 0$ and $p(v_9) = 0$ and so $p(v_6) + p(v_7) \geq 12$. Thus we can move four pebbles to v_9 from the vertices v_5, v_6 and v_7. Assume $p(v_5) \leq 1$. In a similar way, we assume that $p(v_7) \leq 1$. Thus we have $p(v_6) \geq 9$. If $p(v_1) = 1$ or $p(v_3) = 1$ then clearly we are done. Let $p(v_1) = 0$ and $p(v_3) = 0$ and so $p(v_6) \geq 11$. Let $p(v_5) = 1$. We move five pebbles to v_5 from v_6. Clearly, we are done if $p(v_4) = 1$ or $p(v_9) = 1$. Assume $p(v_4) = p(v_9) = 0$ and so $p(v_6) \geq 13$. If $p(v_7) = 1$ then we move one pebble to v_7 and then we move three pebbles to v_5 from v_6 and hence we are done since v_9 receives four pebbles from v_5 and v_7. Let $p(v_7) = 0$ and so $p(v_6) \geq 14$. We move seven pebbles to v_5 from v_6 and hence we are done easily. Assume $p(v_5) = 0$. In a similar way, we may assume that $p(v_7) = 0$. Thus, $p(v_6) \geq 13$. If $p(v_4) = 1$ or $p(v_8) = 1$ or $p(v_9) = 1$ then we move three pebbles to v_4 or v_8 or v_9 and hence we are done. Assume $p(v_4) = p(v_8) = p(v_9) = 0$ and so $p(v_6) = 16$. Thus we can move one pebble to v_2 easily. □

Theorem 2.3 *For the Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.*

Proof Put fifteen pebbles at v_6 and one pebble each at v_8 and v_{10}. Clearly we cannot move a pebble to v_2. Thus $f(J_{2,5}) \geq 18$.

To prove that $f(J_{2,5}) \leq 18$, we have the following cases:

Case 1. Let v_{11} be the target vertex.

Clearly, $p(v_{11}) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq 13$, there exists a vertex, say v_2, such that $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_{11} easily. If $p(v_{10}) \geq 2$ or $p(v_4) \geq 2$ then also we can move one pebble to v_{11}. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$ and $p(v_{10}) \leq 1$. Thus, we can move one pebble to v_{11} easily, since $p(v_2) \geq 4$.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $i \in \{2, 10, 11\}$ by Remark 1.24. Let $p(v_3) \geq 2$. If $p(v_3) \geq 4$ or a vertex of $S_1 - \{v_1, v_3\}$ contains two or more pebbles then we can move one pebble to v_1 easily through v_{11}. So, assume $p(v_3) = 2$ or 3 and no vertex of $S_1 - \{v_1, v_3\}$ contain more than one pebble. Clearly, $p(v_6) + p(v_8) \geq 7$ and hence we can move one pebble to v_{11} from v_6 or v_8 and hence we are done, since $p(v_3) \geq 2$. Assume $p(v_3) \leq 1$. In a similar way, we assume that $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3\}$. Clearly, $p(v_4) + p(v_6) + p(v_8) \geq 11$. Let $p(v_4) \geq 4$. If $p(v_6) \geq 4$ or $p(v_8) \geq 4$ or $p(v_6) \geq 2$ and $p(v_8) \geq 2$ then we can move one pebble to v_{11}. Since $p(v_4) \geq 4$, we can move another one pebble to v_{11} from v_4 and hence one pebble can be moved to v_1. Assume $p(v_6) \leq 3$ and $p(v_8) \leq 3$ such that we cannot move two pebbles to v_7. Thus $p(v_6) + p(v_8) \leq 4$ and so $p(v_4) \geq 8$ and hence we can move one pebble to v_1 from v_4. Assume $p(v_4) \leq 3$. Similarly, $p(v_8) \leq 3$. We have $p(v_6) \geq 6$. Clearly, we are done if $p(v_5) = 1$ or $p(v_7) = 1$. Otherwise, $p(v_6) \geq 8$ and hence we can move one pebble to v_1 easily.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$ by Remark 1.24. Let $p(v_5) \geq 4$. If $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$ or $p(v_{11}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{10}) \leq 1$ and $p(v_{11}) \leq 1$. Also we assume that $p(v_7) + p(v_9) \leq 4$ such that we cannot move two pebbles to v_{11}. Let $p(v_7) \geq 2$ and so $p(v_9) \leq 1$. If $p(v_{11}) = 1$ or $p(v_5) \geq 6$ then clearly, we are done. Assume $p(v_{11}) = 0$ and $p(v_5) = 4$ or 5. Thus $p(v_6) + p(v_8) \geq 7$ and we can move one pebble to v_{11} from v_6 or v_8 and hence we are done. Assume $p(v_7) \leq 1$. In a similar way, we may assume that $p(v_9) \leq 1$. Let $p(v_5) = 6$ or 7 and so $p(v_6) + p(v_8) \geq 6$. Thus we can move one pebble to v_{11} from v_6 and v_8. Assume $p(v_5) = 4$ or 5 and so $p(v_6) + p(v_8) \geq 8$. If $p(v_7) = 1$ or $p(v_{11}) = 1$ then we can move one pebble to v_2 easily through v_{11}. Let $p(v_7) = p(v_{11}) = 0$ and so $p(v_6) + p(v_8) \geq 10$. Clearly, we can move two pebbles to v_{11} from v_6 and v_8 and hence we are done since $p(v_5) \geq 4$. Assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_7) \leq 3$ and $p(v_9) \leq 3$.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_{11}) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$. Assume $p(v_1) = p(v_3) = p(v_{11}) = 0$ and $p(v_4) \leq 1$, $p(v_{10}) \leq 1$. Clearly, $p(v_6) + p(v_8) \geq 7$ and hence we can move one pebble to v_{11} from v_6 or v_8. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1 - \{v_1, v_3\}$.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$ or $p(v_{11}) \geq 2$. Let $p(v_{11}) = 1$ and so we can move three pebbles to v_{11} from the two vertices of $S_1 - \{v_1, v_3\}$ and v_6 or v_8. Assume $p(v_{11}) = 0$ and so $p(v_6) + p(v_8) \geq 9$. Thus we can move two pebbles to v_{11} from the vertices v_6 and v_8 and then we move two more pebbles to v_{11} from the two vertices of $S_1 - \{v_1, v_3\}$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$ or $p(v_{11}) \geq 2$. Let $p(v_{11}) = 1$ and so $p(v_6) + p(v_8) \geq 10$. Thus we can move three pebbles to v_{11} from the vertex of $S_1 - \{v_1, v_3\}$ and the vertices v_6 and v_8. Assume $p(v_{11}) = 0$ and let v_5 is the vertex of $S_1 - \{v_1, v_3\}$ contains more than one pebble on it. So $p(v_6) + p(v_8) \geq 12$. If $p(v_7) = 1$ then we can move three pebbles to v_{11} from v_6 and v_8 and hence we are done since $p(v_5) \geq 2$. Assume $p(v_7) = 0$ and so we can move three pebbles to v_{11} from v_6 and v_8 and hence we are done. In a similar way, we can move one pebble to v_2 if $p(v_9) \geq 2$ and $p(v_7) \geq 2$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{10}) \geq 2$ or $p(v_{11}) \geq 2$. Thus we have $p(v_6) + p(v_8) \geq 12$. Let $p(v_{11}) = 1$. Clearly we can move three pebbles to v_{11} if $p(v_7) = 1$. Assume $p(v_7) = 0$ and so we can move three pebbles to v_{11} since $p(v_6) + p(v_8) \geq 13$ and hence we are done. Assume $p(v_{11}) = 0$. Without loss of generality, we let $p(v_6) \geq 7$. If $p(v_4) = 1$ or $p(v_5) = 1$ or $p(v_7) = 1$ then we can move two pebbles to v_3 and hence we are done. Assume $p(v_4) = p(v_5) = p(v_7) = 0$. Let $p(v_8) \geq 2$. If $p(v_9) = 1$ then we move one pebble to v_{11} and then we move another three pebbles to v_{11} from v_6 and v_8 since $p(v_6) + p(v_8) - 2 \geq 14$ and hence we are done. Assume $p(v_9) = 0$ and so $p(v_6) + p(v_8) \geq 17$. Clearly we can move one pebble to v_2 from v_6 and v_8. □

Theorem 2.4 *For the Jahangir graph $J_{2,6}$, $f(J_{2,6}) = 21$.*

Proof Put fifteen pebbles at v_6, three pebbles at v_{10} and one pebble each at v_8 and v_{12}. Then, we cannot move a pebble v_2. Thus, $f(J_{2,6}) \geq 21$.

To prove that $f(J_{2,6}) \leq 21$, we have the following cases:

Case 1. Let v_{13} be the target vertex.

Clearly, $p(v_{13}) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq 15$, there exists a vertex, say v_2, such that $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_{13} easily. If $p(v_{12}) \geq 2$ or $p(v_4) \geq 2$ then also we can move one pebble to v_{13}. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$ and $p(v_{10}) \leq 1$. Thus, we can move one pebble to v_{13} easily, since $p(v_2) \geq 4$.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $i \in \{2, 12, 13\}$ by Remark 1.24. Let $p(v_3) \geq 2$. If $p(v_2) = 1$ or $p(v_{13}) = 1$ or a vertex of $S_1 - \{v_1, v_3\}$ has more than one pebble then we can move one pebble to v_1 easily. Otherwise, there exists a vertex, say v_6, of $S_2 - \{v_2, v_{12}\}$, contains more than three pebbles and hence we are done. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1\}$. Clearly, $S_2 - \{v_2, v_{12}\} \geq 13$, and so we can move two pebbles to v_{13} and hence we are done.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$ by Remark 1.24. Let $p(v_5) \geq 4$. If $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$ or $p(v_{13}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{12}) \leq 1$ and $p(v_{13}) \leq 1$. Also we assume that $p(v_7) + p(v_9) + p(v_{11}) \leq 5$ such that we cannot move two pebbles to v_{13}. Let $p(v_7) \geq 2$ and so $p(v_9) \leq 1$ and $p(v_{11}) \leq 1$. If $p(v_{13}) = 1$ or $p(v_5) \geq 6$ then clearly, we are done. Assume $p(v_{13}) = 0$ and $p(v_5) = 4$ or 5. Thus $p(v_6) + p(v_8) + p(v_{10}) \geq 9$ and we can move one pebble to v_{13} from v_6, v_8 and v_{10} and hence we are done. Assume $p(v_7) \leq 1$. In a similar way, we may assume that $p(v_9) \leq 1$ and $p(v_{11}) \leq 1$. Let $p(v_5) = 6$ or 7 and so $p(v_6) + p(v_8) + p(v_{10}) \geq 8$. Thus we can move one pebble to v_{13} from v_6, v_8 and v_{10}. Assume $p(v_5) = 4$ or 5 and so $p(v_6) + p(v_8) + p(v_{10}) \geq 10$. If $p(v_7) = 1$ or $p(v_9) = 1$ or $p(v_{13}) = 1$ then we can move one pebble to v_2 easily through v_{13}. Let $p(v_7) = p(v_9) = p(v_{13}) = 0$ and so $p(v_6) + p(v_8) + p(v_{10}) \geq 13$. Clearly, we can move two pebbles to v_{13} from v_6, v_8 and v_{10} and hence we are done since $p(v_5) \geq 4$. Assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_{11}) \leq 3$, $p(v_7) \leq 3$, and $p(v_9) \leq 3$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move four pebbles to v_{13} and hence one pebble can be moved to v_2 from v_{13}.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_{13}) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$. Assume $p(v_1) = p(v_3) = p(v_{13}) = 0$ and $p(v_4) \leq 1$, $p(v_{12}) \leq 1$. Clearly, $p(v_6)+p(v_8)+p(v_{10}) \geq 9$ and hence we can move one pebble to v_{13} from v_6, v_8 and v_{10}. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1 - \{v_1, v_3\}$.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$ or $p(v_{13}) \geq 2$. Let $p(v_{13}) = 1$ and so we can move three pebbles to v_{13} from the two vertices of $S_1 - \{v_1, v_3\}$ and v_6, v_8 and v_{10}. Assume $p(v_{13}) = 0$ and so $p(v_6) + p(v_8) + p(v_{10}) \geq 11$. Thus we can move two pebbles to v_{13} from the vertices v_6, v_8 and v_{10} and then we move two more pebbles to v_{13} from the two vertices of $S_1 - \{v_1, v_3\}$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$ or $p(v_{13}) \geq 2$. Let $p(v_{13}) = 1$ and so $p(v_6) + p(v_8) + p(v_{10}) \geq 12$. Thus we can move three pebbles to v_{13} from the vertex of $S_1 - \{v_1, v_3\}$ and the vertices v_6, v_8 and v_{10}. Assume $p(v_{13}) = 0$ and let v_5 is the vertex of $S_1 - \{v_1, v_3\}$ contains more than one pebble on it. So $p(v_6) + p(v_8) + p(v_{10}) \geq 13$. If $p(v_7) = 1$ or $p(v_9) = 1$ then we can move three pebbles to v_{13} from v_6, v_8 and v_{10} and hence we are done since $p(v_5) \geq 2$. Assume $p(v_7) = p(v_9) = 0$ and so we can move three pebbles to v_{13} from v_6, v_8 and v_{10} and hence we are done. In a similar way, we can move one pebble to v_2 if $p(v_{11}) \geq 2$, $p(v_7) \geq 2$ and $p(v_9) \geq 2$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{12}) \geq 2$ or $p(v_{13}) \geq 2$. Thus we have $p(v_6) + p(v_8) + p(v_{10}) \geq 14$. Let $p(v_{13}) = 1$. Clearly we can move three pebbles to v_{13} if $p(v_7) = 1$ or $p(v_9) = 1$. Assume $p(v_7) = p(v_9) = 0$ and so we can move three pebbles to v_{13} since $p(v_6) + p(v_8) + p(v_{10}) \geq 15$ and hence we are done. Assume $p(v_{13}) = 0$. Without loss of generality, we let $p(v_6) \geq 5$. If $p(v_4) = 1$ or $p(v_5) = 1$ or $p(v_7) = 1$ then we can move two pebbles to v_3 and hence we are done. Assume $p(v_4) = p(v_5) = p(v_7) = 0$. Let $p(v_8) \geq 2$. If $p(v_9) = 1$ then we move one pebble to v_{13} and then we move another three pebbles to v_{13} from v_6, v_8 and v_{10} since $p(v_6) + p(v_8) + p(v_{10}) - 2 \geq 16$ and hence we are done. Assume $p(v_9) = 0$ and so $p(v_6) + p(v_8) + p(v_{10}) \geq 20$. Clearly we can move one pebble to v_2 from v_6, v_8 and v_{10}.□

Theorem 2.5 *For the Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.*

Proof Put fifteen pebbles at v_6, three pebbles at v_{10} and one pebble each at v_8, v_{14}, v_{12}, and v_{13}. Then, we cannot move a pebble to v_2. Thus, $f(J_{2,7}) \geq 23$.

To prove that $f(J_{2,7}) \leq 23$, we have the following cases:

Case 1. Let v_{15} be the target vertex.

Clearly, $p(v_{15}) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq 16$, there exists a vertex, say v_2, such that $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_{15} easily. If $p(v_{14}) \geq 2$ or $p(v_4) \geq 2$ then also we can move one pebble to v_{15}. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$ and $p(v_{14}) \leq 1$. Thus, we can move one pebble to v_{15} easily, since $p(v_2) \geq 4$.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $i \in \{2, 14, 15\}$ by Remark 1.24. Let $p(v_3) \geq 2$. If

$p(v_2) = 1$ or $p(v_{15}) = 1$ or a vertex of $S_1 - \{v_1, v_3\}$ has more than one pebble then we can move one pebble to v_1 easily. Otherwise, there exists a vertex, say v_6, of $S_2 - \{v_2, v_{14}\}$, contains more than two pebbles and hence we are done if $p(v_5) = 1$ or $p(v_7) = 1$. Let $p(v_5) = p(v_7) = 0$ and so $p(v_6) \geq 4$ and hence we are done. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1\}$. Clearly, $p(S_2 - \{v_2, v_{14}\}) \geq 14$, and so we can move two pebbles to v_{15} and hence we are done.

Case 3. Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$ by Remark 1.24. Let $p(v_5) \geq 4$. If $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$ or $p(v_{15}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{14}) \leq 1$ and $p(v_{15}) \leq 1$.Also we assume that $p(v_7)+p(v_9)+p(v_{11})+p(v_{13}) \leq 6$ such that we cannot move two pebbles to v_{15}. Let $p(v_7) \geq 2$ and so $p(v_9) \leq 1$, $p(v_{11}) \leq 1$ and $p(v_{13}) \leq 1$. If $p(v_{15}) = 1$ or $p(v_5) \geq 6$ then clearly, we are done. Assume $p(v_{15}) = 0$ and $p(v_5) = 4$ or 5. Thus $p(v_6) + p(v_8) + p(v_{10}) + p(v_{12}) \geq 10$ and we can move one pebble to v_{15} from v_6, v_8, v_{10} and v_{12} and hence we are done. Assume $p(v_7) \leq 1$. In a similar way, we may assume that $p(v_9) \leq 1$, $p(v_{11}) \leq 1$ and $p(v_{13}) \leq 1$. Let $p(v_5) = 6$ or 7 and so $p(v_6) + p(v_8) + p(v_{10}) + p(v_{12}) \geq 10$. Thus we can move one pebble to v_{15} from v_6, v_8, v_{10} and v_{12}. Assume $p(v_5) = 4$ or 5 and so $p(v_6) + p(v_8) + p(v_{10}) + p(v_{12}) \geq 12$. If $p(v_7) = 1$ or $p(v_9) = 1$ or $p(v_{11}) = 1$ or $p(v_{15}) = 1$ then we can move one pebble to v_2 easily through v_{15}. Let $p(v_7) = p(v_9) = p(v_{11}) = p(v_{15}) = 0$ and so $p(v_6) + p(v_8) + p(v_{10}) + p(v_{12}) \geq 15$. Clearly, we can move two pebbles to v_{15} from v_6, v_8, v_{10} and v_{12} and hence we are done since $p(v_5) \geq 4$. Assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_{13}) \leq 3$, $p(v_7) \leq 3$, $p(v_9) \leq 3$ and $p(v_{11}) \leq 3$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move four pebbles to v_{15} and hence one pebble can be moved to v_2 from v_{15}.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_{15}) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$. Assume $p(v_1) = p(v_3) = p(v_{15}) = 0$ and $p(v_4) \leq 1$, $p(v_{14}) \leq 1$. Clearly, $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 10$ and hence we can move one pebble to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1 - \{v_1, v_3\}$.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$ or $p(v_{15}) \geq 2$. Let $p(v_{15}) = 1$ and so we can move three pebbles to v_{15} from the two vertices of $S_1 - \{v_1, v_3\}$ and the vertices of $S_2 - \{v_2, v_4, v_{14}\}$. Assume $p(v_{15}) = 0$ and so $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 12$. Thus we can move two pebbles to v_{15} from the vertices $p(S_2 - \{v_2, v_4, v_{14}\})$ and then we move two more pebbles to v_{15} from the two vertices of $S_1 - \{v_1, v_3\}$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$ or $p(v_{15}) \geq 2$. Let $p(v_{15}) = 1$ and so $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 13$. Thus we can move three pebbles to v_{15} from the vertex of $S_1 - \{v_1, v_3\}$ and the vertices $S_2 - \{v_2, v_4, v_{14}\}$. Assume $p(v_{15}) = 0$ and let v_5 is the vertex of $S_1 - \{v_1, v_3\}$ contains more than one pebble on it. So $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 14$. If $p(v_7) = 1$ or $p(v_9) = 1$ or $p(v_{11}) = 1$ then we can move three pebbles to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$ and hence we are done since $p(v_5) \geq 2$. Assume $p(v_7) = p(v_9) = p(v_{11}) = 0$

and so we can move three pebbles to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$ and hence we are done. In a similar way, we can move one pebble to v_2 if $p(v_i) \geq 2$, where $v_i \in S_1 - \{v_1, v_3, v_5\}$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{14}) \geq 2$ or $p(v_{15}) \geq 2$. Thus we have $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 15$. Let $p(v_{15}) = 1$. Clearly we can move three pebbles to v_{15} if $p(v_7) = 1$ or $p(v_9) = 1$ or $p(v_{11}) = 1$. Assume $p(v_7) = p(v_9) = p(v_{11}) = 0$ and so we can move three pebbles to v_{15} since $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 18$ and hence we are done. Assume $p(v_{15}) = 0$. Without loss of generality, we let $p(v_6) \geq 5$. If $p(v_4) = 1$ or $p(v_5) = 1$ or $p(v_7) = 1$ then we can move two pebbles to v_3 and hence we are done. Assume $p(v_4) = p(v_5) = p(v_7) = 0$. Let $p(v_8) \geq 2$. If $p(v_9) = 1$ then we move one pebble to v_{15} and then we move another three pebbles to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$, since $p(S_2 - \{v_2, v_4, v_{14}\}) - 2 \geq 17$ and hence we are done. Assume $p(v_9) = 0$ and so $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 20$. Clearly we can move one pebble to v_2 from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$. □

Theorem 2.6 *For the Jahangir graph $J_{2,m}$ where $m \geq 8$, $f(J_{2,m}) = 2m + 10$.*

Proof If m is even, then consider the following configuration C_1 such that $C_1(v_2) = 0$, $C_1(v_{m+2}) = 15$, $C_1(v_{m-2}) = 3$, $C_1(v_{m+6}) = 3$, $C_1(x) = 1$ where $x \notin N[v_2]$, $x \notin N[v_{m+2}]$, $x \notin N[v_{m-2}]$, and $x \notin N[v_{m+6}]$ and $C_1(y) = 0$ for all other vertices of $J_{2,m}$. If m is odd, then consider the following configuration C_2 such that $C_2(v_2) = 0$, $C_2(v_{m+1}) = 15$, $C_2(v_{m-3}) = 3$, $C_2(v_{m+5}) = 3$, $C_2(x) = 1$ where $x \notin N[v_2]$, $x \notin N[v_{m+1}]$, $x \notin N[v_{m-3}]$, and $x \notin N[v_{m+5}]$ and $C_1(y) = 0$ for all other vertices of $J_{2,m}$. Then, we cannot move a pebble to v_2. The total number of pebbles placed in both configurations is $15+2(3)+(m-4)(1)+(m-8)(1) = 2m+9$. Therefore, $f(J_{2,m}) \geq 2m + 10$.

To prove that $f(J_{2,m}) \leq 2m + 10$, for $m \geq 8$, we have the following cases:

Case 1. Let v_{2m+1} be the target vertex.

Clearly, $p(v_{2m+1}) = 0$ and $p(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.24. Since, $p(S_2) \geq m+10$, there exists a vertex, say v_2, such that $p(v_2) \geq 2$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_{2m+1} easily. If $p(v_{2m}) \geq 2$ or $p(v_4) \geq 2$ then also we can move one pebble to v_{2m+1}. Assume $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$ and $p(v_{2m}) \leq 1$. Thus, we can move one pebble to v_{2m+1} easily, since $p(v_2) \geq 4$.

Case 2. Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $i \in \{2, 2m, 2m+1\}$ by Remark 1.24. Let $p(v_3) \geq 2$. If $p(v_2) = 1$ or $p(v_{2m+1}) = 1$ or a vertex of $S_1 - \{v_1, v_3\}$ has more than one pebble then we can move one pebble to v_1 easily. Otherwise, there exists a vertex, say v_6, of $S_2 - \{v_2, v_{2m}\}$, contains more than one pebble and hence we are done if $p(v_5) = 1$ or $p(v_7) = 1$. Let $p(v_5) = p(v_7) = 0$ and so $p(v_6) \geq 4$ and hence we are done. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1\}$. Clearly, $p(S_2 - \{v_2, v_{2m}\}) \geq m + 8$, and so we can move two pebbles to v_{2m+1} and hence we are done.

Case 3: Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$ by Remark 1.24. Let $p(v_5) \geq 4$. If $p(v_1) = 1$ or

$p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$ or $p(v_{2m+1}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $p(v_1) = 0$, $p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{2m}) \leq 1$ and $p(v_{2m+1}) \leq 1$. Also we assume that $p(S_1 - \{v_1, v_3, v_5\}) \leq m-1$ such that we cannot move two pebbles to v_{2m+1}. Let $p(v_7) \geq 2$ and so $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3, v_5, v_7\}$. If $p(v_{2m+1}) = 1$ or $p(v_5) \geq 6$ then clearly, we are done. Assume $p(v_{2m+1}) = 0$ and $p(v_5) = 4$ or 5. Thus $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+4$ and we can move one pebble to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done. Assume $p(v_7) \leq 1$. In a similar way, we may assume that $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3, v_5, v_7\}$. Let $p(v_5) = 6$ or 7 and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+4$. Thus we can move one pebble to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$. Assume $p(v_5) = 4$ or 5 and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+6$. If $p(v_j) = 1$, a vertex v_j of $S_1 - \{v_1, v_3, v_5, v_{2m-1}\}$ then we can move one pebble to v_2 easily through v_{2m+1}. Let $p(S_1 - \{v_1, v_3, v_5, v_{2m-1}\}) = 0$ and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq 2m+2$. Clearly, we can move two pebbles to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done since $p(v_5) \geq 4$. Assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_k) \leq 3$, for all $v_k \in S_1 - \{v_1, v_3, v_5\}$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move four pebbles to v_{2m+1} and hence one pebble can be moved to v_2 from v_{2m+1}

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_{2m+1}) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$. Assume $p(v_1) = p(v_3) = p(v_{2m+1}) = 0$ and $p(v_4) \leq 1$, $p(v_{2m}) \leq 1$. Clearly, $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+4$ and hence we can move one pebble to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1 - \{v_1, v_3\}$.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$ or $p(v_{2m+1}) \geq 2$. Let $p(v_{2m+1}) = 1$ and so we can move three pebbles to v_{2m+1} from the two vertices of $S_1 - \{v_1, v_3\}$ and the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$. Assume $p(v_{2m+1}) = 0$ and so $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+6$. Thus we can move two pebbles to v_{2m+1} from the vertices $S_2 - \{v_2, v_4, v_{2m}\}$ and then we move two more pebbles to v_{2m+1} from the two vertices of $S_1 - \{v_1, v_3\}$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1) = 1$ or $p(v_3) = 1$ or $p(v_4) \geq 2$ or $p(v_{2m}) \geq 2$ or $p(v_{2m+1}) \geq 2$. Let $p(v_{2m+1}) = 1$ and so $p(S_2 - \{v_2, v_4, v_{14}\}) \geq m+7$. Thus we can move three pebbles to v_{2m+1} from the vertex of $S_1 - \{v_1, v_3\}$ and the vertices $S_2 - \{v_2, v_4, v_{2m}\}$. Assume $p(v_{2m+1}) = 0$ and let v_5 is the vertex of $S_1 - \{v_1, v_3\}$ contains more than one pebble on it. So $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m+8$. If $p(v_j) = 1$, a vertex v_j of $S_1 - \{v_1, v_3, v_5, v_{2m-1}\}$ then we can move three pebbles to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done since $p(v_5) \geq 2$. Assume $p(S_1 - \{v_1, v_3, v_5, v_{2m-1}\}) = 0$ and so we can move three pebbles to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done. In a similar way, we can move one pebble to v_2 if $p(v_i) \geq 2$, where $v_i \in S_1 - \{v_1, v_3, v_5\}$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles.

Clearly, we are done if $p(v_1)=1$ or $p(v_3)=1$ or $p(v_4)\geq 2$ or $p(v_{2m})\geq 2$ or $p(v_{2m+1})\geq 2$. Thus we have $p(S_2-\{v_2,v_4,v_{2m}\})\geq m+9$. Let $p(v_{2m+1})=1$. Clearly we can move three pebbles to v_{2m+1} if a vertex v_j of $S_1-\{v_1,v_3,v_5,v_{2m-1}\}$ such that $p(v_j)=1$. Assume $p(S_1-\{v_1,v_3,v_5,v_{2m-1}\})=0$ and so we can move three pebbles to v_{2m+1} since $p(S_2-\{v_2,v_4,v_{2m}\})\geq m+12$ and hence we are done. Assume $p(v_{2m+1})=0$. Without loss of generality, we let $p(v_6)\geq 2$. If $p(v_5)=1$ or $p(v_7)=1$ then we can move two pebbles to v_3 and hence we are done. Assume $p(v_5)=p(v_7)=0$. Let $p(v_8)\geq 2$. If $p(v_9)=1$ then we move one pebble to v_{2m+1} and then we move another three pebbles to v_{2m+1} from the vertices of $S_2-\{v_2,v_4,v_{2m}\}$, since $p(S_2-\{v_2,v_4,v_{2m}\})-2\geq m+11$ and hence we are done. Assume $p(v_9)=0$ and so $p(S_2-\{v_2,v_4,v_{2m}\})\geq m+12$. Clearly we can move one pebble to v_2 from the vertices of $S_2-\{v_2,v_4,v_{2m}\}$. □

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On 4-Total Product Cordiality of Some Corona Graphs

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Abstract: Let f be a map from $V(G)$ to $\{0, 1, \cdots, k-1\}$ where k is an integer, $2 \le k \le |V(G)|$. For each edge uv, assign the label $f(u)f(v) \pmod k$. f is called a k-total product cordial labeling of G if $|ev_f(i) - ev_f(j)| \le 1$, $i, j \in \{0, 1, \cdots, k-1\}$ where $ev_f(x)$ denotes the total number of vertices and edges labelled with x ($x = 0, 1, 2, \cdots, k-1$). We investigate the 4-Product cordial labeling behaviour of comb, double comb and subdivision of some corona graphs.

Key Words: Labelling, k-total product cordial labeling, Smarandachely k-total product cordial labeling, comb, double comb, crown.

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§1. Introduction

Throughout this paper we have considered finite, undirected and simple graphs only. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The graph obtained by subdividing each edge of a graph G by a new vertex is denoted by $S(G)$. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is obtained by taking one copy of G_1(which has p_1 vertices) and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy G_2 The notion of k-Total Product cordial labeling of graphs was introduced in [2]. In this paper we investigate the 4-Total Product cordial labeling behaviour of $P_n \odot K_1$, $P_n \odot 2K_1$, $S(P_n \odot K_1)$, $S(P_n \odot 2K_1)$, $S(C_n \odot K_1)$ and $S(C_n \odot 2K_1)$ Terms not defined here are used in the sense of Harary [1].

§2. k-Total Product Cordial Labeling

Definition 2.1 *Let f be a map from $V(G)$ to $\{0, 1, \cdots, k-1\}$ where k is an integer, $2 \le k \le |V(G)|$. For each edge uv, assign the label $f(u)f(v) \pmod k$. f is called a k-total product cordial labeling of G if $|ev_f(i) - ev_f(j)| \le 1$, otherwise, a Smarandachely k-total product cordial labeling of G if $|ev_f(i) - ev_f(j)| \ge 2$ for $i, j \in \{0, 1, \cdots, k-1\}$, where $ev_f(x)$ denotes the total number of vertices and edges labelled with x ($x = 0, 1, 2, \cdots, k-1$).*

A graph with k-total product cordial labeling is called k-total product cordial graph.

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Now we investigate the 4-Total product cordiality of $P_n \odot K_1$ and $P_n \odot 2K_1$.

Theorem 2.2 *$P_n \odot K_1$ is 4-total product cordial.*

Proof Let $u_1 u_2 \cdots u_n$ be the path P_n and let v_i be the pendant vertices adjacent to u_i $(1 \leq i \leq n)$.

Case 1. n is even.

Define $f : V(P_n \odot K_1) \to \{0, 1, 2, 3\}$ by $f(u_1) = 0$,

$$\begin{array}{lcll} f(u_i) & = & 2, & 2 \leq i \leq \frac{n-2}{2} \\ f(u_{\frac{n-2}{2}+i}) & = & 3, & 1 \leq i \leq \frac{n}{2} \\ f(v_i) & = & 2 & 1 \leq i \leq \frac{n}{2} \\ f(v_{\frac{n}{2}+i}) & = & 3, & 1 \leq i \leq \frac{n}{2}. \end{array}$$

Clearly $ev_f(0) = ev_f(2) = ev_f(3) = n$ and $ev_f(1) = n - 1$. Hence f is a 4-total product cordial labeling.

Case 2. n is odd.

Define $f : V(P_n \odot K_1) \to \{0, 1, 2, 3\}$ by $f(u_1) = f(u_2) = 0$,

$$\begin{array}{lcll} f(u_i) & = & 2, & 3 \leq i \leq \frac{n-1}{2} \\ f(u_{\frac{n-1}{2}+i}) & = & 3, & 1 \leq i \leq \frac{n-1}{2} \\ f(v_i) & = & 2, & 1 \leq i \leq \frac{n+1}{2} \\ f(v_{\frac{n+1}{2}+i}) & = & 3, & 1 \leq i \leq \frac{n-1}{2}. \end{array}$$

Values of i	$ev_f(i)$
0	n
1	$n-1$
2	n
3	n

Table 1

Table 1 establish that f is a 4-total product cordial labeling. □

Theorem 2.3 *$P_n \odot 2K_1$is 4-total product cordial.*

Proof Let $u_1 u_2 \ldots u_n$ be the path P_n and let v_i and w_i be the pendant vertices adjacent to u_i $(1 \leq i \leq n)$.

Case 1. n is even.

Define $f : V(P_n \odot 2K_1) \to \{0,1,2,3\}$ by $f(u_1) = 0$,

$$\begin{array}{lcll} f(u_i) & = & 2, & 2 \le i \le \frac{n-2}{2} \\ f(u_{\frac{n-2}{2}+i}) & = & 3 & 1 \le i \le \frac{n+2}{2} \\ f(v_i) & = & 2, & 1 \le i \le \frac{n}{2} \\ f(v_{\frac{n}{2}+i}) & = & 3 & 1 \le i \le \frac{n}{2} \\ f(w_i) & = & 2, & 1 \le i \le \frac{n}{2} \\ f(w_{\frac{n}{2}+i}) & = & 3, & 1 \le i \le \frac{n}{2}. \end{array}$$

Clearly $ev_f(0) = ev_f(2) = ev_f(3) = \frac{3n}{2}$ and $ev_f(1) = \frac{3n}{2} - 1$. Hence f is a 4-total product cordial labeling.

Case 2. n is odd.

Define $f : V(P_n \odot 2K_1) \to \{0,1,2,3\}$ by $f(u_1) = f(u_2) = 0$,

$$\begin{array}{lcll} f(u_i) & = & 2, & 3 \le i \le \frac{n-1}{2} \\ f(u_{\frac{n-1}{2}+i}) & = & 3, & 1 \le i \le \frac{n}{2} \\ f(v_i) & = & 2, & 1 \le i \le \frac{n+1}{2} \\ f(v_{\frac{n-1}{2}+i}) & = & 3, & 1 \le i \le \frac{n-1}{2} \\ f(w_i) & = & 2, & 1 \le i \le \frac{n-1}{2} \\ f(w_{\frac{n-1}{2}+i}) & = & 3, & 1 \le i \le \frac{n+1}{2}. \end{array}$$

Values of i	$ev_f(i)$
0	$\frac{3n-1}{2}$
1	$\frac{3n-1}{2}$
2	$\frac{3n-1}{2}$
3	$\frac{3n+1}{2}$

Table 2

Table 2 shows that f is a 4-total product cordial labeling. □

Now we look in to the subdivision graphs.

Theorem 2.4 *$S(P_n \odot K_1)$ is 4-total product cordial.*

Proof Let $V(S(P_n \odot K_1)) = \{u_i, v_i, w_i, z_j : 1 \le i \le n,\ 1 \le j \le n-1\}$ and $E(S(P_n \odot K_1)) = \{u_iv_i, v_iw_i, u_iz_j, z_ju_{j+1} : 1 \le i \le n,\ 1 \le j \le n-1\}$.

Case 1. $n \equiv 0 (\text{mod } 4)$.

Let $n = 4t$. Define $f(u_1) = 0$,

$$\begin{array}{lcl} f(u_i) & = & 2, \quad 2 \leq i \leq \frac{n-2}{2} \\ f(u_{\frac{n-2}{2}+i}) & = & 3, \quad 1 \leq i \leq \frac{n}{2} \\ f(v_i) & = & 2 \quad 1 \leq i \leq \frac{n}{2} \\ f(v_{\frac{n}{2}+i}) & = & 3, \quad 1 \leq i \leq \frac{n}{2} \\ f(w_i) & = & 2 \quad 1 \leq i \leq \frac{n}{2} \\ f(w_{\frac{n}{2}+i}) & = & 3, \quad 1 \leq i \leq \frac{n}{2} \\ f(z_i) & = & 2, \quad 1 \leq j \leq \frac{n-2}{2} \\ f(z_{\frac{n-2}{2}+i}) & = & 3, \quad 1 \leq j \leq \frac{n}{2}. \end{array}$$

Clearly $ev_f(0) = ev_f(1) = ev_f(2) = 4t - 1$ and $ev_f(3) = 4t$. Hence f is a 4-total product cordial labeling.

Case 2. $n \equiv 1(\text{mod } 4)$.

Let $n = 4t + 1$. Assign the label to the vertices u_i, v_i, w_i, z_j $1 \leq i \leq n-1,\ 1 \leq j \leq n-1$ as in case 1. Then label $3, 3, 2, 0$ to the vertices $z_n,\ u_n,\ v_n,\ w_n$ respectively. Here $ev_f(0) = ev_f(1) = ev_f(2) = 4t + 1$ and $ev_f(3) = 4t + 2$. Hence f is a 4-total product cordial labeling.

Case 3. $n \equiv 2(\text{mod } 4)$.

Let $n = 4t + 2$. Assign the label to the vertices u_i, v_i, w_i, z_j $1 \leq i \leq n-1,\ 1 \leq j \leq n-1$ as in case 2. Then label $3, 3, 2, 0$ to the vertices $z_n,\ u_n,\ v_n,\ w_n$ respectively. Here $ev_f(0) = ev_f(1) = ev_f(2) = 4t + 3$ and $ev_f(3) = 4t + 4$. Hence f is a 4-total product cordial labeling.

Case 4. $n \equiv 3(\text{mod } 4)$.

Let $n = 4t + 3$. Assign the label to the vertices u_i, v_i, w_i, z_j $1 \leq i \leq n-1,\ 1 \leq j \leq n-1$ as in case 3. Then label $3, 3, 2, 0$ to the vertices $z_n,\ u_n,\ v_n,\ w_n$ respectively. Here $ev_f(0) = ev_f(1) = ev_f(2) = 4t + 5$ and $ev_f(3) = 4t + 6$. Hence f is a 4-total product cordial labeling. □

Theorem 2.5 *$S(P_n \odot 2K_1)$ is 4-total product cordial.*

Proof Let $V(S(P_n \odot 2K_1)) = \{u_i, v_i, w_i, a_j, b_i, c_i : 1 \leq i \leq n,\ 1 \leq j \leq n-1\}$ and $E(S(P_n \odot 2K_1)) = \{u_ia_j, u_ib_i, u_ic_i, b_iv_i, c_iw_i, a_ju_{j+1} : 1 \leq i \leq n,\ 1 \leq j \leq n-1\}$.

Case 1. $n \equiv 0(\text{mod } 4)$.

Let $n = 4t$ and let $f(u_1) = 0$,

$$\begin{array}{lcl} f(u_i) & = & 2, \quad 2 \leq i \leq \frac{n-2}{2} \\ f(u_{\frac{n-2}{2}+i}) & = & 3, \quad 1 \leq i \leq \frac{n}{2} \\ f(v_i) & = & 2, \quad 1 \leq i \leq \frac{n}{2} \\ f(v_{\frac{n}{2}+i}) & = & 3, \quad 1 \leq i \leq \frac{n}{2} \end{array}$$

$$\begin{aligned}
f(w_i) &= 2, \quad 1 \le i \le \tfrac{n}{2}\\
f(w_{\frac{n}{2}+i}) &= 3, \quad 1 \le i \le \tfrac{n}{2}\\
f(a_j) &= 2, \quad 1 \le j \le \tfrac{n-2}{2}\\
f(a_{\frac{n-2}{2}+1}) &= 1\\
f(a_{\frac{n-2}{2}+1+j}) &= 1, \quad 1 \le j \le \tfrac{n-2}{2}\\
f(b_i) &= 2, \quad 1 \le i \le \tfrac{n}{2}\\
f(b_{\frac{n}{2}+i}) &= 3, \quad 1 \le i \le \tfrac{n}{2}\\
f(c_i) &= 2, \quad 1 \le i \le \tfrac{n}{2}\\
f(c_{\frac{n}{2}+i}) &= 3, \quad 1 \le i \le \tfrac{n}{2}.
\end{aligned}$$

Clearly $ev_f(0) = ev_f(1) = ev_f(2) = 4t+7$ and $ev_f(3) = 4t+8$. Hence f is a 4-total product cordial labeling.

Case 2. $n \equiv 1(\mathrm{mod}\ 4)$.

Let $n = 4t+1$ and assign the label to the vertices $u_i, v_i, w_i, a_j, b_i, c_i$ $1 \le i \le n-1,\ 1 \le j \le n-2$ as in case 1. Then label $3, 3, 2, 2, 1, 0$ to the vertices $a_n,\ u_n,\ b_n,\ v_n,\ c_n,\ w_n$ respectively. Here $ev_f(0) = ev_f(1) = ev_f(2) = 4t+10$ and $ev_f(3) = 4t+11$. Hence f is a 4-total product cordial labeling.

Case 3. $n \equiv 2(\mathrm{mod}\ 4)$.

Let $n = 4t+2$. Assign the label to the vertices $u_i, v_i, w_i, a_j, b_i, c_i$ $1 \le i \le n-2,\ 1 \le j \le n-3$ as in case 2. Then label $3, 3, 2, 2, 2, 2, 3, 3, 2, 3, 0, 3$ to the vertices $a_{n-2}, u_{n-1}, b_{n-1}, v_{n-1}, c_{n-1}, w_{n-1}$, $a_{n-1},\ u_n,\ b_n,\ v_n,\ c_n,\ w_n$ respectively. Here $ev_f(0) = ev_f(1) = ev_f(2) = 4t+13$ and $ev_f(3) = 4t+14$. Hence f is a 4-total product cordial labeling.

Case 4. $n \equiv 3(\mathrm{mod}\ 4)$.

Let $n = 4t+3$. We assign the label to the vertices $u_i, v_i, w_i, a_j, b_i, c_i$ $1 \le i \le n-3,\ 1 \le j \le n-4$ as in case 3. Then label $3, 3, 2, 0, 2, 2, 3, 3, 2, 2, 2, 2, 3, 3, 3, 2, 3, 3$ to the vertices $a_{n-3}, u_{n-2}, b_{n-2}, v_{n-2}, c_{n-2}, w_{n-2},\ a_{n-2},\ u_{n-1},\ b_{n-1},\ v_{n-1},\ c_{n-1},\ w_{n-1},\ a_{n-1},\ u_n,\ b_n,\ v_n,\ c_n,$ w_n respectively. Here $ev_f(0) = ev_f(1) = ev_f(2) = 4t+22$ and $ev_f(3) = 4t+23$. Hence f is a 4-total product cordial labeling. □

Theorem 2.6 *$S(C_n \odot K_1)$ is 4-total product cordial.*

Proof Let $V(S(C_n \odot K_1)) = \{u_i, v_i, w_i, z_i : 1 \le i \le n\}$ and $E(S(C_n \odot K_1)) = \{u_i z_i, u_i w_i, w_i v_j, z_i u_{i+1} : 1 \le i \le n\}$.

Case 1. $n \equiv 0, 2(\mathrm{mod}\ 4)$.

Let $n = 4t$ and let $f(u_1) = 0$,

$$\begin{array}{lllll} f(u_i) & = & f(z_i) & = 3 & 1 \leq i \leq n \\ f(v_i) & = & 2 & & 1 \leq i \leq n \\ f(w_i) & = & 2 & & 1 \leq i \leq \frac{n}{2} \\ f(v_{\frac{n}{2}+i}) & = & 0 & & 1 \leq i \leq \frac{n}{2}. \end{array}$$

In this case, $ev_f(0) = ev_f(1) = ev_f(2) = ev_f(3) = 2n$. Hence f is a 4-total product cordial labeling.

Case 2. $n \equiv 1 (\text{mod } 4)$.

Let $n = 4t + 1$. We assign the label to the vertices $u_i, v_i, w_i, z_i,\ 1 \leq i \leq n - 1$ as in case 1. Then label $3, 3, 2, 0$ to the vertices u_n, z_n, w_n, v_n respectively. Hence $ev_f(0) = ev_f(1) = ev_f(2) = ev_f(3) = 2n$. Hence f is a 4-total product cordial labeling.

Case 3. $n \equiv 3 (\text{mod } 4)$.

Let $n = 4t + 3$ and assign the label to the vertices $u_i, v_i, w_i, z_i,\ 1 \leq i \leq n - 1$ as in case 1. Then label $3, 3, 2, 0$ to the vertices u_n, z_n, w_n, v_n respectively. Hence $ev_f(0) = ev_f(1) = ev_f(2) = ev_f(3) = 2n$. Therefore f is a 4-total product cordial labeling. □

Theorem 2.7 *$S(C_n \odot 2K_1)$ is 4-total product cordial.*

Proof Let $V(S(C_n \odot 2K_1)) = \{u_i, v_i, w_i, a_i, b_i, c_i : 1 \leq i \leq n,\}$ and $E(S(C_n \odot 2K_1)) = \{u_i u_{i+1 \ (\text{mod } n)}, u_i a_i, u_i b_i, b_i v_i, u_i c_i, c_j w_i : 1 \leq i \leq n\}$.

Case 1. $n \equiv 0 (\text{mod } 4)$

Define

$$\begin{array}{lllll} f(u_i) & = f(a_i) & = & 3 & 1 \leq i \leq n \\ f(v_i) & = f(b_i) & = & 2 & 1 \leq i \leq \frac{n}{2} \\ f(b_{\frac{n}{2}+i}) & = f(v_{\frac{n}{2}+i}) & = & 0 & 1 \leq i \leq \frac{n}{4} \\ f(b_{\frac{3n}{4}+i}) & = f(v_{\frac{3n}{4}+i}) & = & 0 & 1 \leq i \leq \frac{n}{4} \\ f(w_i) & = f(c_i) & = & 2 & 1 \leq i \leq \frac{n}{2} \\ f(c_{\frac{n}{2}+i}) & = f(w_{\frac{n}{2}+i}) & = & 0 & 1 \leq i \leq \frac{n}{4} \\ f(c_{\frac{3n}{4}+i}) & = f(w_{\frac{3n}{4}+i}) & = & 0 & 1 \leq i \leq \frac{n}{4} \end{array}$$

Therefore $ev_f(0) = ev_f(1) = ev_f(2) = ev_f(3) = 3n$. Hence f is a 4-total product cordial labeling.

Case 2. $n \equiv 1 (\text{mod } 4)$

Let $n = 4t + 1$. We assign the label to the vertices $u_i, v_i, w_i, a_i, b_i, c_i$ $1 \leq i \leq n - 1$ as in case 1. Then label $3, 3, 3, 2, 2, 2$ to the vertices $u_n,\ a_n,\ b_n,\ v_n,\ w_n,\ c_n$ respectively. Hence $ev_f(0) = ev_f(1) = ev_f(2) = ev_f(3) = 3n$. Hence f is a 4-total product cordial labeling.

Case 3. $n \equiv 2(\bmod 4)$

Let $n = 4t + 2$. Define

$$\begin{array}{lllll} f(u_i) & = & f(a_i) & = & 3 \quad 1 \le i \le n \\ f(v_i) & = & f(b_i) & = & 2 \quad 1 \le i \le \frac{n}{2} \\ f(b_{\frac{n}{2}+i}) & = & f(v_{\frac{n}{2}+i}) & = & 0 \quad 1 \le i \le \frac{n}{2} - 2 \\ f(b_{n-2+i}) & = & f(v_{n-2+i}) & = & 3 \quad 1 \le i \le \frac{n}{2} - 3 \\ f(w_i) & = & f(c_i) & = & 2 \quad 1 \le i \le \frac{n}{2} \\ f(c_{\frac{n}{2}+i}) & = & f(w_{\frac{n}{2}+i}) & = & 0 \quad 1 \le i \le \frac{n}{2} - 3 \\ f(c_{n-3+i}) & = & f(w_{n-3+i}) & = & 3 \quad 1 \le i \le \frac{n}{2} - 2. \end{array}$$

Therefore $ev_f(0) = ev_f(1) = ev_f(2) = ev_f(3) = 3n$. Hence f is a 4-total product cordial labeling.

Case 4. $n \equiv 3(\bmod 4)$

Let $n = 4t + 3$ and let

$$\begin{array}{lllll} f(u_i) & = & f(a_i) & = & 3 \quad 1 \le i \le n \\ f(v_i) & = & f(b_i) & = & 2 \quad 1 \le i \le \frac{n-1}{2} \\ f(v_{\frac{n+1}{2}}) & = & 2 & & \\ f(w_{\frac{n+1}{2}}) & = & 0 & & \\ f(v_n) & = & 3 & & \\ f(w_n) & = & 2 & & \\ f(v_{\frac{n+1}{2}+i}) & = & f(w_{\frac{n+1}{2}+i}) & = & 0 \quad 1 \le i \le \frac{n-3}{4} \\ f(v_{\frac{3n-1}{4}+i}) & = & f(w_{\frac{3n-1}{4}+i}) & = & 3 \quad 1 \le i \le \frac{n-3}{4} \\ f(b_i) & = & f(c_i) & = & 2 \quad 1 \le i \le \frac{n-1}{2} \\ f(b_{\frac{n-1}{2}+i}) & = & f(c_{\frac{n-1}{2}+i}) & = & 0 \quad 1 \le i \le \frac{n+1}{4} \\ f(b_{\frac{3n-1}{4}+i}) & = & f(c_{\frac{3n-1}{4}+i}) & = & 3 \quad 1 \le i \le \frac{n+1}{4} \end{array}$$

Therefore $ev_f(0) = ev_f(1) = ev_f(2) = ev_f(3) = 3n$. Hence f is a 4-total product cordial labeling. □

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On m-Neighbourly Irregular Instuitionistic Fuzzy Graphs

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Abstract: In this paper, m-neighbourly irregular intuitionistic fuzzy graphs and m-neighbourly totally irregular intuitionistic fuzzy graphs are defined. Relation between m-neighbourly irregular intuitionistic fuzzy graph and m-neighbourly totally irregular intuitionistic fuzzy graph are discussed. An m-neighbourly irregularity on intuitionistic fuzzy graphs whose underlying crisp graphs are cycle C_n, a path P_n are studied.

Key Words: d_m-degree and total d_m-degree of a vertex in intuitionistic fuzzy graph, irregular intuitionistic fuzzy graph, neighbourly irregular intuitionistic fuzzy graph, neighbourly totally irregular intuitionistic fuzzy graph.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

In 1965, Lofti A. Zadeh [20] introduced the concept of fuzzy subset of a set as method of representing the phenomena of uncertainty in real life situation. K.T.Attanassov [1] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. K.T.Atanassov added a new component(which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set(and the non-membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than one. Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry and economics [1,2].

Azriel Rosenfeld introduced the concept of fuzzy graph in 1975 ([12]). It has been growing fast and has numerous application in various fields. Bhattacharya [5] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Morderson and Peng [11]. Krassimir T Atanassov [2] introduced the intuitionistic fuzzy graph theory. R.Parvathi and M.G.Karunambigai [10] introduced intuitionistic fuzzy graphs as a special case of Atanassov's

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IFG and discussed some properties of regular intuitionistic fuzzy graphs [7]. M. G. Karunambigai and R. Parvathi and R. Buvaneswari introduced constant intuitionistic fuzzy graphs [8].

M. Akram, W. Dudek [3] introduced the regular intuitionistic fuzzy graphs. M.Akram and Bijan Davvaz [4] introduced the notion of strong intuitionistic fuzzy graphs and discussed some of their properties. R.Jahirhussain and S.Yahyu Mohammed discussed Properties on intuitionistic fuzzy graphs [6]. A.Nagoorgani and S.Shajitha Begum introduced the degree, order and size in intuitionistic fuzzy graphs [9].

N.R.Santhi Maheswari and C.Sekar introduced d_2- degree of a vertex in fuzzy graphs and introduced 2-neighbourly irregular fuzzy graphs and 2-neighbourly totally irregular fuzzy graphs [13]. Also, they introduced d_m-degree, total d_m-degree, of a vertex in fuzzy graphs and introduced an m-neighbourly irregular fuzzy graphs [14, 17]. S.Ravinarayanan and N.R.Santhi Maheswari introduced m-neighbourly irreular bipolar fuzzy graphs [15].

N.R.Santhi Maheswari and C.Sekar introduced d_m- degree of a vertex in intuitionistic fuzzy graphs and introduced $(m,(c_1,c_2))$-regular fuzzy graphs and totally $(m,(c_1,c_2))$-regular fuzzy graphs [19]. These motivates us to introduce m-neighbourly irregular intuitionistic fuzzy graphs and totally m-neighbourly irregular intuitionistic fuzzy graphs.

§2. Preliminaries

We present some known definitions related to fuzzy graphs and intuitionistic fuzzy graphs for ready reference to go through the work presented in this paper.

Definition 2.1([11]) *A fuzzy graph $G:(\sigma,\mu)$ is a pair of functions (σ,μ), where $\sigma: V\rightarrow$[0,1] is a fuzzy subset of a non empty set V and $\mu: VXV\rightarrow$[0, 1] is a symmetric fuzzy relation on σ such that for all u,v in V, the relation $\mu(u,v)\leq\sigma(u)\wedge\sigma(v)$ is satisfied. A fuzzy graph G is called complete fuzzy graph if the relation $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ is satisfied.*

Definition 2.2([14]) *Let $G:(\sigma,\mu)$ be a fuzzy graph. The d_m-degree of a vertex u in G is $d_m(u)=\sum\mu^m(uv)$, where $\mu^m(uv)=sup\{\mu(uu_1)\wedge\mu(u_1u_2)\wedge\ldots,\mu(u_{m-1}v): u,u_1,u_2,\ldots,u_{m-1},v$ is the shortest path connecting u and v of length $m\}$. Also, $\mu(uv)=0$, for uv not in E.*

Definition 2.3([14]) *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. The total d_m-degree of a vertex $u\in V$ is defined as $td_m(u)=\sum\mu^m(uv)+\sigma(u)=d_m(u)+\sigma(u)$.*

Definition 2.4([14]) *Let $G:(V,E)$ be a fuzzy graph on $G^*(V,E)$. Then G is said to be an m-neighbourly irregular fuzzy graph if every two adjacent vertices in G have distinct d_m-degrees.*

Definition 2.5([14]) *Let $G:(V,E)$ be a bipolar fuzzy graph on $G^*(V,E)$. Then G is said to be an m-neighbourly totally irregular fuzzy graph if every two adjacent vertices in G have distinct total d_m-degrees.*

Definition 2.6([8]) *An intuitionistic fuzzy graph with underlying set V is defined to be a pair $G=(V,E)$ where*

(i) $V=\{v_1,v_2,v_3,\cdots,v_n\}$ such that $\mu_1: V\rightarrow[0,1]$ and $\gamma_1: V\rightarrow[0,1]$ denote the

degree of membership and nonmembership of the element $v_i \in V$, $(i = 1, 2, 3, \cdots, n)$ *such that* $0 \le \mu_1(v_i) + \gamma_1(v_i) \le 1$;

(*ii*) $E \subseteq V \times V$, *where* $\mu_2 : V \times V \to [0,1]$ *and* $\gamma_2 : V \times V \to [0,1]$ *are such that* $\mu_2(v_i, v_j) \le min\{\mu_1(v_i), \mu_1(v_j)\}$ *and* $\gamma_2(v_i, v_j) \le \max\{\gamma_1(v_i), \gamma_1(v_j)\}$ *and* $0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \le 1$ *for every* $(v_i, v_j) \in E$, $(i, j = 1, 2, \cdots, n)$.

Definition 2.7([8]) *If* $v_i, v_j \in V \subseteq G$, *the* μ*-strength of connectedness between two vertices* v_i *and* v_j *is defined as* $\mu_2^{\infty}(v_i, v_j) = \sup\{\mu_2^k(v_i, v_j) : k = 1, 2, \cdots, n\}$ *and* γ*-strength of connectedness between two vertices* v_i *and* v_j *is defined as* $\gamma_2^{\infty}(v_i, v_j) = \inf\{\gamma_2^k(v_i, v_j) : k = 1, 2, \cdots, n\}$.

If u *and* v *are connected by means of paths of length* k *then* $\mu_2^k(u, v)$ *is defined as* $\sup\{\mu_2(u, v_1) \wedge \mu_2(v_1, v_2) \wedge \cdots \wedge \mu_2(v_{k-1}, v) \colon (u, v_1, v_2, \cdots, v_{k-1}, v) \in V\}$ *and* $\gamma_2^k(u, v)$ *is defined as* $\inf\{\gamma_2(u, v_1) \vee \gamma_2(v_1, v_2) \vee \cdots \vee \gamma_2(v_{k-1}, v) \colon (u, v_1, v_2, \cdots, v_{k-1}, v) \in V\}$.

Definition 2.8([8]) *Let* $G = (V, E)$ *be an Intuitionistic fuzzy graph on* $G^*(V, E)$. *Then the degree of a vertex* $v_i \in G$ *is defined by* $d(v_i) = (d_{\mu_1}(v_i), d_{\gamma_1}(v_i))$, *where* $d_{\mu_1}(v_i) = \sum \mu_2(v_i, v_j)$ *and* $d_{\gamma_1}(v_i) = \sum \gamma_2(v_i, v_j)$ *for* $v_i, v_j \in E$ *and* $\mu_2(v_i, v_j) = 0$ *and* $\gamma_2(v_i, v_j) = 0$ *for* $v_i, v_j \notin E$.

Definition 2.9([8]) *Let* $G = (V, E)$ *be an Intuitionistic fuzzy graph on* $G^*(V, E)$. *Then the total degree of a vertex* $v_i \in G$ *is defined by* $td(v_i) = (td_{\mu_1}(v_i), td_{\gamma_1}(v_i))$, *where* $td_{\mu_1}(v_i) = d\mu_1(v_i) + \mu_1(v_i)$ *and* $td_{\gamma_1}(v_i) = d\gamma_1(v_i) + \gamma_1(v_i)$.

Definition 2.10([19]) *Let* $G = (V, E)$ *be an intuitionistic fuzzy graph on* $G^*(V, E)$. *Then the* d_m *- degree of a vertex* $v \in G$ *is defined by* $d_{(m)}(v) = (d_{(m)\mu_1}(v), d_{(m)\gamma_1}(v))$, *where* $d_{(m)\mu_1}(v) = \sum \mu_2^{(m)}(u, v)$ *and* $\mu_2^{(m)}(u, v) = \sup\{\mu_2(u, u_1) \wedge \mu_2(u_1, u_2) \wedge \cdots \wedge \mu_2(u_{m-1}, v) : u, u_1, u_2, \cdots, u_{m-1}v$ *is the shortest path connecting* u *and* v *of length* $m\}$ *and* $d_{(m)\gamma_1}(v) = \sum \gamma_2^{(m)}(u, v)$, *where* $\gamma_2^{(m)}(u, v) = \inf\{\gamma_2(u, u_1) \vee \gamma_2(u_1, u_2) \vee \cdots \vee \gamma_2(u_{m-1}, v) : u, u_1, u_2, \cdots, u_{m-1}v$ *is the shortest path connecting* u *and* v *of length* $m\}$. *The minimum* d_m*-degree of* G *is* $\delta_m(G) = \wedge\{(d_{(m)\mu_1}(v), d_{(m)\gamma_1}(v)) : v \in V\}$ *and the maximum* d_m*-degree of* G *is* $\Delta_m(G) = \vee\{(d_{(m)\mu_1}(v), d_{(m)\gamma_1}(v)) : v \in V\}$.

Definition 2.11([19]) *Let* $G : (V, E)$ *be an intuitionistic fuzzy graph on* $G^*(V, E)$. *If all the vertices of* G *have same* d_m*- degree then* G *is said to be an* $(m, (c_1, c_2))$*- regular intuitionistic fuzzy graph.*

Definition 2.12([19]) *Let* $G = (V, E)$ *be an intuitionistic fuzzy graph on* $G^*(V, E)$. *Then the total* d_m*-degree of a vertex* $v \in G$ *is defined by* $td_{(m)}(v) = (td_{(m)\mu_1}(v), td_{(m)\gamma_1}(v))$, *where* $td_{(m)\mu_1}(v) = d_{(m)\mu_1}(v) + \mu_1(v)$ *and* $td_{(m)\gamma_1}(v) = d_{(m)\gamma_1}(v) + \gamma_1(v)$. *The minimum* td_m*-degree of* G *is* $t\delta_m(G) = \wedge\{(td_{(m)\mu_1}(v),\ td_{(\gamma_1}(v)) : v \in V\}$. *The maximum* td_m*-degree of* G *is* $t\Delta_m(G) = \vee\{(td_{(m)\mu_1}(v),\ td_{(m)\gamma_1}(v)) : v \in V\}$.

§3. m-Neighbourly Irregular intuitionistic Fuzzy Graphs

Definition 3.1 *let* G *be an intuitionistic fuzzy graph on* $G^*(V, E)$. *Then* G *is said to be an* m*-neighbourly irregular intuitionistic fuzzy graph if every two adjacent vertices in* G *have distinct*

d_m- degrees.

Example 3.2 Consider an intuitionistic fuzzy graph on $G^* : (V, E)$.

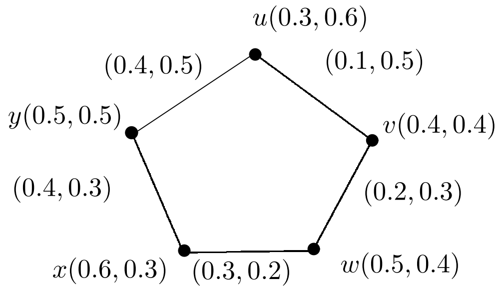

Figure 1

Here, $d(u) = (0.5, 1)$; $d(v) = (0.3, 0.8)$; $d(w) = (0.5, 0.5)$; $d(x) = (0.7, .5)$; $d(y) = (0.8, 0.8)$; $d_{(2)\mu_1}(u) = (0.1 \wedge 0.2) + (0.4 \wedge 0.4) = 0.1 + 0.4 = 0.5$; $d_{(2)\gamma_1}(u) = (0.5 \vee 0.2) + (0.4 \vee 0.4) = (0.5) + (0.4) = 0.9$; $d_{(2)}(u) = (0.5, 0.9)$; $d_{(2)}(v) = (0.3, 0.8)$; $d_{(2)}(w) = (0.4, 0.8)$; $d_{(2)}(x) = (0.6, 0.8)$; $d_{(2)}(y) = (0.4, 0.8)$.

Every pair of adjacent vertices in G have distinct degrees and distinct d_2- degrees. Hence G is m -neighbourly irregular intuitionistic fuzzy graph for m=1,2.

§4. m-Neighbourly Totally Irregular Intuitionistic Fuzzy Graphs

Definition 4.1 *Let G be a intuitionistic fuzzy graph on $G^*(V, E)$. Then G is said to be m-neighbourly totally irregular intuitionistic fuzzy graph if every two adjacent vertices in G have distinct total d_m- degrees.*

Example 4.2 Consider a intuitionistic fuzzy graph on $G^* : (V, E)$ in Figure 1, $td_2(u) = (0.8, 1.5), td_2(v) = (0.7, 1.2), td_2(w) = (0.9, 1.2), td_2(x) = (1.2, 1.1), td_2(y) = (0.9, 1.3)$. Every Pair of adjacent vertices in G have distinct total degrees and distinct total d_2- degrees. Hence G is m-neighbourly totally irregular intuitionistic fuzzy graph for m=1,2.

Remark 4.3 An m-neighbourly irregular instuitionistic fuzzy graph need not be m- neighbourly totally irregular intuitionistic fuzzy graph.

Example 4.4 For example consider $G = (V, E)$ be an intuitionistic fuzzy graph such that $G^*(V, E)$ is path on 6 vertices.

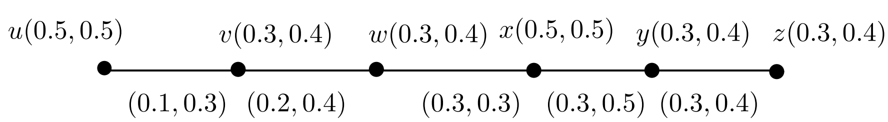

Figure 2

Here, $d_{(3)}(u) = (0.1, 0.4)$; $d_{(3)}(v) = (0.2, 0.5)$; $d_{(3)}(w) = (0.3, 0.5)$;$d_{(3)}(x) = (0.1, 0.4)$;$d_{(3)}(y) = (0.2, 0.5)$;$d_{(3)}(z) = (0.3, 0.5)$. Hence G is m- neighbourly irregular intuitionistic fuzzy graph. Here, $td_{(3)}(w) = td_{(3)}(x) = td_{(3)}(y) = (0.6, 0.9)$. Hence G is not m-neighbourly totally irregular intuitionistic fuzzy graph.

Remark 4.5 An m- neighbourly totally irregular intuitionistic fuzzy graph need not be m-neighbourly irregular intuitionistic fuzzy graph.

Example 4.6 Consider a intuitionistic fuzzy graph on $G^*(V, E)$.

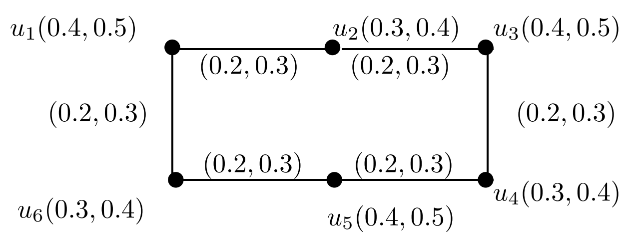

Figure.3

Here, $d_{(2)}(u_1) = d_{(2)}(u_2) = d_{(2)}(u_3) = d_{(2)}(u_4) = d_{(2)}(u_5) = d_{(2)}(u_6) = (0.4, 0.6)$. Hence G is not an m- neighbourly irregular intuitionistic fuzzy graph.

$td_{(2)}(u_1) = (0.8, 1.1)$; $td_{(2)}(u_2) = (0.7, 1)$; $td_{(2)}(u_3) = (0.8, 1.1)$; $td_{(2)}(u_4) = (0.7, 1)$; $td_{(2)}(u_5) = (0.8, 1.1)$; $td_{(2)}(u_6) = (0.7, 1)$.

Here, all adjacent vertices have distinct total d_m-degrees. Hence G is m-neighbourly totally irregular intuitionistic fuzzy graph.

Theorem 4.7 *If the membership values of adjacent vertices are distinct then $(m, (c_1, c_2))$-regular intuitionistic fuzzy graph is an m- neighbourly totally irregular intuitionistic fuzzy graph.*

Proof Let $G : (V, E)$ be an intuitionistic fuzzy graph on $G^*(V, E)$. If $(m, (c_1, c_2))$- regular intuitionistic fuzzy graph and the membership values of adjacent vertices are distinct, then d_m-degree of all vertices are the same $\Rightarrow d_m(v) = (c_1, c_2)$ for all $v \in G \Rightarrow$ total degrees of adjacent vertices are distinct. So G is an m-neighbourly totally irregular intuitionistic fuzzy graph. $\square$

Theorem 4.8 *Let $G : (V, E)$ be an intuitionistic fuzzy graph on $G^*(V, E)$. If G is an m-neighbourly irregular intuitionistic fuzzy graph and A is a constant function then G is an m-neighbourly totally irregular intuitionistic fuzzy graph.*

Proof Let G be m- neighbourly irregular intuitionistic fuzzy graph. Then the d_m degree of every two adjacent vertices are distinct. Let u and v be two adjacent vertices of G with distinct degrees. Then $d_m(u) = (k_1, k_2)$ and $d_m(v) = (c_1, c_2)$, where $k_1 \neq c_1, k_2 \neq c_2$. Assume that $A(u) = A(v) = (r_1, r_2)$. Suppose $td_m(u) = td_m(v) \Rightarrow d_m(u) + A(u) = d_m(v) + A(v)$ $\Rightarrow (k_1, k_2 + (r_1, r_2) = (c_1, c_2) + (r_1, r_2) \Rightarrow (k_1 + r_1, k_2 + r_2 = (c_1 + r_1, c_2 + r_2) \Rightarrow k_1 + r_1 = c_1 + r_1$

and $k_2 + r_2 = c_2 + r_2 \Rightarrow k_1 = c_1$ and $k_2 = c_2$, which is a contradiction. So $td_m(u) \neq td_m(v)$. Thus every pair of adjacent vertices have distinct total $d_m$degree provided A is a constant function. This is true for every pair of adjacent vertices in G. Hence G is an m- neighbourly totally irregular intuitionistic fuzzy graph. □

Theorem 4.9 *Let G be an intuitionistic fuzzy graph on $G^*(V,E)$. If G is an m- neighbourly totally irregular intuitionistic fuzzy graph and A is constant function then G is an m- neighbourly irregular intuitionistic fuzzy graph.*

Proof The proof is similar to above theorem 4.8. □

Remark 4.10 The above two theorems jointly yield the following result. let $G : (V,E)$ be a intuitionistic fuzzy graph on $G^*(V,E)$. If A is constant function then G is m- neighbourly irregular intuitionistic fuzzy graph if and only if G is m- neighbourly totally irregular intuitionistic fuzzy graph.

Remark 4.11 Let $G : (V,E)$ be an intuitionistic fuzzy graph on $G^*(V,E)$. If G is both m-neighbourly irregular intuitionistic fuzzy graph and m- neighbourly totally irregular intuitionistic fuzzy graph then A need not be a constant function.

Theorem 4.12 *Let G be intuitionistic fuzzy graph on $G^*(V,E)$, a cycle of length n. If the edges takes positive membership values $c_1, c_2, \cdots, c_n$ and negative membership values $k_1, k_2, \cdots, k_n$ such that $c_1 < c_2 < \cdots < c_n$ and $k_1 > k_2 > \cdots > k_n$ then G is m- neighbourly irregular intuitionistic fuzzy graph.*

Proof Let the edges take membership values $c_1, c_2, \cdots, c_n$ and $k_1, k_2, \cdots, k_n$ such that $c_1 < c_2 < \cdots < c_n$ and $k_1 > k_2 > \cdots > k_n$. Then,

$$\begin{aligned}
d_{m\mu_1}(v_1) &= \mu_2(e_1) \wedge \mu_2(e_2) \wedge \cdots \mu_2(e_m) + \mu_2(e_{m+1}) \wedge \mu_2(e_{n-1})) \wedge \cdots \wedge \mu_2(e_{n-(m-1)}) \\
&= (c_1 \wedge c_2 \wedge \cdots \wedge c_m) + (c_n \wedge c_{n-1} \wedge \cdots \wedge c_{n-(m-1)}) \\
&= c_1 + c_{n-(m-1)}, \\
d_{(m)\gamma_1}(v_1) &= \gamma_2(e_1) \vee \gamma_2(e_2) \vee \cdots \vee \gamma_2(e_m) + \gamma_2(e_n) \vee \gamma_2(e_{n-1}) \vee \cdots \vee \gamma_2(e_{n-(m-1)}) \\
&= k_1 + k_{n-(m-1)}, \\
d_m(v_1) &= (c_1 + c_{n-(m-1)}, k_1 + k_{n-(m-1)}).
\end{aligned}$$

Similarly, $d_m(v_2) = (c_1+c_2, k_1+k_2)$. For $i = 3, 4, \cdots, n-1$, $d_{(m)\mu_1}(v_i) = \mu_2(e_i) \wedge \mu_2(e_{i+1}) \wedge \cdots \wedge \mu_2(e_{i+m}) + \mu_2(e_{i-1}) \wedge \mu_2(e_{i-2}) \wedge \cdots \mu_2(e_{n-(m-3)})$

$$\begin{aligned}
&\Rightarrow d_{(m)\mu_1}(v_i) = (c_i \wedge c_{i+1} \wedge \cdots \wedge c_{i+m}) + (c_{i-1} \wedge c_{i-2} \wedge \cdots \wedge c_{n-(m-3)}) \\
&\Rightarrow d_{(m)\mu_1}(v_i) = c_i + c_{n-(m-3)} \\
&\Rightarrow d_{(m)\gamma_1}(v_i) = \gamma_2(e_i) \vee \gamma_2(e_{i+1}) \vee \cdots \vee \gamma_2(e_{i+m}) + \gamma_2(e_{i-1}) \vee \gamma_2(e_{i-2}) \vee \cdots \vee \gamma_2(e_{n-(m-3)}) \\
&\Rightarrow d_{(m)\gamma_1}(v_i) = k_i \vee k_{i+1} \vee \cdots \vee k_{i+m} + k_{i-1} \vee k_{i-2} \vee \cdots \vee k_{n-(m-3)} \\
&\Rightarrow d_{(m)\gamma_1}(v_i) = k_i + k_{n-(m-3)} \\
&\Rightarrow d_m(v_i) = (c_i + c_{n-(m-3)}, (k_i + k_{n-(m-3)}) \\
&\Rightarrow d_{(m)\mu_1}(v_n) = \mu_2(e_n) \wedge \mu_2(e_1) \wedge \cdots \wedge \mu_2(e_{m-1}) + \mu_2(e_{n-1}) \wedge \mu_2(e_{n-2}) \wedge \cdots \wedge \mu_2(e_{n-m}) \\
&\Rightarrow d_{(m)\mu_1}(v_n) = (c_n \wedge c_1 \wedge \cdots \wedge c_{m-1}) + (c_{n-1} \wedge c_{n-2} \wedge \cdots \wedge c_{n-m}) \\
&\Rightarrow d_{(m)\mu_1}(v_n) = c_1 + c_{n-1}
\end{aligned}$$

$\Rightarrow d_{(m)\gamma_1}(v_n) = \gamma_2(e_n) \vee \gamma_2(e_1) \vee \cdots \vee \gamma_2(e_{m-1}) + \gamma_2(e_{n-1}) \vee \gamma_2(e_{n-2}) \vee \cdots \vee \gamma_2(e_{n-m})$
$\Rightarrow d_{(m)\gamma_1}(v_n) = k_n \vee k_1 \vee \cdots \vee k_{m-1} + k_{n-1} \vee k_{n-2} \vee \cdots \vee k_{n-m}$
$\Rightarrow d_{(m)\gamma_1}(v_n) = k_1 + k_{n-1}$
$\Rightarrow d_m(v_n) = (c_1 + c_{n-1}, k_1 + k_{n-1})$.

Hence G is m- neighbourly irregular intuitionistic fuzzy graph. □

Remark 4.13 The above theorem 4.12 does not hold for m- neighbourly totally irregular intuitionistic fuzzy graph.

Theorem 4.14 *Let G be an intuitionistic fuzzy Graph on $G^*(V, E)$, a path on n vertices. If the edges takes positive membership values $c_1, c_2, \ldots, c_n$ and negative membership values $k_1, k_2, \ldots, k_n$ such that $c_1 < c_2 < \cdots < c_n$ and $k_1 > k_2 > \cdots > k_n$ then G is m - neighbourly irregular intuitionistic fuzzy graph.*

Proof Let the edges take membership values $c_1, c_2, \cdots, c_n$ and $k_1, k_2, \cdots, k_n$ such that $c_1 < c_2 < \cdots < c_n$ and $k_1 > k_2 > \cdots > k_n$. Then,

$d_{(m)\mu_1}(v_1) = \mu_2(e_1) \wedge \mu_2(e_2) \wedge \cdots \wedge \mu_2(e_m) = c_1 \wedge c_2 \wedge \cdots \wedge c_m = c_1$.
$d_{(m)\gamma_1}(v_1) = \gamma_2(e_1) \vee \gamma_2(e_2) \vee \cdots \vee \gamma_2(e_m) = k_1 \vee k_2 \vee \cdots \vee k_m = k_1$.
$d_m(v_1) = (c_1, k_1)$.

Similarly, $d_m(v_2) = (c_2, k_2)$. For $i = 3, 4, \cdots, n-2$ $d_{(m)\mu_1}(v_i) = \mu_2(e_{i-1}) \wedge \mu_2(e_{i-2}) \wedge \cdots \wedge \mu_2(e_{i-m}) + \mu_2(e_i) \wedge \mu_2(e_{i+1}) \wedge \cdots \wedge \mu_2(e_{i+(m-1)}$

$\Rightarrow d_{(m)\mu_1}(v_i) = c_{i-1} \wedge c_{i-2} \wedge \cdots \wedge c_{i-m} + c_i \wedge c_{i+1} \wedge \cdots \wedge c_{i+m-1}$
$\Rightarrow d_{(m)\mu_1}(v_i) = c_{i-m} + c_i$
$\Rightarrow d_{(m)\gamma_1}(v_i) = \gamma_2(e_{i-1}) \vee \gamma_2(e_{i-2}) \vee \cdots \vee \gamma_2(e_{i-m}) + \gamma_2(e_i) \vee \gamma_2(e_{i+1}) \vee \cdots \vee \gamma_2(e_{i+(m-1)})$
$\Rightarrow d_{(m)\gamma_1}(v_i) = k_{i-1} \vee k_{i-2} \vee \cdots \vee k_{i-m} + k_i \vee k_{i+1} \vee \cdots \vee k_{i+m-1}$
$\Rightarrow d_{(m)\gamma_1}(v_i) = k_{i-m} + k_i$
$\Rightarrow d_m(v_i) = (c_{i-m} + c_i, k_{i-m} + k_i)$
$\Rightarrow d_{(m)\mu_1}(v_{n-1}) = \mu_2(e_{n-2}) \wedge \mu_2(e_{n-3}) \wedge \cdots \wedge \mu_2(e_{n-(m+1)})$
$\Rightarrow d_{(m)\mu_1}(v_{n-1}) = c_{n-2} \wedge c_{n-3} \wedge \cdots \wedge c_{n-(m+1)}$
$d_{(m)\mu_1}(v_{n-1}) = c_{n-(m+1)}$
$\Rightarrow d_{(m)\gamma_1}(v_{n-1}) = \gamma_2(e_{n-2}) \vee \gamma_2(e_{n-3}) \vee \cdots \vee \gamma_2(e_{n-(m+1)})$
$\Rightarrow d_{(m)\gamma_1}(v_{n-1}) = k_{n-2} \vee k_{n-3} \vee \cdots \vee k_{n-(m+1)} = k_{n-(m+1)}$
$\Rightarrow d_m(v_{n-1}) = (c_{n-(m+1)}, k_{n-(m+1)})$
$\Rightarrow d_{(m)\mu_1}(v_n) = \mu_2(e_{n-1}) \wedge \mu_2(e_{n-2}) \wedge \cdots \wedge \mu_2(e_{n-m})$
$d_{(m)\mu_1}(v_n) = c_{n-1} \wedge c_{n-2} \wedge \cdots \wedge c_{n-m}$
$\Rightarrow d_{(m)\mu_1}(v_n) = c_{n-m}$
$\Rightarrow d_{(m)\gamma_1}(v_n) = \gamma_2(e_{n-1}) \vee \gamma_2(e_{n-2}) \vee \cdots \vee \gamma_2(e_{n-m})$
$\Rightarrow d_{(m)\gamma_1}(v_n) = k_{n-1} \vee k_{n-2} \vee \cdots \vee k_{n-m}$
$\Rightarrow d_{(m)\gamma_1}(v_n) = k_{n-m}$
$\Rightarrow d_m(v_n) = (c - n - m, k_{n-m})$.

Hence G is m - neighbourly irregular intuitionistic fuzzy graph. □

Remark 4.15 Theorem 4.14 does not hold for m-neighbourly totally irregular intuitionistic fuzzy graph.

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Star Edge Coloring of Corona Product of Path with Some Graphs

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Abstract: A star edge coloring of a graph G is a proper edge coloring of G, such that any path of length 4 in G is not bicolored, denoted by $\chi'_{st}(G)$, is the smallest integer k for which G admits a star edge coloring with k colors. In this paper, we obtain the star edge chromatic number of $P_m \circ P_n$, $P_m \circ S_n$, $P_m \circ K_{1,n,n}$ and $P_m \circ K_{m,n}$.

Key Words: Star edge coloring, Smarandachely subgraph edge coloring, corona product, path, sunlet graph, double star and complete bipartite.

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§1. Introduction

All graphs considered in this paper are finite and simple, i.e., undirected, loopless and without multiple edges.

The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

The $n-$sunlet graph on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.

Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $2n+1$ vertices and $2n$ edges.

A star edge coloring of a graph G is a proper edge coloring where at least three distinct colors are used on the edges of every path and cycle of length four, i.e., there is neither bichro-

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matic path nor cycle of length four. The minimum number of colors for which G admits a star edge coloring is called the star edge chromatic index and it is denoted by $\chi'_{st}(G)$. Generally, a Smarandachely subgraphs edge coloring of G for $H_1, H_2, \cdots, H_m \prec G$ is such a proper edge coloring on G with at least three distinct colors on edges of each subgraph H_i, where $1 \leq i \leq m$.

The star edge coloring was initiated in 2008 by Liu and Deng [8], motivated by the vertex version (see [1, 3, 4, 6, 7, 10]). Dvořák, Mohar and Šámal [5] determined upper and lower bounds for complete graphs. Additional graph theory terminology used in this paper can be found in [2].

§2. Preliminaries

Theorem 2.1([5]) *The star chromatic index of the complete graph K_n satisfies*

$$2n\left(1+\mathcal{O}(n)\right) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+\mathcal{O}(1))\sqrt{\log n}}}{\left(\log n^{\frac{1}{4}}\right)}$$

In particular, for every $\in > 0$ there exists a constant c such that $\chi'_{st}(K_n) \leq cn^{1+c}$ for every $n \geq 1$.

They asked what is true order of magnitude of $\chi'_{st}(K_n)$, in particular, if $\chi'_{st}(K_n) = \mathcal{O}(n)$. From Theorem 2.1, they also derived the following near-linear upper bound in terms of the maximum degree Δ for general graphs.

Theorem 2.2([5]) *Let G be an arbitrary graph of maximum degree Δ. Then*

$$\chi'_{st}(G) \leq \chi'_{st}(K_{n+1}) \cdot \mathcal{O}\left(\frac{\log \Delta}{\log\log \Delta}\right)^2$$

and therefore $\chi'_{st}(G) \leq \Delta \cdot 2^{\mathcal{O}(1)\sqrt{\log \Delta}}$.

Theorem 2.3([5])

(a) If G is a subcubic graph, then $\chi'_{st}(G) \leq 7$.

(b) If G is a simple cubic graph, then $\chi'_{st}(G) \geq 4$, and the equality holds if and only if G covers the graph of the 3-cube.

A graph G covers a graph H if there is a locally bijective graph homomorphism from G to H. While there exist cubic graphs with the star chromatic index equal to 6. e.g., $K_{3,3}$ or Heawood graph, no example of a subcubic graph that would require 7 colors is known. Thus, Dvořák et al. proposed the following conjecture.

Conjecture 2.4([5]) *If G is a subcubic graph, then $\chi'_{st}(G) \leq 6$.*

Theorem 2.5([9]) *Let T be a tree with maximum degree Δ. Then*

$$\chi'_{st}(T) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor.$$

Moreover, the bound is tight.

Theorem 2.6([9]) *Let G be an outerplaner graph with maximum degree Δ. Then*

$$\chi'_{st}(G) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 12.$$

Lemma 2.7([9]) *Every outerplanar embedding of a light cactus graph admits a proper 4-edge coloring such that no bichromatic 4-path exists on the boundary of the outer face.*

Theorem 2.8([9]) *Let G be an subcubic outerplaner graph. Then,*

$$\chi'_{st}(G) \leq 5.$$

Conjecture 2.9([9]) *Let G be an outerplaner graph with maximum degree $\Delta \geq 3$. Then*

$$\chi'_{st}(G) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1.$$

For graphs with maximum degree $\Delta = 2$, i.e. for paths and cycles, there exist star edge coloring with at most 3 colors except for C_5 which requires 4 colors. In case of subcubic outerplanar graphs the conjecture is confirmed by Theorem 2.8.

§3. Main Results

Theorem 3.1 *For any positive integer m and n, then*

$$\chi'_{st}(P_m \circ P_n) = \begin{cases} n & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n+2 & \text{if } m \geq 3 \end{cases}$$

Proof Let $V(P_m) = \{u_i : i = 1, 2, \cdots, m\}$ and $V(P_n) = \{v_j : j = 1, 2, \cdots, n\}$. Let $E(P_m) = \{u_i u_{i+1} : i = 1, 2, \cdots, m-1\}$ and $E(P_n) = \{v_j v_{j+1} : j = 1, 2, \cdots, n-1\}$. By the definition of corona product,

$$\begin{aligned} V(P_m \circ P_n) &= V(P_m) \bigcup_{i=1}^{m} V\left(P_n^i\right), \\ E(P_m \circ P_n) &= E(P_m) \bigcup_{i=1}^{m} E\left(P_n^i\right) \bigcup_{i=1}^{m} \{u_i v_{i,j} : 1 \leq j \leq n\}. \end{aligned}$$

Let σ be a mapping from $E(P_m \circ P_n)$ as follows:

Case 1. For $m = 1$,

$$\begin{cases} \sigma(u_i v_{i,j}) = i + j - 2 \,(\mathrm{mod}\ n), 1 \le j \le n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\mathrm{mod}\ n), 1 \le j \le n-1; \end{cases}$$

Case 2. For $m = 2$,

$$\begin{cases} \text{For } i = 1, 2, \\ \sigma(u_i v_{i,j}) = i + j - 2 \,(\mathrm{mod}\ n+1), 1 \le j \le n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\mathrm{mod}\ n+1), 1 \le j \le n-1; \\ \sigma(u_1 u_2) = n; \end{cases}$$

Case 3 For $m \ge 3$, $\sigma(u_i u_{i+1}) = n + 2 \,(\mathrm{mod}\ n+3), 1 \le i \le m-1;$

$$\begin{cases} \text{For } 1 \le i \le m, \\ \sigma(u_i v_{i,j}) = i + j - 2 \,(\mathrm{mod}\ n+3), 1 \le j \le n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\mathrm{mod}\ n+1), 1 \le j \le n-1; \end{cases}$$

It is easy to see that σ is satisfied length of path-4 are not bicolored. To prove

$$\chi'_{st}(P_m \circ P_n) \le \begin{cases} n & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n+2 & \text{if } m \ge 3. \end{cases}$$

we have

$$\chi'_{st}(P_m \circ P_n) \ge \chi'(P_m \circ P_n) \ge \Delta(P_m \circ P_n) \ge \begin{cases} n & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n+2 & \text{if } m \ge 3. \end{cases}$$

Thus the conclusion is true. □

Theorem 3.2 *For any positive integer m and n, then*

$$\chi'_{st}(P_m \circ S_n) = \begin{cases} 2n & \text{if } m = 1 \\ 2n+1 & \text{if } m = 2 \\ 2n+2 & \text{if } m \ge 3. \end{cases}$$

Proof Let $V(P_m) = \{u_i : i = 1, 2, \cdots, m\}$ and $V(S_n) = \{v_j : j = 1, 2, \cdots, n\} \cup \{v_{n+j} : j = 1, 2, \cdots, n\}$. Let $E(P_m) = \{u_i u_{i+1} : i = 1, 2, \cdots, m-1\}$ and $E(S_n) = \{v_j v_{j+1} : j = 1, 2,$

$\cdots, n-1\} \cup \{v_{n-1}v_n\} \cup \{v_j v_{n+j} : j = 1, 2, \cdots, n\}$, where v_{n+j}'s are pendent edges of v_j. By the definition of corona product,

$$\begin{aligned} V(P_m \circ S_n) &= V(P_m) \bigcup_{i=1}^{m} V(S_n^i), \\ E(P_m \circ S_n) &= E(P_m) \bigcup_{i=1}^{m} E(S_n^i) \bigcup_{i=1}^{m} \{u_i v_{i,j} : 1 \leq j \leq 2n\} \end{aligned}$$

Let σ be a mapping from $E(P_m \circ S_n)$ as follows:

Case 1. For $m = 1$,

$$\begin{cases} \sigma(u_i v_{i,j}) = j - 1 \,(\text{mod } 2n), 1 \leq j \leq 2n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\text{mod } 2n), 1 \leq j \leq n-1; \\ \sigma(v_{i,j} v_{i,n+j}) = n + i + j \,(\text{mod } 2n), 1 \leq j \leq n; \\ \sigma(v_{i,n-1} v_{i,n}) = n + 1; \end{cases} \tag{1}$$

Case 2. For $m = 2$,

$f(u_1 u_2) = 2n$ and using Equation (1).

Case 3. For $m \geq 3$, $\sigma(u_i u_{i+1}) = 2n + i \,(\text{mod } 2n+2), 1 \leq i \leq m-1$;

$$\begin{cases} \text{For } 1 \leq i \leq m, \\ \sigma(u_i v_{i,j}) = i + j - 2 \,(\text{mod } 2n+2), 1 \leq j \leq 2n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\text{mod } 2n+2), 1 \leq j \leq n-1; \\ \sigma(v_{i,j} v_{i,n+j}) = n + i + j \,(\text{mod } 2n+2), 1 \leq j \leq n; \\ \sigma(v_{i,n-1} v_{i,n}) = n + i \,(\text{mod } 2n+2); \end{cases}$$

It is easy to see that σ is satisfied length of path-4 are not bicolored. To prove

$$\chi'_{st}(P_m \circ S_n) \leq \begin{cases} 2n & \text{if } m = 1 \\ 2n+1 & \text{if } m = 2 \\ 2n+2 & \text{if } m \geq 3. \end{cases}$$

we have

$$\chi'_{st}(P_m \circ S_n) \geq \chi'(P_m \circ S_n) \geq \Delta(P_m \circ S_n) \geq \begin{cases} 2n & \text{if } m = 1 \\ 2n+1 & \text{if } m = 2 \\ 2n+2 & \text{if } m \geq 3. \end{cases}$$

Thus the conclusion is true. $\square$

Theorem 3.3 *For any positive integer m and n, then*

$$\chi'_{st}\left(P_m \circ K_{1,n,n}\right) = \begin{cases} 2n+1 & if \;\; m=1 \\ 2n+2 & if \;\; m=2 \\ 2n+3 & if \;\; m \geq 3 \end{cases}$$

Proof Let $V\left(P_m\right) = \{u_i : i = 1, 2, \cdots, m\}$ and $V\left(K_{1,n,n}\right) = \{v_0\} \cup \{v_{2j-1} : j = 1, 2, \cdots, n\}$ $\cup \{v_{2j} : j = 1, 2, \cdots, n\}$. Let $E\left(P_m\right) = \{u_i u_{i+1} : i = 1, 2, \cdots, m-1\}$, $E\left(K_{1,n,n}\right) = \{v_0 v_{2j-1} : j = 1, 2, \cdots, n\} \cup \{v_{2j-1} v_{2j} : j = 1, 2, \cdots, n\}$, where v_0 is adjacent to v_{2j-1} and v_{2j} are pendent vertices of v_{2j-1}. By the definition of corona product,

$$\begin{aligned} V\left(P_m \circ K_{1,n,n}\right) &= V\left(P_m\right) \bigcup_{i=1}^{m} V\left(K^i_{1,n,n}\right), \\ E\left(P_m \circ K_{1,n,n}\right) &= E\left(P_m\right) \bigcup_{i=1}^{m} E\left(K^i_{1,n,n}\right) \bigcup_{i=1}^{m} \{u_i v_{i,j} : 0 \leq j \leq 2n\} \end{aligned}$$

Let σ be a mapping from $E\left(P_m \circ K_{1,n,n}\right)$ as follows:

Case 1. For $m = 1$,

$$\begin{cases} \sigma\left(u_i v_{i,j}\right) = j \bmod 2n, 0 \leq j \leq 2n; \\ \sigma\left(v_{i,0} v_{i,2j-1}\right) = 2j+2 \pmod{2n+1}, 1 \leq j \leq n; \\ \sigma\left(v_{i,2j-1} v_{i,2j}\right) = 2j+3 \pmod{2n+1}, 1 \leq j \leq n; \end{cases} \tag{2}$$

Case 2. For $m = 2$,

$\sigma\left(u_1 u_2\right) = 2n+1$; and using Equation (2).

Case 3. For $m \geq 3$,

$\sigma\left(u_i u_{i+1}\right) = 2n+i \pmod{2n+3}, 1 \leq i \leq m-1$;

$$\begin{cases} \text{For } 1 \leq i \leq m, \\ \sigma\left(u_i v_{i,j}\right) = i+j-1 \pmod{2n+3}, 0 \leq j \leq 2n; \\ \sigma\left(v_{i,0} v_{i,2j-1}\right) = i+2j-1 \pmod{2n+3}, 1 \leq j \leq n; \\ \sigma\left(v_{i,2j-1} v_{i,2j}\right) = i+2j \pmod{2n+3}, 1 \leq j \leq n; \end{cases}$$

It is easy to see that σ is satisfied length of path-4 are not bicolored. To prove

$$\chi'_{st}\left(P_m \circ K_{1,n,n}\right) \leq \begin{cases} 2n+1 & \text{if } \;\; m=1 \\ 2n+2 & \text{if } \;\; m=2 \\ 2n+3 & \text{if } \;\; m \geq 3. \end{cases}$$

we have

$$\chi'_{st}\left(P_m \circ K_{1,n,n}\right) \geq \chi'\left(P_m \circ K_{1,n,n}\right) \geq \Delta\left(P_m \circ K_{1,n,n}\right) \geq \begin{cases} 2n+1 & \text{if } m=1 \\ 2n+2 & \text{if } m=2 \\ 2n+3 & \text{if } m \geq 3. \end{cases}$$

So the conclusion is true. □

Theorem 3.4 *For any positive integer $l \geq 3$, $m \geq 3$ and $n \geq 3$, then*

$$\chi'_{st}\left(P_l \circ K_{m,n}\right) = m+n+2.$$

Proof Let $V\left(P_l\right) = \{u_i : 1 \leq i \leq l\}$ and $V\left(K_{m,n}\right) = \{v_j : 1 \leq j \leq m\} \cup \{v'_k : 1 \leq k \leq n\}$. Let $E\left(P_l\right) = \{u_i u_{i+1} : 1 \leq i \leq l-1\}$ and $E\left(K_{m,n}\right) = \bigcup\limits_{j=1}^{m} \{v_j v'_k : 1 \leq k \leq n\}$. By the definition of corona product,

$$\begin{aligned} V\left(P_l \circ K_{m,n}\right) &= V\left(P_l\right) \bigcup_{i=1}^{l} \{v_{ij} : 1 \leq j \leq m\} \bigcup_{i=1}^{l} \{v'_{ik} : 1 \leq k \leq n\}, \\ E\left(P_l \circ K_{m,n}\right) &= E\left(P_l\right) \bigcup_{i=1}^{l} E\left(K^i_{m,n}\right) \bigcup_{i=1}^{l} \{u_i v_{ij} : 1 \leq j \leq m\} \bigcup_{i=1}^{l} \{u_i v'_{ik} : 1 \leq k \leq n\}. \end{aligned}$$

Let σ be a mapping from $P_l \circ K_{m,n}$ as follows:

$\sigma\left(u_{2i-1}u_{2i}\right) = n-1, 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$; $\sigma\left(u_{2i}u_{2i+1}\right) = n, 1 \leq i \leq \left\lceil \frac{l}{2} \right\rceil$ and

$$\begin{cases} \text{For } 1 \leq i \leq l, \\ \sigma\left(v_{ij}v'_{ik}\right) = j+k-1, 1 \leq j \leq m, 1 \leq k \leq n; \\ \sigma\left(u_i v_{ij}\right) = n+j, 1 \leq j \leq m; \\ \sigma\left(u_i v'_{ik+2}\right) = k, 1 \leq k \leq n-2; \\ \sigma\left(u_i v'_{i1}\right) = m+n+1; \\ \sigma\left(u_i v'_{i2}\right) = m+n+2. \end{cases}$$

Clearly above color partitions are satisfied length of path-4 are not bicolored. We assume that $\chi'_{st}\left(P_m \circ K_{m,n}\right) \leq m+n+2$. We know that $\chi'_{st}\left(P_m \circ K_{m,n}\right) \geq \chi'\left(P_m \circ K_{m,n}\right) \geq m+n+2$, since $\chi'_{st}\left(P_m \circ K_{m,n}\right) \geq m+n+2$. Therefore $\chi'_{st}\left(P_m \circ K_{m,n}\right) = m+n+2$. □

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Balance Index Set of Caterpillar and Lobster Graphs

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Consider the set $A = \{0, 1\}$. A labeling $f : V(G) \to A$ induces a partial edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x)$, if and only if $f(x) = f(y)$ for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_{f^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \leq 1$. A friendly labeling is balanced if $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$. The balance index set of the graph $G, BI(G)$, is defined as $\{|e_{f^*}(0) - e_{f^*}(1)| :$ the vertex labeling f is friendly$\}$. In this paper, we obtain the balance index set of caterpillar graphs and lobster graphs.

Key Words: Friendly labeling, Smarandache friendly labeling, partial edge labeling and balance index set.

AMS(2010): 05C78.

§1. Introduction

We begin with simple, finite, connected and undirected graph G=(V, E). Here the elements of set V and E are known as vertices and edges respectively with $|V| = p$ and $|E| = q$. For all other terminologies and notations we follow Harary [1].

Definition 1.1 *A path graph or linear graph is a tree with two or more vertices that contains only vertices of degree* 2 *and* 1*.*

Definition 1.2 *A caterpillar is a tree in which all the vertices are within distance* 1 *of a central path.*

Definition 1.3 *The graph $B_{l,m,k}$ is a tree obtained from a path of length k by attaching the stars $K_{1,l}$ and $K_{1,m}$ with its pendent vertices.*

Definition 1.4 *A coconut Tree $CT(m, l)$ is the graph obtained from the path P_m by appending l new pendent edges at an end vertex of P_m.*

Definition 1.5 *A lobster graph is a tree in which all the vertices are within distance* 2 *of a central path.*

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Definition 1.6 *A mapping $f : V(G) \to \{0,1\}$ is called friendly labeling of G if*

$$|v_f(0) - v_f(1)| \leq 1,$$

otherwise, a Smarandache friendly labeling of G, i.e., $|v_f(0) - v_f(1)| \geq 2$.

Lee, Liu and Tan [5] considered a new labeling problem of graph theory. A vertex labeling of G is a mapping f from $V(G)$ into the set $\{0,1\}$. For each vertex labeling f of G, a partial edge labeling f^* of G is defined in the following way.

For each edge uv in G,

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0 \\ 1, & \text{if } f(u) = f(v) = 1 \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge uv is not labeled by f^*. Thus f^* is a partial function from $E(G)$ into the set $\{0,1\}$. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_{f^*}(0)$ and $e_{f^*}(1)$ denote the number of edges of G that are labeled by 0 and 1 under the induced partial function f^* respectively.

In [3] Kim, Lee, and Ng define the balance index set of a graph G as BI$(G) = \{|e_{f^*}(0) - e_{f^*}(1)| : f^*$ runs over all friendly labelings f of $G\}$.

Example 1.7 Figure 1 shows a graph G with $BI(G) = \{0,1\}$.

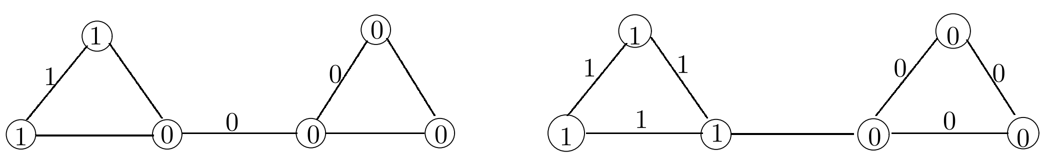

Figure 1 The friendly labelings of graph G with $BI(G) = \{0,1\}$.

For a graph with a vertex labeling f, we denote $e_{f^*}(X)$ to be the subset of $E(G)$ containing all the unlabeled edges. In [4] Kwong and Shiu developed an algebraic approach to attack the balance index set problems. It shows that the balance index set depends on the degree sequence of the graph.

Lemma 1.8([6]) *For any graph G,*

(1) $2e_{f^*}(0) + e_{f^*}(X) = \sum_{v \epsilon v(0)} deg(v)$;
(2) $2e_{f^*}(1) + e_{f^*}(X) = \sum_{v \epsilon v(1)} deg(v)$;
(3) $2|E(G)| = \sum_{v \epsilon v(G)} deg(v) = \sum_{v \epsilon v(0)} deg(v) + \sum_{v \epsilon v(1)} deg(v)$.

Corollary 1.9([6]) *For any friendly labeling f, the balance index is*

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right).$$

More details of known results of graph labelings are given in Gallian [2].

In number theory and combinatorics, a partition of a positive integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered to be the same partition; if order matters then the sum becomes a composition. For example, 4 can be partitioned in five distinct ways:

$$4+0, 3+1, 2+2, 2+1+1, 1+1+1+1.$$

Let G be any graph with p vertices. Partition of p in to $(p_0,\, p_1)$, where p_0 and p_1 are the number of vertices labeled by 0 and 1 respectively.

In [6] Lee, Su and Wang gave the results for balance index set of trees of diameter four. In this paper we obtain balance index set of caterpillar and lobster graphs of diameter n. To prove our result we are using Lemma 1.8 and Corollary 1.9.

§2. Balance Index Set of Caterpillar Graphs

Consider the caterpillar graph $CT(a_1, a_2, a_3, \cdots, a_{n-1})$, where a_i, i=1, 2, 3, $\cdots$,$n-1$ are the number of vertices adjacent to i^{th} spine vertices. We name $n-1$ vertices on the spine as $u_{a_1}, u_{a_2}, u_{a_3}, \cdots, u_{a_{n-1}}$. Thus for a caterpillar graph there are $(a_1 + a_2 + a_3 + \cdots + a_{n-1})$ number of pendant vertices. The degrees of $u_{a_1}, u_{a_2}, u_{a_3}, \cdots, u_{a_{n-1}}$ are $a_1+1, a_2+2, a_3+2, \cdots, a_{n-2}+2, a_{n-1}+1$ respectively. We also name non-spinal vertices adjacent to u_{a_1} by $u_{a_1,1}$, $u_{a_1,2}$, $u_{a_1,3}$, $\cdots$, u_{a_1,a_1}. Similarly we name non spinal vertices adjacent to u_{a_2}, u_{a_3}, u_{a_4}, $\cdots$, $u_{a_{n-1}}$.

Theorem 2.1 *For $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ of order p and diameter n, the balance index is,*

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \left\{\dfrac{1}{2}\left(l + \sum\limits_{i=1}^{n-1}(-1)^{f(u_{a_i})}a_i\right)\right\}, & \text{if } p \text{ is even} \\[2ex] \left\{\dfrac{1}{2}\left(l + 1 + \sum\limits_{i=1}^{n-1}(-1)^{f(u_{a_i})}a_i\right)\right\}, & \text{if } p \text{ is odd} \end{cases}$$

where

$$l = \begin{cases} n-2j-3, & \text{if } j = i, i-1, i-2, \text{ where } i = 2,3,4,\cdots, \left\lfloor \dfrac{n}{2} \right\rfloor \\ & \text{and j number of coefficients of } a_i \text{ are negative} \\ n-3, & \text{if } f(u_{a_i}) = 0 \text{ for all i or } f(u_{a_i}) = \begin{cases} 1, & \text{if } \ i = 1, n-1 \\ 0, & otherwise \end{cases} \end{cases}$$

Proof Consider the caterpillar graph $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ of order p and diameter n.

Case 1. n is even.

Subcase 1.1 If $a_1+a_2+a_3+\cdots+a_{n-1}$ is odd, then the number of vertices of $CT(a_1, a_2, a_3,$

$\cdots, a_{n-1})$ is $a_1+a_2+a_3+\cdots+a_{n-1}+n-1$ which is even. Let $(a_1+a_2+a_3+\cdots+a_{n-1})+n-1 = 2M$. For a friendly labeling, M vertices are labeled 0 and remaining M vertices are labeled 1.

We first consider the case that $u_{a_1}, u_{a_2}, u_{a_3}, \cdots, u_{a_{n-1}}$ are all labeled 0, i.e. $n-1$ spine vertices are partitioned in to $(n-1, 0)$. Then $M-(n-1)$ pendant vertices are labeled 0 and M pendant vertices are labeled 1. Therefore by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M-(n-1)+(a_1+1)+(a_2+2)+(a_3+2)+\cdots+(a_{n-1}+1)-M] \\ &= \frac{1}{2}[a_1+a_2+a_3+\cdots+a_{n-1}+n-3]. \end{aligned}$$

If $n-1$ spine vertices are partitioned in to $(n-2, 1)$, then $M-(n-2)$ pendant vertices are labeled 0 and $M-1$ pendant vertices are labeled 1. Two possibilities arise.

(a) If the vertex u_{a_1} is labeled 1, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M-(n-2)-(a_1+1)+(a_2+2)+(a_3+2)+\cdots \\ &\quad +(a_{n-1}+1)-(M-1)] \\ &= \frac{1}{2}[-a_1+a_2+a_3+\cdots+a_{n-1}+n-3]. \end{aligned}$$

Similarly If $u_{a_{n-1}}$ is labeled 1, then

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2}[a_1+a_2+a_3+\cdots+a_{n-2}-a_{n-1}+n-3].$$

(b) If one vertex u_{a_i}, $i = 2, 3, 4, \cdots, n-2$ is labeled 1, then $M-(n-2)$ pendant vertices are labeled 0 and $M-1$ pendant vertices are labeled 1.

Therefore,

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M-(n-2)+(a_1+1)+(a_2+2)+(a_3+2)+\cdots-(a_i+2)+\cdots \\ &\quad +(a_{n-1}+1)-(M-1)] \\ &= \frac{1}{2}[a_1+a_2+a_3+\cdots+a_{i-1}-a_i+a_{i+1}+\cdots+a_{n-1}+n-5], \end{aligned}$$

where $i = 2, 3, 4, \cdots, n-2$.

If $n-1$ spine vertices are partitioned in to $(n-1-i, i)$, where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$. Then $M-(n-1-i)$ pendant vertices are labeled 0 and $M-i$ pendant vertices are labeled 1. Three possibilities arise.

(a) If $f(u_{a_1}) = f(u_{a_{n-1}}) = 0$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) + (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2i - 3], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and i coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

(b) If $f(u_{a_1}) = 0$ and $f(u_{a_{n-1}}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i - 1], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and $i-1$ coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

(c) If $f(u_{a_1}) = f(u_{a_{n-1}}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M - (n-1-i) - (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[-a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i + 1], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and $i-2$ coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

Subcase 1.2 If $a_1+a_2+a_3+\cdots+a_{n-1}$ is even, then the number of vertices of $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ is $a_1+a_2+a_3+\cdots+a_{n-1}+n-1$ which is odd. Let $(a_1+a_2+a_3+\cdots+a_{n-1})+n-1 = 2M+1$.

For a friendly labeling, without loss of generality, there are $M+1$ vertices labeled 0 and M vertices labeled 1.

We first consider the case that $u_{a_1}, u_{a_2}, u_{a_3}, \cdots, u_{a_{n-1}}$ are all labeled 0, i.e. $n-1$ spine vertices are partitioned in to $(n-1, 0)$. Then $(M+1)-(n-1)$ pendant vertices are labeled 0 and M pendant vertices are labeled 1.

Therefore,

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-1) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-1}+1) - M] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2]. \end{aligned}$$

If $n-1$ spine vertices are partitioned in to $(n-2, 1)$, then $(M+1) - (n-2)$ pendant vertices are labeled 0 and $M-1$ pendant vertices are labeled 1. Two possibilities arise.

(a) If $f(u_{a_1}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-2) - (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-1}+1) - (M-1)] \\ &= \frac{1}{2}[-a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2]. \end{aligned}$$

Similarly, if $f(u_{a_{n-1}}) = 1$ then

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2].$$

(b) If one spine vertex of degree $a_i, i = 2, 3, 4, \cdots, n-2$ is labeled 1, then $M - (n-2)$ pendant vertices are labeled 0 and $M-1$ pendant vertices are labeled 1. Therefore,

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-2) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots - (a_i+2) + \cdots + (a_{n-1}+1) - (M-1)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{i-1} - a_i + a_{i+1} + \cdots + a_{n-1} + n - 4], \end{aligned}$$

where $i = 2, 3, 4, \cdots, n-2$.

If $n-1$ spine vertices are partitioned in to $(n-1-i, i)$, where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$, then $(M+1) - (n-1-i)$ pendant vertices are labeled 0 and $M-i$ pendant vertices are labeled 1. Three possibilities arise.

(a) If $f(u_{a_1}) = f(u_{a_{n-1}}) = 0$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) + (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2i - 2], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and i coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

(b) If $f(u_{a_1}) = 0$ and $f(u_{a_{n-1}}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and $i-1$ coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

(c) If $f(u_{a_1}) = f(u_{a_{n-1}}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-1-i) - (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[-a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i + 2], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and $i-2$ coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

Case 2. n is odd.

Subcase 2.1 If $(a_1 + a_2 + a_3 + \cdots + a_{n-1})$ is odd, then the number of vertices of $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ is $a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1$ which is odd. Therefore the proof is similar to Subcase 1.2.

Subcase 2.2 If $a_1+a_2+a_3+\cdots+a_{n-1}$ is even, then the number of vertices of $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ is $a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1$ which is even. Therefore the proof is similar to Subcase 1.1. □

Example 2.2 Figure 2 shows the caterpillar $CT(2, 1, 1, 2, 1)$ of diameter 6 and order 12 with balance index set $\{0, 1, 2, 3, 4, 5\}$.

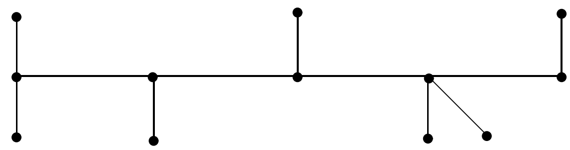

Figure 2 The caterpillar $CT(2, 1, 1, 2, 1)$ of diameter 6 and order 12.

Corollary 2.3 *The balance index set of the graph $B_{l,m,k}$,*

$$\begin{aligned} BI(B_{l,m,k}) &= \left\{ \left\lfloor \frac{l+m+k}{2} \right\rfloor, \left\lfloor \frac{|l-m+k|}{2} \right\rfloor, \left\lfloor \frac{|-l+m+k|}{2} \right\rfloor, \left\lfloor \frac{l+m+k-2}{2} \right\rfloor \right\} \bigcup \\ & \left\{ \left\lfloor \frac{l+m+k-2i}{2} \right\rfloor, \left\lfloor \frac{|l-m+k-2i+2|}{2} \right\rfloor, \left\lfloor \frac{|-l-m+k-2i+4|}{2} \right\rfloor : \right. \\ & \left. i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor \right\}. \end{aligned}$$

Proof The graph $B_{l,m,k}$ is a caterpillar $CT(l, 0, 0, \cdots, m)$ of diameter $k+2$. Therefore substituting $n = k+2$, $a_1 = l$, $a_{n-1} = m$ and $a_2 = a_3 = a_4 = \cdots = a_{n-2} = 0$ in the Theorem 2.1, we get

$$\begin{aligned} BI(B_{l,m,k}) &= \left\{ \left\lfloor \frac{l+m+k}{2} \right\rfloor, \left\lfloor \frac{|l-m+k|}{2} \right\rfloor, \left\lfloor \frac{|-l+m+k|}{2} \right\rfloor, \left\lfloor \frac{l+m+k-2}{2} \right\rfloor \right\} \\ & \bigcup \left\{ \left\lfloor \frac{l+m+k-2i}{2} \right\rfloor, \left\lfloor \frac{|l-m+k-2i+2|}{2} \right\rfloor, \left\lfloor \frac{|-l-m+k-2i+4|}{2} \right\rfloor : \right. \\ & \left. i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor \right\}. \end{aligned}$$ □

Example 2.4 Figure 3 shows the graph $B_{3,3,3}$ of diameter 5 and order 10 with balance index set $\{0, 1, 2, 3, 4\}$.

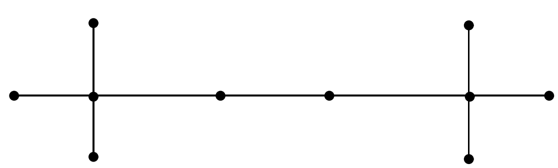

Figure 3 The graph $B_{3,3,3}$ of diameter 5 and order 10.

Corollary 2.5 *The balance index set of coconut tree $CT(m,l)$,*

$$\begin{aligned}BI(CT(m,l)) &= \left\{\left\lfloor\frac{l+m-2}{2}\right\rfloor, \left\lfloor\frac{|-l+m-2|}{2}\right\rfloor, \left\lfloor\frac{|l+m-4|}{2}\right\rfloor\right\}\bigcup\\ &\left\{\left\lfloor\frac{l+m-2i-2}{2}\right\rfloor, \left\lfloor\frac{|-l+m-2i|}{2}\right\rfloor, \left\lfloor\frac{|-l-m-2i+2|}{2}\right\rfloor :\right.\\ &\left. i=2,3,4,\cdots,\left\lfloor\frac{m}{2}\right\rfloor\right\}.\end{aligned}$$

Proof The coconut tree $CT(m,l)$ is a caterpillar graph $CT(0,0,0,\cdots,m)$ of diameter m. Therefore substituting $n=m$, $a_{n-1}=l$ and $a_1=a_2=a_3=\cdots=a_{n-2}=0$ in the Theorem 2.1, we get

$$\begin{aligned}BI(CT(m,l)) &= \left\{\left\lfloor\frac{l+m-2}{2}\right\rfloor, \left\lfloor\frac{|-l+m-2|}{2}\right\rfloor, \left\lfloor\frac{|l+m-4|}{2}\right\rfloor\right\}\bigcup\\ &\left\{\left\lfloor\frac{l+m-2i-2}{2}\right\rfloor, \left\lfloor\frac{|-l+m-2i|}{2}\right\rfloor, \left\lfloor\frac{|-l-m-2i+2|}{2}\right\rfloor :\right.\\ &\left. i=2,3,4,\cdots,\left\lfloor\frac{m}{2}\right\rfloor\right\}.\end{aligned}$$

□

Example 2.6 Figure 4 shows coconut tree of diameter 5 and order 9 with balance index set $\{0,1,2,3,5\}$.

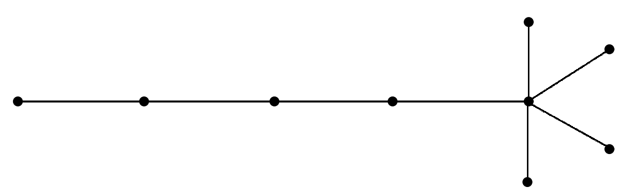

Figure 4 The coconut tree of diameter 5 and order 9.

§3. Balance Index Set of Lobster Graphs

In a caterpillar graph $CT(a_1,a_2,a_3,\cdots,a_{n-1})$, if $a_i\neq 0$ for $i=2,3,\cdots,n-2$, then we have $a_i, i=2,3,4,\cdots,n-2$ number of P_3 paths contained the vertex u_{a_i}. Since P_3 is of length 2, after adding more adjacent edges and vertices to the two end vertices of these paths, the new graph is a lobster graph of diameter n. We denote the new graph as

$$\begin{aligned}&CT(a_1,a_2,a_3,\cdots,a_{n-1})(u_{a_2}(t_{2,1},t_{2,2},t_{2,3},\cdots,t_{2,a_2}),(u_{a_3}(t_{3,1},t_{3,2},t_{3,3},\cdots,t_{3,a_3})),\\&u_{a_4}(t_{4,1},t_{4,2},t_{4,3},\cdots,t_{4,a_4}),\cdots,u_{a_{n-2}}(t_{n-2,1},t_{n-2,2},t_{n-2,3},\cdots,t_{n-2,a_{n-2}})),\end{aligned}$$

where $t_{i,j}$ is the number of edges and vertices added to the vertex $u_{a_i,j}$, $i=2,3,4,\cdots,n-2$,

$j = 1, 2, 3, \cdots, a_i$. Here we have

$$a_1 + a_{n-1} + \sum_{i=2}^{n-2} \sum_{j=1}^{a_i} t_{i,j}$$

pendant vertices.

In order to write the results in an uniform manner we name this lobster graph as

$CT(d_1, d_2, d_3, \cdots, d_{n-2}, d_0)(u_{a_2}(d_{n-1}, d_n, d_{n+1}, \cdots, d_{d_2+n-2}),$
$u_{a_3}(d_{d_2+n-1}, d_{d_2+n}, d_{d_2+n+1}, \cdots, d_{d_2+d_3+n-2}),$
$u_{a_4}(d_{d_2+d_3+n-1}, d_{d_2+d_3+n}, d_{d_2+d_3+n+1}, \cdots, d_{d_2+d_3+d_4+n-2}), \cdots,$
$u_{a_{n-2}}(d_{d_2+d_3+\cdots+d_{n-3}+n-1}, d_{d_2+d_3+\cdots+d_{n-3}+n}, d_{d_2+d_3+\cdots+d_{n-3}+n+1}, \cdots, d_{d_2+d_3+\cdots+d_{n-2}+n-2}))$.

We also name $n-1$ spine vertices by $v_1, v_2, v_3, \cdots, v_{n-2}, v_0$, the vertices adjacent to v_2 by $v_{n-1}, v_n, v_{n+1}, \cdots, v_{d_2+n-2}$, adjacent to v_3 by v_{d_2+n-1}, v_{d_2+n}, v_{d_2+n+1}, $\cdots, v_{d_2+d_3+n-2}$, etc. and adjacent to v_{n-2} by $v_{d_2+d_3+d_4+\cdots+d_{n-3}+n-1}$, $v_{d_2+d_3+d_4+\cdots+d_{n-3}+n}$, $v_{d_2+d_3+d_4+\cdots+d_{n-3}+n+1}$, $\cdots, v_{d_2+d_3+d_4+\cdots+d_{n-2}+n-2}$.

Thus in this lobster, we have $d_0 + d_1 + \sum\limits_{i=n-1}^{m} d_i$ pendant vertices where $m = \sum\limits_{j=2}^{n-2} d_j + n - 2$, the degree of v_i for $n-1 \leq i \leq \sum\limits_{j=2}^{n-2} d_j + n - 2$ is $d_i + 1$ and the degree of v_k is $d_i + 2$ for $k = 2, 3, 4, \cdots, n-2$.

Theorem 3.1 *For a lobster graph of diameter n and order p, the balance index is*

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \dfrac{1}{2}\left[\sum\limits_{i=0}^{m}(-1)^{f(v_i)} d_i + \sum\limits_{i=2}^{n-2}(-1)^{f(v_i)}\right], & \text{if } p \text{ is even,} \\ \pm\dfrac{1}{2}\left[1 + \sum\limits_{i=0}^{m}(-1)^{f(v_i)} d_i + \sum\limits_{i=2}^{n-2}(-1)^{f(v_i)}\right], & \text{if } p \text{ is odd,} \end{cases}$$

where $m = \sum\limits_{j=2}^{n-2} d_j + n - 2$.

Proof Let G be a lobster graph of order p and diameter n.

Case 1. n is even.

Subcase 1.1 If $\sum\limits_{i=0}^{m} d_i$ is odd, then the number of vertices equal to $\sum\limits_{i=0}^{m} d_i + n - 1$ is even. Let it be $2M$. For a friendly labeling, there are M vertices labeled 0 and remaining M vertices labeled 1. We first consider the case that v_i for all i are labeled 0. Then there are $M - (n - 1) - \sum\limits_{i=2}^{n-2} d_i$ pendant vertices labeled 0 and remaining M pendant vertices labeled 1. Then by

Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left[M - (n-1) + 2(n-3) + d_0 + 1 + d_1 + 1 + \sum_{i=n-1}^{m}(d_i+1) - M\right] \\ &= \frac{1}{2}\left[\sum_{i=n-1}^{m} d_i + \sum_{j=0}^{n-2} d_j + n - 3\right] = \frac{1}{2}\left[\sum_{i=0}^{m} d_i + n - 3\right]. \end{aligned}$$

Similarly we assume that there are k vertices among v_i for all $0 \le i \le \sum\limits_{j=2}^{n-2} d_j + n - 2$ labeled 0. Then there are $M-k$ pendant vertices labeled 0 and $M - \left[\sum\limits_{j=2}^{n-2} d_j + n - 1 - k\right]$ pendant vertices labeled 1. We define P to be the set containing all the 0-vertices among v_i for all $0 \le i \le \sum\limits_{j=2}^{n-2} d_j + n - 2$. We also name N to be the set containing all the 1-vertices among v_i for all $0 \le i \le \sum\limits_{j=2}^{n-2} d_j + n - 2$. Then by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left[M - k + \sum_{v\in P} deg(v) - \left(M - \left[\sum_{j=2}^{n-2} d_j + n - 1 - k\right] + \sum_{v\in N} deg(v)\right)\right] \\ &= \frac{1}{2}\left[M - k + \left(\sum_{v\in P}(deg(v)-1) + 1\right) - M + \sum_{j=2}^{n-2} d_j\right] \\ &\quad + \left[n - 1 - k - \left(\sum_{v\in N}(deg(v)-1) + 1\right)\right] \\ &= \frac{1}{2}\left[M - k + \left(\sum_{v\in P}(deg(v)-1)\right) + k - M + \sum_{j=2}^{n-2} d_j\right] \\ &\quad + \left[n - 1 - k - \left(\sum_{v\in N}(deg(v)-1)\right) - \left(\sum_{j=2}^{n-2} d_j + n - 1 - k\right)\right] \\ &= \frac{1}{2}\left[\sum_{v\in P}(deg(v)-1) - \sum_{v\in N}(deg(v)-1)\right]. \end{aligned}$$

Also note that

$$deg(v) - 1 = \begin{cases} d_i, & \text{if } i = 0,\ 1 \text{ and } n-3 \le i \le \sum\limits_{j=2}^{n-2} d_j + n - 2 \\ d_i + 1, & \text{if } 2 \le i \le n-2 \end{cases}$$

Therefore

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left[(-1)^{f(v_0)} d_0 + (-1)^{f(v_1)} d_1 + \sum_{i=n-3}^{m}(-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2}(-1)^{f(v_i)}(d_i+1)\right] \\ &= \frac{1}{2}\left[\sum_{i=0}^{m}(-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2}(-1)^{f(v_i)}\right]. \end{aligned}$$

Subcase 1.2. If $\sum\limits_{i=0}^{m} d_i$ is even, then the number of vertices equal to $\sum\limits_{i=0}^{m} d_i + n - 1$ is odd. Let it be $2M+1$. For a friendly labeling, there are $M+1$ vertices labeled 0 and remaining M vertices labeled 1. We first consider the case that v_i for all i are labeled 0. Then there are $(M+1)-(n-1)-\sum\limits_{i=2}^{n-2} d_i$ pendant vertices labeled 0 and remaining M pendant vertices labeled 1. Then again by Corollary 1.9, we get

$$\begin{aligned}
e_{f^*}(0)-e_{f^*}(1) &= \frac{1}{2}\left[(M+1)-(n-1)+2(n-3)+d_0+1+d_1+1+\sum_{i=n-1}^{m}(d_i+1)-M\right] \\
&= \frac{1}{2}\left[\sum_{i=n-1}^{m} d_i+\sum_{j=0}^{n-2} d_j+n-2\right]=\frac{1}{2}\left[\sum_{i=0}^{m} d_i+n-2\right].
\end{aligned}$$

Similarly we assume that there are k vertices among v_i for all $0\le i\le \sum\limits_{j=2}^{n-2} d_j+n-2$ labeled 0. Then there are $M+1-k$ pendant vertices labeled 0 and $M-\left[\sum\limits_{j=2}^{n-2} d_j+n-1-k\right]$ pendant vertices labeled 1. We define P to be the set containing all the 0-vertices among v_i for all $0\le i\le \sum\limits_{j=2}^{n-2} d_j+n-2$. We also name N to be the set containing all the 1-vertices among v_i for all $0\le i\le \sum\limits_{j=2}^{n-2} d_j+n-2$. Then by Corollary 1.9, we get

$$\begin{aligned}
e_{f^*}(0)-e_{f^*}(1) &= \frac{1}{2}\left[\left(M+1-k+\sum_{v\in P} deg(v)\right)-\left(M-\left[\sum_{j=2}^{n-2} d_j+n-1-k\right]+\sum_{v\in N} deg(v)\right)\right] \\
&= \frac{1}{2}\left[M+1-k+\left(\sum_{v\in P}(deg(v)-1)+1\right)-M+\sum_{j=2}^{n-2} d_j\right] \\
&\quad +\left[n-1-k-\left(\sum_{v\in N}(deg(v)-1)+1\right)\right] \\
&= \frac{1}{2}\left[M+1-k+\left(\sum_{v\in P}(deg(v)-1)\right)+k-M+\sum_{j=2}^{n-2} d_j\right] \\
&\quad +\left[n-1-k-\left(\sum_{v\in N}(deg(v)-1)\right)-\left(\sum_{j=2}^{n-2} d_j+n-1-k\right)\right] \\
&= \frac{1}{2}\left[1+\sum_{v\in P}(deg(v)-1)-\sum_{v\in N}(deg(v)-1)\right].
\end{aligned}$$

Also note that

$$deg(v)-1=\begin{cases} d_i, & \text{if } i=0,\,1 \text{ and } n-3\le i\le \sum_{j=2}^{n-2} d_j+n-2 \\ d_i+1, & \text{if } 2\le i\le n-2 \end{cases}$$

Therefore,

$$\begin{aligned}
e_{f^*}(0)-e_{f^*}(1) &= \frac{1}{2}\left[1+(-1)^{f(v_0)}d_0+(-1)^{f(v_1)}d_1+\sum_{i=n-3}^{m}(-1)^{f(v_i)}d_i+\sum_{j=2}^{n-2}(-1)^{f(v_i)}(d_i+1)\right] \\
&= \frac{1}{2}\left[1+\sum_{i=0}^{m}(-1)^{f(v_i)}d_i+\sum_{j=2}^{n-2}(-1)^{f(v_i)}\right].
\end{aligned}$$

When a friendly labeling with $v_f(1) > v_f(0)$, it produces the negative values of the above balance indexes. Therefore,

$$e_{f^*}(0) - e_{f^*}(1) = \pm\frac{1}{2}\left[1 + \sum_{i=0}^{m}(-1)^{f(v_i)}d_i + \sum_{j=2}^{n-2}(-1)^{f(v_i)}\right].$$

Case 2. n is odd.

Subcase 2.1 If $\sum\limits_{i=0}^{m} d_i$ is odd, then the number of vertices equal to $\left[\sum\limits_{i=0}^{m} d_i\right] + n - 1$ is odd and proof is similar to Subcase 1.2.

Subcase 2.2 If $\sum\limits_{i=0}^{m} d_i$ is even, then the number of vertices equal to $\left[\sum\limits_{i=0}^{m} d_i\right] + n - 1$ is even and proof is similar to Subcase 1.1.

Therefore, for a lobster graph of diameter n and order p, the balance index is

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \frac{1}{2}\left[\sum\limits_{i=0}^{m}(-1)^{f(v_i)}d_i + \sum\limits_{i=2}^{n-2}(-1)^{f(v_i)}\right], & \text{if } p \text{ is even} \\ \\ \pm\frac{1}{2}\left[1 + \sum\limits_{i=0}^{m}(-1)^{f(v_i)}d_i + \sum\limits_{i=2}^{n-2}(-1)^{f(v_i)}\right], & \text{if } p \text{ is odd} \end{cases} \qquad \square$$

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Lagrange Space and Generalized Lagrange Space Arising From Metric $e^{\sigma(x)}g_{ij}(x,y)+\dfrac{1}{c^2}y_iy_j$

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Abstract: Some properties of Lagrange space with metric tensor $g_{ij}(x,y)+\frac{1}{c^2}y_iy_j$ where $g_{ij}(x,y)$ is metric tensor of Finsler space (M^n,F), and associated generalized Lagrange space has been studied by U. P. Singh in his paper [6]. In the present paper some properties of Lagrange space with metric tensor $e^{\sigma(x)}g_{ij}(x,y)+\frac{1}{c^2}y_iy_j$, where $g_{ij}(x,y)$ is metric tensor of Finsler space (M^n,F), $e^{\sigma(x)}$ is conformal factor and associated generalized Lagrange space has been studied.

Key Words: Lagrange space, generalized Lagrange space, $C-$reducible space.

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§1. Introduction

Various authors like R. Miron, M. Anastasiei, H. Shimada, T. Kawaguchi, U. P. Singh have studied Lagrange space and generalized Lagrange space in their papers [3], [2], [4], [5]. A generalized Lagrange space with metric tensor $\gamma_{ij}(x)+\dfrac{1}{c^2}y_iy_j$, where $\gamma_{ij}(x)$ is metric tensor of Riemannian space and c is velocity of light has been studied by Beil in his paper [1]. In this chapter $\gamma_{ij}(x)$ has been replaced by $e^{\sigma(x)}g_{ij}(x,y)$, where $g_{ij}(x,y)$ is metric tensor of Finsler space (M^n,F).

Let M^n is n-dimensional smooth manifold and F is Finsler function, the metric tensor $g_{ij}(x,y)$ is given by

$$g_{ij}(x,y)=\frac{\partial^2F^2}{\partial y^i\,\partial y^j}. \tag{1.1}$$

Since F is Finsler function of homogeneity one, so $g_{ij}(x,y)$ is homogeneous function of degree zero. The angular metric tensor of Finsler space (M^n,F), $h_{ij}(x,y)$ is given by

$$h_{ij}(x,y)=g_{ij}(x,y)-l_il_j, \tag{1.2}$$

where l_i is unit vector given by

$$l_i=\frac{y_i}{F}. \tag{1.3}$$

§2. Generalized Lagrange Space L^n and Associated Lagrange Space L^{*n}

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Consider a generalized Lagrange space $L^n = (M^n, G_{ij}(x,y))$ with metric tensor

$$G_{ij} = e^{\sigma} g_{ij}(x,y) + \frac{1}{c^2} y_i \, y_j. \tag{2.1}$$

The reciprocal metric tensor G^{ij} of G_{ij} is

$$G^{ij} = e^{-\sigma}\left(g^{ij} - \frac{1}{a_1 c^2} y^i y^j\right), \tag{2.2}$$

where

$$a_1 = e^{\sigma} + \frac{F^2}{C^2}, \qquad F^2 = g_{ij} y^i y^j. \tag{2.3}$$

The d-tensor field $\overline{C}_{ijk}$ of L^n is defined as

$$\overline{C}_{jhk} = \frac{1}{2}\left(\frac{\partial G_{jh}}{\partial y^k} + \frac{\partial G_{hk}}{\partial y^j} - \frac{\partial G_{jk}}{\partial y^h}\right). \tag{2.4}$$

Since $\dfrac{\partial y_i}{\partial y^j} = g_{ij}$ from (2.1) and (2.4), we have

$$\overline{C}_{jhk} = e^{\sigma} C_{jhk} + \frac{1}{c^2} g_{jk} y_h, \tag{2.5}$$

$$\overline{C}^i_{jk} = G^{ih}\overline{C}_{jhk} = C^i_{jk} + \frac{1}{a_1 c^2}\, g_{jk} y^i. \tag{2.6}$$

The metric tensor G_{ij} is used to define the Lagrangian L^* is given by

$$L^{*2} = G_{ij} y^i \, y^j. \tag{2.7}$$

The Lagrangian gives a metric tensor G^*_{ij}, is given by

$$G^*_{ij} = \frac{1}{2}\frac{\partial^2 L^{*2}}{\partial y^i \, \partial y^j}. \tag{2.8}$$

From (2.7) and (2.1), we have

$$L^{*2} = e^{\sigma} F^2 + \frac{F^4}{c^2} = a_1 F^2, \tag{2.9}$$

and from (2.8) and (2.9), we have

$$G^*_{ij} = a_2 g_{ij}(x,y) + \frac{4}{c^2}\, y_i \, y_j, \tag{2.10}$$

$$G^{*ij} = \frac{1}{a_2}\left(g^{ij} - \frac{1}{a_2 c^2} y^i \, y^j\right), \tag{2.11}$$

$$C^*_{jhk} = a_2 C_{jhk} + \frac{2}{c^2}(g_{hk} y_j + g_{jh} y_k + g_{jk} y_h). \tag{2.12}$$

From (2.12) and (2.11), we have

$$C^{*i}_{jk} = C^i_{jk} + \frac{2}{a_2 c^2}\left(\delta^i_j y_k + \delta^i_k y_j + \frac{a_2}{a_6} g_{jk} y^i - \frac{8}{a_6 c^2} y^i y_k y_j\right), \tag{2.13}$$

where $a_2 = e^\sigma + \dfrac{F^2}{c^2}$ and $a_6 = e^\sigma + \dfrac{6F^2}{c^2}$. In general,

$$a_\gamma = e^\sigma + \frac{\gamma F^2}{c^2}.$$

Theorem 2.1 *If the metric tensor of generalized Lagrange space given by G_{ij} in (2.1) then the metric tensor of associated Lagrange space G^*_{ij} is given by (2.10) and reciprocal metric tensor of generalized Lagrange space and associated Lagrange space are given by (2.2) and (2.11) respectively.*

§3. Angular Metric Tensor of L^n and L^{*n}

For a Finsler space F^n the angular metric tensor h_{ij} is

$$h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j} = g_{ij} - l_i l_j, \tag{3.1}$$

where $l_i = \dfrac{y_i}{L}$.

The generalized Lagrange space is not obtained from a Lagrangian therefore its angular metric tensor H_{ij}

$$H_{ij} = G_{ij} - L_i L_j. \tag{3.2}$$

Now,

$$L_i = G_{ij} L^j = \left\{ e^\sigma g_{ij}(x,y) + \frac{1}{c^2} y_i y_j \right\} \frac{y^j}{L^*}. \tag{3.3}$$

From (2.9)

$$\begin{aligned} L_i = G_{ij} L^j &= \left(e^\sigma g_{ij}(x,y) + \frac{1}{c^2} y_i y_j \right) \frac{y^j}{\sqrt{a_1} F} = \left(e^\sigma l_i + \frac{F^2}{c^2}\frac{y_i}{F} \right) \frac{1}{\sqrt{a_1}} \\ &= \left(e^\sigma + \frac{F^2}{c^2} \right) \frac{l_i}{\sqrt{a_1}} = a_1 \frac{l_i}{\sqrt{a_1}} = \sqrt{a_1}\, l_i. \end{aligned} \tag{3.4}$$

From (3.4) and (3.2) and (2.1)

$$H_{ij} = e^\sigma g_{ij}(x,y) + \frac{1}{c^2}\, y_i\, y_j - a_1 l_i l_j. \tag{3.5}$$

Putting the value of a_1 in (3.5), we have

$$H_{ij} = e^\sigma h_{ij}. \tag{3.6}$$

The angular metric tensor of Lagrange space L^{*n} is given by

$$H^*_{ij} = L^* \frac{\partial^2 L^*}{\partial y^i\, \partial y^j}.$$

The successive differentiation of (2.9) w.r.t. y^j and y^i gives

$$L^* \frac{\partial L^*}{\partial y^j} = a_1 y_j + \frac{F^2}{c^2}\, y_j, \tag{3.7}$$

$$L^* \frac{\partial^2 L^*}{\partial y^i\, \partial y^j} + \frac{\partial L^*}{\partial y^i} \frac{\partial L^*}{\partial y^j} = \left(\frac{2}{c^2} y_i \right) y_j + a_1 g_{ij} + \frac{F^2}{c^2} g_{ij} + \frac{2}{c^2}\, y_i y_j, \tag{3.8}$$

or

$$L^*\frac{\partial^2 L^*}{\partial y^i \partial y^j} + \frac{\partial L^*}{\partial y^i}\frac{\partial L^*}{\partial y^j} = \frac{4}{c^2}y_i y_j + a_2 g_{ij},$$

or

$$L^*\frac{\partial^2 L^*}{\partial y^i \partial y^j} = \frac{4F^2}{c^2}l_i l_j - L^*_i L^*_j + a_2 g_{ij}, \tag{3.9}$$

Now, from (3.7)

$$L^* L^*_j = a_2 y_j \qquad \Rightarrow \qquad L^*_j = \frac{a_2 y_j}{L^*}. \tag{3.10}$$

From (3.9) and (3.10), we get

$$H^*_{ij} = (a_4 - e^\sigma)\, l_i l_j - \frac{a_2^2}{a_1}\, l_i\, l_j + a_2 g_{ij},$$

$$H^*_{ij} = a_2\, h_{ij} + \left(a_6 - \frac{a_2^2}{a_1}\right) l_i\, l_j. \tag{3.11}$$

Theorem 3.1 *If the metric tensor of generalized Lagrange space given by G_{ij} in (2.1), the angular metric tensor of generalized Lagrange space and associated Lagrange space are given by (3.6) and (3.11) respectively.*

§4. C-Reducibility of L^n and L^{*n}

Definition 4.1 *A generalized Lagrange space L^n is called C-reducible space if*

$$\overline{C}_{jhk} = (M_j H_{hk} + M_h H_{jk} + M_k H_{jh}), \tag{4.1}$$

where M_j are component of a covariant vector field.

Suppose generalized Lagrange space L^n is C-reducible, then (4.1) holds. Using (2.5) and (3.6) and relation $y_h = Fl_h$, (4.1) can be written as

$$e^\sigma C_{jhk} + \frac{F}{c^2} g_{jk} l_h = (M_j h_{hk} + M_h h_{jk} + M_k h_{jh})\, e^\sigma. \tag{4.2}$$

Contracting (4.2) by $l^h l^j l^k$ and using (4.1), we get

$$\frac{F}{c^2} = 0 \qquad \Rightarrow \qquad F = 0,$$

which is contradiction.

Theorem 4.1 *The generalized Lagrange space $L^n = (M^n, G_{ij})$ can not be C-reducible.*

Now consider the space L^{*n}, its C-reducibility is given by

$$C^*_{jhk} = (M^*_j H^*_{hk} + M^*_h H^*_{jk} + M^*_k H^*_{jh}), \tag{4.3}$$

where M^*_h are component of covariant vector field using (2.12), (3.11), (4.3) and $y_h = Fl_h$ in (4.3), we get

$$a_2 C_{jhk} + \frac{2F}{c^2}(g_{hk} l_j + g_{jh} l_k + g_{jk} l_h) = a_2(M^*_j h_{hk} + M^*_h h_{jk} + M^*_k h_{jh})$$
$$+ \left(a_6 - \frac{a_2^2}{a_1}\right)(M^*_j l_h l_k + M^*_h l_j l_k + M^*_k l_h l_j), \tag{4.4}$$

Contracting (4.4) by l^j and putting $\rho^* = M_i^* l^i$, we have

$$\frac{2F^2}{c^2}(g_{hk} + 2l_h l_k) = a_2 \rho^* h_{hk} + \left(a_6 - \frac{a_2^2}{a_1}\right)(\rho^* l_h l_k + M_h^* l_k + M_k^* l_h). \tag{4.5}$$

Contracting (4.5) by l^h, we have

$$\frac{6F^2}{c^2} l_k = \left(a_6 - \frac{a_2^2}{a_1}\right)(\rho^* l_k + \rho^* l_k + M_k^*). \tag{4.6}$$

Again contracting (4.6) by l^k, which gives

$$\rho^* = \frac{2F^2}{c^2}\left(\frac{a_1}{a_1 a_6 - a_2^2}\right). \tag{4.7}$$

From (4.6) and (4.7), we have

$$\frac{2F^2}{c^2} l_k = \left(a_6 - \frac{a_2^2}{a_1}\right) M_k^*. \tag{4.8}$$

From (4.8) and (4.5), we have

$$\frac{2F^2}{c^2} g_{hk} = a_2 \rho^* h_{hk} + \frac{2F^2}{c^2} l_h l_k. \tag{4.9}$$

Using $g_{hk} = h_{hk} + l_h l_k$ in (4.9), we get

$$\left(\frac{2F^2}{c^2} - a_2 \rho^*\right) h_{hk} = 0.$$

It gives $\rho^* = \dfrac{2F^2}{c^2 a_2}$, which contradict (4.7). Hence

Theorem 4.2 *The Lagrange space $L^{*n} = (M^n, L^*)$ can not be C-reducible.*

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A Study on Hamiltonian Property of Cayley Graphs Over Non-Abelian Groups

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Abstract: The hamiltonian cycles and paths in Cayley graphs naturally arise in computer science in the study of word hyperbolic groups and automatic groups. All Cayley graphs over abelian groups are always hamiltonian. However , for Cayley graphs over non-abelian groups, Chen and Quimpo prove in [1] that Cayley graphs over group of order pq, where p and q primes are Hamiltonian and in [2] that Cayley graphs over hamiltonian groups (i.e., non-abelian groups in which every subgroup is normal) are always hamiltonian. In this paper we investigate the existence of complete hamiltonian cycles and hamiltonian paths in the vertex induced subgraphs of Cayley graphs over non-abelian groups.

Key Words: Cayley graphs, hamiltonian cycles and paths,complete graph,orbit and centralizer of an element in a group,dihedral group.

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§1. Introduction

Let G be a finite non-abelian group and S be a non-empty subset of G. The graph $Cay(G,S)$ is defined as the graph whose vertex set is G and whose edges are the pairs (x,y) such that $sx=y$ for some $s\in S$ and $x\neq y$. Such a graph is called the Cayley graph of G relative to S. The definition of Cayley graphs of groups was introduced by Arthur Cayley in 1978 and the Cayley graphs of groups have received serious attention since then.Since the 1984 survey of results on hamiltonian cycles and paths in Cayley graphs by Witte and Gallian [6], many advances have been made. In this paper, we present a short survey of various results in that direction and make some observations.

§2. Preliminaries

In this section deals with the basic definitions and terminologies of group theory in [4] and [5] and graph theory in [3] which are needed in sequel.

Let G be a group. The orbit of an element x under G is usually denoted by $\bar{x}$ and is defined as $\bar{x}=\{gx/g\in G\}$. Let x be a fixed element of G. The centralizer of an element x in G, $C_G(x)$ is the set of all elements in G that commute with x. In symbols, $C_G(x)=\{g\in G/gx=xg\}$.

A group G act on G by conjugation means $gx=gxg^{-1}$ for all $x\in G$. An element $x\in G$ is called an involution if $x^2=e$, where e is the identity.Let H be a subgroup of a group G. The subset $aH=\{ah/h\in H\}$ is the left coset of H containing a, while the subset $Ha=\{ha/h\in H\}$ is the right

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coset of H containing a. The notations D_n and Z_n are the dihedral group of order $2n$ and the group of integers modulo n respectively.

A partition of a set S is a collection of non-empty disjoint subset of S whose union is S.

A graph $G = (V, E)$ is said to be connected if there is a path between any two vertices of G. If for each pair of vertices of G there exist a directed path, then it is strongly connected.

Each pair of arbitrary vertices in G can be joined by a directed edge, then it is complete. A subgraph $H = (U, F)$ of a graph $G = (V, E)$ is said to be vertex induced subgraph if F consist of all the edges of G joining pairs of vertices of U.

A hamiltonian path is a path in $G = (V, E)$ which goes through all the vertices in G exactly once. A hamiltonian cycle is a closed hamiltonian path.

§3. Main Results

Theorem 3.1 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ has two disjoint hamiltonian cycles,provided $\bar{x}$ has an element a of order 3 which do not generate $C_G(x)$ but it generates a proper cylic subgroup $\{e, a, b\}$ of $C_G(x)$.*

Proof Since $\bar{x}$ has an element a of order 3 which do not generates $C_G(x)$, we see that $x \neq e$. Let $u \in \{e, a, b\}$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation,we can choose an element $s \in \bar{x}$ such that $s = (ua)a(ua)^{-1}$. Now $su = (ua)a(ua)^{-1}u = (ua)(aa^{-1})(u^{-1}u) = ((ua)e)e = ua$, then there is an edge from u to ua. Again $s(ua) = (ua)a(ua)^{-1}(ua) = ((ua)a)e = ua^2 = ub$, then there is an edge from ua to ub , so there exist a path from u to ub. Again $s(ub) = (ua)a(ua)^{-1}(ub) = (ua)a(a^{-1}u^{-1})(ub) = (ua)(aa^{-1})(u^{-1}u)b = ((ua)e)eb = (ua)eb = uab = ue = u$, then there exist an edge from ub to u.Thus we get a hamiltonian cycle $C_1 : u \to ua \to ub \to u$ in $Cay(C_G(x), \bar{x})$.

Since $a \in \bar{x}$ which do not generate $C_G(x)$, then $C_G(x)$ contains at least one element other than $\{e, a, b\}$. Let $u_1 \notin \{e, a, b\}$. Then $su_1 = (ua)a(ua)^{-1}u_1 = (ua)a(a^{-1}u^{-1})u_1 = (ua)(aa^{-1})u^{-1}u_1 = ((ua)e)u^{-1}u_1 = (ua)u^{-1}u_1$.Since u belongs to the subgroup {e,a,b}, we have $ua = au$, therefore $(ua)u^{-1}u_1 = (au)u^{-1}u_1 = a(uu^{-1})u_1 = (ae)u_1 = au_1$. Clearly $au_1 \notin \{e, a, b\}$, for if $au_1 \in \{e, a, b\}$, then $au_1 = u_2 \in \{e, a, b\}$ which implies $u_1 = a^{-1}u_2 \in \{e, a, b\}$ which is a contradiction to our assumption that $u_1 \notin \{e, a, b\}$. So there exist an edge from u_1 to au_1. Again $s(au_1) = (ua)a(ua)^{-1}(au_1) = (ua)a(a^{-1}u^{-1})(au_1) = (ua)(aa^{-1})u^{-1}(au_1) = (ua)eu^{-1}(au_1) = (ua)u^{-1}(au_1) = (au)u^{-1}(au_1) = a(uu^{-1})au_1 = (ae)au_1 = aau_1 = a^2u_1 = bu_1$, as above we can show that $bu_1 \notin \{e, a, b\}$. Thus there exist an edge from au_1 to bu_1 and consequently a path from u_1 to bu_1. Again $s(bu_1) = (ua)a(ua)^{-1}(bu_1) = (ua)a(a^{-1}u^{-1})bu_1 = (ua)eu^{-1}(bu_1) = (au)u^{-1}(bu_1) = (ae)bu_1 = abu_1 = eu_1 = u_1$, then there exist an edge from bu_1 to u_1 . Thus we get another hamiltonian cycle $C_2 : u_1 \to au_1 \to bu_1 \to u_1$ in $Cay(C_G(x),\bar{x})$ which is disjoint from C_1. □

Theorem 3.2 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ has two complete hamiltonian cycles, one with vertex set P_1 and other with vertex set P_2, provided $C_G(x)$ has a partition (P_1, P_2), where $\bar{x}$ has an element a which generates a proper cyclic subgroup $P_1 = \{e, a, b\}$ of $C_G(x)$ and P_2 is the generating set of P_1.*

Proof Since $a \in \bar{x}$ which generates a proper cyclic subgroup $P_1 = \{e, a, b\}$ of $C_G(x)$, by Theorem 3.1, for every $u \in P_1$,we get a complete hamiltonian cycle $C_1 : u \to au \to bu \to u$ in $Cay(P_1, \bar{x})$.Since P_2 is the generating set of P_1, we have $P_2P_2 = P_1, P_2P_1 = P_2, P_1P_2 = P_2$ and $P_1P_1 = P_1$. Let

u_1 be an element in P_2. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose an element $s \in \bar{x}$ such that $s = (ua)a(ua)^{-1}$ for $u \in P_1$. Now $su_1 = (ua)a(ua)^{-1}u_1 = (ua)a(a^{-1}u^{-1})u_1 = (ua)(aa^{-1})(u^{-1}u_1) = (ua)e(u^{-1}u_1) = (ua)(u^{-1}u_1) = (au)(u^{-1}u_1) = a(uu^{-1})u_1 = (ae)u_1 = au_1$. Clearly $au_1 \notin P_1$ since $P_1P_2 = P_2$. So there exist an edge from u_1 to au_1. Again $s(au_1) = (ua)a(ua)^{-1}(au_1) = (ua)a(a^{-1}u^{-1})(au_1) = (ua)(aa^{-1})u^{-1}(au_1) = (ua)eu^{-1}(au_1) = (au)u^{-1}(au_1) = a(uu^{-1})au_1 = (ae)au_1 = (aa)u_1 = a^2u_1 = bu_1$. Here also $bu_1 \notin P_1$,so there exist an edge from au_1 to bu_1 and consequently a path from u_1 to bu_1.Again $s(bu_1) = (ua)a(ua)^{-1}(bu_1) = (ua)a(a^{-1}u^{-1})(bu_1) = (ua)(aa^{-1})u^{-1}(bu_1) = (ua)eu^{-1}(bu_1) = (ua)u^{-1}(bu_1) = (au)u^{-1}(bu_1) = a(uu^{-1})bu_1 = (ae)bu_1 = (ab)u_1 = eu_1 = u_1$, so there exist an edge from bu_1 to u_1. Thus we get another complete hamiltonian cycle $C_2 : u_1 \to au_1 \to bu_1 \to u_1$ in $Cay(P_2, \bar{x})$, which is disjoint from C_1. □

Lemma 3.3 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$,the induced subgraph $Cay(H_1, \bar{x})$of the Cayley graph $Cay(C_G(x), \bar{x})$ is hamiltonian provided $\bar{x}$ has three involutions a, b, c with $ab = ba$, which generates $C_G(x)$ and $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3.*

Proof Since $\bar{x}$ has three involutions a, b, c with $ab = ba$, which generates $C_G(x)$, we see that $C_G(x) = \{e, a, b, ab, cb, a(cb), b(cb), ab(cb), (cb)^2, a(cb)^2, b(cb)^2, ab(cb)^2\}$. Also ,since $H_1 =< b, c >$ is a subgroup of $C_G(x)$,we have $H_1 = \{e, b, cb, b(cb), (cb)^2, b(cb)^2\}$. Let $u \in H_1$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose two involutions s_1 and $s_2 \in \bar{x}$ such that $s_1 = (ub)b(ub)^{-1}$and $s_2 = (uc)c(uc)^{-1}$. Now $s_1u = (ub)b(ub)^{-1}u = (ub)b(b^{-1}u^{-1})u = (ub)(bb^{-1})(u^{-1}u) = ((ub)e)e = ub$, then there is an edge from u to ub. Again $s_2(ub) = (uc)c(uc)^{-1}(ub) = (uc)c(c^{-1}u^{-1})ub = (uc)(cc^{-1})(u^{-1}u)b = (uc)e(eb) = (uc)b = u(cb)$, then there is an edge from ub to $u(cb)$, so there exist a path from u to $u(cb)$. Again $s_1(u(cb)) = (ub)b(ub)^{-1}(u(cb)) = (ub)b(b^{-1}u^{-1})(u(cb)) = (ub)(bb^{-1})(u^{-1}u)(cb) = (ub)e(cb) = ub(cb)$, so there exist an edge from $u(cb)$to $ub(cb)$ and consequently a path from u to $ub(cb)$. Again $s_2(ub(cb)) = (uc)c(uc)^{-1}(ub(cb)) = (uc)(cc^{-1})u^{-1}(ub(cb)) = (uc)c(c^{-1}u^{-1})(ub(cb)) = (uc)e(u^{-1}u)b(cb) = (uc)eb(cb) = u(cb)^2$,then there exist an edge from $ub(cb)$ to $u(cb)^2$ and consequently a path from u to $u(cb)^2$. Again $s_1(u(cb)^2) = (ub)b(ub)^{-1}u(cb)^2 = (ub)b(b^{-1}u^{-1})u(cb)^2 = (ub)(bb^{-1})(u^{-1}u)(cb)^2 = (ub)ee(cb)^2 = (ub)(cb)^2 = ub(cb)^2$, so there exist an edge from $u(cb)^2$ to $ub(cb)^2$. Again $s_2(ub(cb)^2) = (uc)c(uc)^{-1}(ub)(cb)^2 = (uc)c(c^{-1}u^{-1})ub(cb)^2 = (uc)(cc^{-1})(u^{-1}u) \times b(cb)^2 = (uc)e(eb)(cb)^2 = ucb(cb)^2 = u(cb)^3$, so there exist an edge from $ub(cb)^2$ to $u(cb)^3$ and consequently a path from u to $u(cb)^3$. Since $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3, we have $(cb)^3 = e$. Therefore $u(cb)^3 = ue = u$. Thus we get a hamiltonian cycle $C_1 : u \to ub \to u(cb) \to ub(cb) \to u(cb)^2 \to ub(cb)^2 \to u(cb)^3 = ue = u$ in $Cay(H_1, \bar{x})$. In particularly, for $u = e$ we get a hamiltonian cycle $e \to b \to cb \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$. □

Lemma 3.4 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(aH_1, \bar{x})$ of the Cayley graph $Cay(C_G(x), \bar{x})$ is hamiltonian provided $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3.*

Proof Since $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3. Then by Lemma 3.3, for every $u \in H_1$, we get a hamiltonian cycle $C_1 : u \to ub \to u(cb) \to ub(cb) \to u(cb)^2 \to ub(cb)^2 \to u(cb)^3 = ue = u$ in $Cay(H_1, \bar{x})$. Since $aH_1 = \{ah/h \in H_1\}$, we see that $aH_1 = \{a, ab, a(cb), ab(cb), a(cb)^2, ab(cb)^2\}$. For if $u = a$ in C_1 , we get another hamiltonian cycle $C_2 : a \to ab \to a(cb) \to ab(cb) \to a(cb)^2 \to ab(cb)^2 \to a(cb)^3 = ae = a$ in $Cay(aH_1, \bar{x})$, which is disjoint from C_1. □

Theorem 3.5 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian provided $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3.*

Proof Since $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$, we see that $C_G(x) = \{e, a, b, ab, cb, a(cb), b(cb), ab(cb), (cb)^2, a(cb)^2, b(cb)^2, ab(cb)^2\}$. Let $H_1 =< b, c >$ be the subgroup of $C_G(x)$ isomorphic to D_3. Then by Lemma 3.3, we get a hamiltonian cycle $C_1 : e \to b \to (cb) \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$ in $Cay(H_1, \bar{x})$. Now, consider $aH_1 = \{ah/h \in H_1\}$. Then by Lemma 3.4, we get another hamiltonian cycle $C_2 : a \to ab \to a(cb) \to ab(cb) \to a(cb)^2 \to ab(cb)^2 \to a(cb)^3 = ae = a$ in $Cay(aH_1, \bar{x})$, which is disjoint from C_1, since $aH_1 \cap H_1 = \phi$.

We have $C_G(x) = H_1 \cup aH_1$. By removing the edges $\{e, b\}$ in $Cay(H_1, \bar{x})$ and $\{a, ab\}$ in $Cay(aH_1, \bar{x})$ and adding $\{e, a\}$ and $\{a, ab\}$ we get a hamiltonian cycle $e \to a \to ab(cb)^2 \to a(cb)^2 \to ab(cb) \to a(cb) \to (ab) \to b \to cb \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$ in $Cay(C_G(x), \bar{x})$. Thus the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian. □

Theorem 3.6 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian provided $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $C_G(x)$ has a partition (H_1, H_2) where $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3 and H_2 is the generating set of H_1.*

Proof Since $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3 , then by Lemma 3.3 we get a hamiltonian cycle $C_1 : e \to b \to cb \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$ in $Cay(H_1, \bar{x})$. We have H_2 is the generating set of H_1 and $H_2H_2 = H_1$, $H_2H_1 = H_2$, $H_1H_2 = H_2, H_1H_1 = H_1$. Since $H_1 =< b, c >$ is a subgroup of $C_G(x)$ and the involutions a, b, c in $\bar{x}$ generates $C_G(x)$, we have $a \in H_2$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose two involutions s_1 and s_2 in $\bar{x}$ such that $s_1 = (ab)b(ab)^{-1}$ and $s_2 = (ac)c(ac)^{-1}$. Now $s_1a = (ab)b(ab)^{-1}a = (ab)b(b^{-1}a^{-1})a = ab(bb^{-1})(a^{-1}a) = ((ab)e)e = ab$, so there exist an edge from a to ab in H_2 , since $H_2H_1 = H_2$. Again $s_2(ab) = (ac)c(ac)^{-1}(ab) = (ac)c(c^{-1}a^{-1})ab = (ac)(cc^{-1})(a^{-1}a)b = (ac)eb = (ac)b = a(cb)$. Clearly $a(cb) \notin H_1$, so there exist an edge from ab to $a(cb)$ in H_2. Again $s_1(a(cb)) = (ab)b(ab)^{-1}acb = (ab)(bb^{-1})(a^{-1}a)cb = (ab)ee(cb) = ab(cb)$, so there exist an edge from $a(cb)$ to $ab(cb)$ in H_2. Again $s_2ab(cb) = (ac)c(ac)^{-1}(ab)(cb) = (ac)(cc^{-1})(a^{-1}a)b(cb) = (ac)eeb(cb) = (ac)b(cb) = a(cb)(cb) = a(cb)^2$. Here also $a(cb)^2 \notin H_1$, since $H_2H_1 = H_2$, so there is an edge from $ab(cb)$ to $a(cb)^2$, consequently a path from a to $a(cb)^2$. Again $s_1a(cb)^2 = (ab)b(ab)^{-1}a(cb)^2 = (ab)(bb^{-1})(a^{-1}a)(cb)^2 = (ab)ee(cb)^2 = (ab)(cb)^2$, so there exist an edge from $a(cb)^2$ to $ab(cb)^2$. Again $s_2ab(cb)^2 = (ac)c(ac)^{-1}(ab)(cb)^2 = (ac)(cc^{-1})(a^{-1}a)b(cb)^2 = a(cb)^3 = ae = a$. Thus we get another cycle $C_2 :\ a \to ab \to a(cb) \to ab(cb) \to a(cb)^2 \to ab(cb)^2 \to a(cb)^3 = ae = a$ in $Cay(H_2, \bar{x})$, which is disjoint from C_1. We have $C_G(x) = H_1 \cup H_2$.By removing the edges $\{e, b\}$ in $Cay(H_1, \bar{x})$ and $\{a, ab\}$ in $Cay(H_2, \bar{x})$ and adding the edges $\{e, a\}$ and $\{a, ab\}$ we get a hamiltonian cycle $e \to a \to ab(cb)^2 \to a(cb)^2 \to ab(cb) \to a(cb) \to ab \to b \to cb \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$ in $Cay(C_G(x), \bar{x})$. □

Theorem 3.7 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian provided $C_G(x)$ isomorphic to $Z_{2n+1}, n = 0, 1, 2, \cdots$.*

Proof Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose an element $s \in \bar{x}$ such that $s = (ua)a(ua)^{-1}$. Now $su =$

$(ua)a(ua)^{-1}u = (ua)a(a^{-1}u^{-1})u = (ua)(aa^{-1})(u^{-1}u) = ((ua)e)e = ua$, then there exist an edge from u to ua. Again $s(ua) = (ua)a(ua)^{-1}(ua) = ((ua)a)e = ua^2$, then there is an edge from ua to ua^2, consequently a path from u to ua^2. Continuing in this way, we get a Hamiltonian cycle $u \to ua \to ua^2 \to ua^3 \to \cdots \to ua^{2n+1} = ue = u$ in $Cay(C_G(x), \bar{x})$. □

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Mean Cordial Labelling of Some Star-Related Graphs

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Abstract: Let f be a map from $V(G)$ to $\{0,1,2\}$. For each edge uv assign the label $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called as a mean cordial labelling if $|v_f(i) - v_f(j)| \leqslant 1$ and $|e^*_f(i) - e^*_f(j)| \leqslant 1$, $i, j \in \{0,1,2\}$, where $v_f(x)$ and $e^*_f(x)$ denote the number of vertices and edges respectively labelled with x $(x = 0,1,2)$. A graph with mean cordial labelling is called mean cordial. In this paper we prove the graph $\langle K_{1,n} : 2\rangle$ and path union of n copies of star $K_{1,m}$ are mean cordial graphs.

Key Words: Mean cordial labelling, Smaarndachely mean cordial labelling, star, path, union graph.

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§1. Introduction

All graphs in this paper are finite, simple and undirected. The vertex set and edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. A graph labelling is an assignment of integers to the vertices or edges or both subject to certain conditions. A useful survey on graph labelling by J. A. Gallian(2014) can be found in [2].

The concept of cordial labelling was introduced by Cahit in the year 1987 in [1]. Here we introduce the notion of mean cordial labelling. We investigate the mean cordial labelling of the graph $\langle K_{1,n} : 2\rangle$ and path union of n copies of star $K_{1,m}$.

Definition 1.1 *Let f be a map from $V(G)$ to $\{0,1,2\}$. For each edge uv assign the label $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called as a mean cordial labelling if $|v_f(i) - v_f(j)| \leq 1$ and $|e^*_f(i) - e^*_f(j)| \leq 1, i, j \in \{0,1,2\}$, otherwise, a Smarandachely mean labeling of G if $|v_f(i) - v_f(j)| \geq 2$ or $|e^*_f(i) - e^*_f(j)| \geq 2, i, j \in \{0,1,2\}$, where $v_f(x)$ and $e^*_f(x)$ denote the number of vertices and edges respectively labelled with $x(x = 0,1,2)$.*

A graph with mean cordial labelling is called a mean cordial graph.

Definition 1.2 *A complete bipartite graph $K_{1,n}$ is called a star graph. The vertex of degree n is called the apex vertex.*

Definition 1.3 *A bistar $B_{n,n}$ is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$ by an edge.*

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Definition 1.4 *Consider two copies of star $K_{1,n}$. Then $\langle K_{1,n}:2\rangle$ is the graph obtained from $B_{n,n}$ by subdividing the middle edge with a new vertex. Such as those shown in Figure 1.*

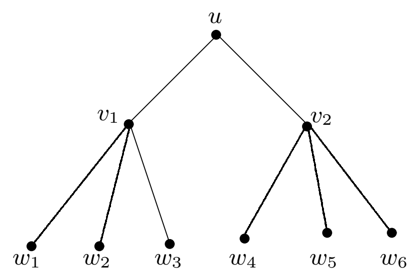

Figure 1

Definition 1.5 *Let G be a graph and $G_1, G_2, \ldots, G_n$ with $n \geqslant 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} for $i = 1, 2, \ldots, n-1$ is called path union of G.*

§2. Results

Theorem 2.1 *The graph $\langle K_{1,n}:2\rangle$ admits a mean cordial labeling.*

Proof Let $G = \langle K_{1,n}:2\rangle$ and let $V(G) = \{u, v_1, v_2, w_i : 1 \leq i \leq 2n\}$, $E(G) = \{uv_1, uv_2, v_1w_i : 1 \leq i \leq n, v_2w_j : n+1 \leq j \leq 2n\}$. Then, $|V(G)| = 2n+3$, $|E(G)| = 2n+2$.

Case 1. $n \equiv 0 \ (\ mod\ 3)$

Let $n = 3t$, $t = 1, 2, \cdots$. Define $f : V(G) \to \{0, 1, 2\}$ as follows:

$f(u) = 2, \qquad f(v_1) = 0, \qquad f(v_2) = 1,$

$$f(w_i) = \begin{cases} 0, & 1 \leqslant i \leqslant 2t \\ 1, & 2t+1 \leqslant i \leqslant 4t \\ 2, & 4t+1 \leqslant i \leqslant 6t \end{cases}$$

The induced edge labelling $f^* : E(G) \to \{0, 1, 2\}$ is found as follows:

$f^*(uv_1) = 1,$
$f^*(uv_2) = 2,$
$f^*(v_1w_i) = 0, \qquad 1 \leq i \leq 2t,$
$f^*(v_1w_i) = 1, \qquad 2t+1 \leq i \leq 3t,$
$f^*(v_2w_i) = 1, \qquad 3t+1 \leq i \leq 4t,$
$f^*(v_2w_i) = 2, \qquad 4t+1 \leq i \leq 6t.$

Then,

$v_f(o) = 2t+1, \qquad v_f(1) = 2t+1 \qquad v_f(2) = 2t+1,$
$e_f^*(0) = 2t, \qquad e_f^*(1) = 2t+1 \qquad e_f^*(2) = 2t+1.$

Thus,

$|v_f(i) - v_f(j)| \leq 1 \qquad \forall\, i, j \in \{0, 1, 2\},$
$|e_f^*(i) - e_f^*(j)| \leq 1 \qquad \forall\, i, j \in \{0, 1, 2\}.$

Hence f is a mean cordial labeling.

Case 2 $n \equiv 1\ (mod)\ 3)$

Let $n = 3t+1, \quad t = 1, 2\cdots$. Define $f : V(G) \to \{0, 1, 2\}$ as follows:

$f(u) = 2, \qquad f(v_1) = 0, \qquad f(v_2) = 1,$

$$f(w_i) = \begin{cases} 0, & 1 \le i \le 2t+1 \\ 1, & 2t+2 \le i \le 4t+2 \\ 2, & 4t+3 \le i \le 6t+2 \end{cases}$$

The induced edge labelling $f^* : E(G) \to \{0, 1, 2\}$ is defined as follows:

$f^*(uv_1) = 1,$
$f^*(uv_2) = 2,$
$f^*(v_1w_i) = 0, \qquad 1 \le i \le 2t+1,$
$f^*(v_1w_i) = 1, \qquad 2t+2 \le i \le 3t+1,$
$f^*(v_2w_i) = 1, \qquad 3t+2 \le i \le 4t+2,$
$f^*(v_2w_i) = 2, \qquad 4t+3 \leqslant i \leqslant 6t+2.$

Then,

$v_f(0) = 2t+2, \qquad v_f(1) = 2t+2, \qquad v_f(2) = 2t+1,$
$e_f^*(0) = 2t+1, \qquad e_f^*(1) = 2t+2, \qquad e_f^*(2) = 2t+1.$

Thus,

$|v_f(i) - v_f(j)| \le 1 \quad \forall \quad i, j \in \{0, 1, 2\},$
$|e_f^*(i) - e_f^*(j)| \le 1 \quad \forall \quad i, j \in \{0, 1, 2\}$

Hence f is a mean cordial labeling.

Case 3. $n \equiv 2(mod3)$

Let $\quad n = 3t+2, \quad t = 1, 2\cdots$. Define $f : V(G) \to \{0, 1, 2\}$ as follows:

$f(u) = 2, \qquad f(v_1) = 0, \qquad f(v_2) = 1,$

$$f(w_i) = \begin{cases} 0, & 1 \leqslant i \leqslant 2t+2 \\ 1, & 2t+3 \le i \le 4t+3 \\ 2, & 4t+4 \le i \le 6t+4 \end{cases}$$

The induced edge labelling $f^* : E(G) \to \{0, 1, 2\}$ is found as follows:

$f^*(uv_1) = 1,$
$f^*(uv_2) = 2,$
$f^*(v_1w_i) = 0, \qquad 1 \le i \le 2t+2,$
$f^*(v_1w_i) = 1, \qquad 2t+3 \le i \le 3t+2,$
$f^*(v_2w_i) = 1, \qquad 3t+3 \le i \le 4t+3,$
$f^*(v_2w_i) = 2, \qquad 4t+4 \le i \le 6t+4.$

Then,

$v_f(0) = 2t+3, \quad v_f(1) = 2t+2, \quad v_f(2) = 2t+2,$
$e_f^*(0) = 2t+2, \quad e_f^*(1) = 2t+2, \quad e_f^*(2) = 2t+2.$

Thus,

$|v_f(i) - v_f(j)| \le 1 \quad \forall\, i, j \in \{0, 1, 2\},$
$|e_f^*(i) - e_f^*(j)| \le 1 \quad \forall \ \ i, j \in \{0, 1, 2\}.$

Thus, f is a mean cordial labelling. Hence, $\langle K_{1,n} : 2\rangle$ is a mean cordial graph. $\square$

Illustration 2.2 The mean cordial labelling of $\langle K_{1,6} : 2\rangle$ is shown in Figure 2.

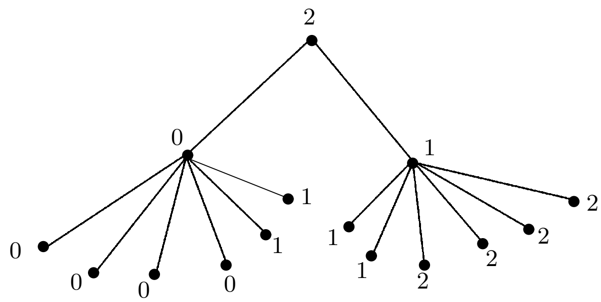

Figure 2

Illustration 2.3 The mean cordial labelling of $\langle K_{1,7} : 2\rangle$ is shown in Figure 3.

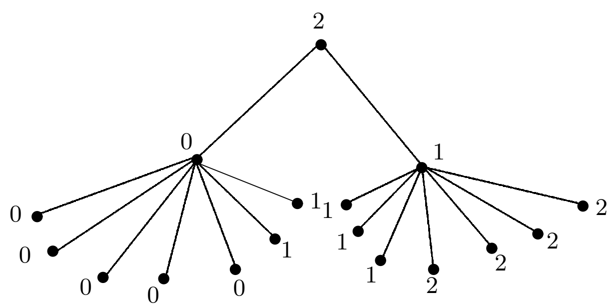

Figure 3

Theorem 2.4 *The path union of n copies of star $K_{1,m}$ is a mean cordial graph.*

Proof Let G be the path union of $'n'$ copies of star$K_{1,m}$ and let $V(G) = \{v_i : i = 1, \cdots n; w_{ij} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$, $E(G) = \{v_i v_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{v_i w_{ij} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$. Then, $|V(G)| = n(m+1)$ *and* $|E(G)| = n(m+1) - 1$.

Case 1. $m \equiv 0(mod3)$

Subcase 1.1 $n \equiv 0(mod3)$

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{3} \\ 1, & \frac{n}{3} + 1 \leqslant i \leqslant \frac{2n}{3} \\ 2, & \frac{2n}{3} + 1 \leqslant i \leqslant n \end{cases}$$

$$f(w_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n}{3} & 1 \leqslant j \leqslant m \\ 1, & \frac{n}{3} + 1 \leqslant i \leqslant \frac{2n}{3} & 1 \leqslant j \leqslant m \\ 2, & \frac{2n}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

The induced edge labelling $f^* : E(G) \to \{0, 1, 2\}$ is known as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n}{3} - 1 \\ 1, & \frac{n}{3} \leqslant i \leqslant \frac{2n}{3} - 1 \\ 2, & \frac{2n}{3} \leqslant i \leqslant n - 1 \end{cases}$$

$$f^*(v_iw_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n}{3} & 1 \leqslant j \leqslant m \\ 1, & \frac{n}{3} + 1 \leqslant i \leqslant \frac{2n}{3} & 1 \leqslant j \leqslant m \\ 2, & \frac{2n}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

Then,

$$v_f(0) = \frac{nm+n}{3}, \quad v_f(1) = \frac{nm+n}{3}, \quad v_f(2) = \frac{nm+n}{3} \text{ and}$$
$$e_f^*(0) = \frac{mn+n-3}{3}, \quad e_f^*(1) = \frac{mn+n}{3}, \quad e_f^*(2) = \frac{mn+n}{3}.$$

Thus,

$$|v_f(i) - v_f(j)| \leqslant 1 \text{ and } |e_f^*(i) - e_f^*(j)| \leqslant 1 \quad \forall \quad i, j \in \{0, 1, 2\}.$$

Hence f is a mean cordial labelling.

Subcase 1.2 $n \equiv 1(mod3)$

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

$$f(v_i) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} \\ 1, & \frac{n+2}{3} + 1 \leqslant i \leqslant \frac{2n+1}{3} \\ 2, & \frac{2n+1}{3} + 1 \leqslant i \leqslant n \end{cases}$$

$$f(w_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} - 1 & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+2}{3} & 1 \leqslant j \leqslant \frac{m}{3} \\ 1, & i = \frac{n+2}{3} & \frac{m}{3} + 1 \leqslant j \leqslant m \\ 1, & \frac{n+5}{3} \leqslant i \leqslant \frac{2n-2}{3} & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+1}{3} & 1 \leqslant j \leqslant \frac{2m}{3} \\ 2, & i = \frac{2n+1}{3} & \frac{2m}{3} + 1 \leqslant j \leqslant m \\ 2, & \frac{2n+4}{3} \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

We get the induced edge labelling $f^* : E(G) \longrightarrow \{0, 1, 2\}$ as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} - 1 \\ 1, & \frac{n+2}{3} \leqslant i \leqslant \frac{2n-2}{3} \\ 2, & \frac{2n-2}{3} + 1 \leqslant i \leqslant n - 1 \end{cases}$$

$$f^*(v_iw_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n-1}{3} & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+2}{3} & 1 \leqslant j \leqslant \frac{m}{3} \\ 1, & i = \frac{n+2}{3} & \frac{m+3}{3} \leqslant j \leqslant m \\ 1, & \frac{n+5}{3} \leqslant i \leqslant \frac{2n-2}{3} & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+1}{3} & 1 \leqslant j \leqslant \frac{2m}{3} \\ 2, & i = \frac{2n+1}{3} & \frac{2m+3}{3} \leqslant j \leqslant m \\ 2, & \frac{2n+4}{3} \leqslant i \leqslant n, & 1 \leqslant j \leqslant m \end{cases}$$

Then,

$$v_f(0)=\frac{n+nm-2}{3},\ v_f(1)=\frac{n+nm-1}{3},\ v_f(2)=\frac{n+nm-1}{3},$$
$$e_f^*(0)=\frac{mn+n-1}{3},\ e_f^*(1)=\frac{mn+n-1}{3},\ e_f^*(2)=\frac{mn+n-1}{3}.$$

Thus,

$$|v_f(i)-v_f(j)|\leqslant 1\ \forall\, i,j\in\{0,1,2\}\ \text{ and }\ |e_f^*(i)-e_f^*(j)|\leqslant 1\ \forall\, i,j\in\{0,1,2\}.$$

Hence f is a mean cordial labelling.

Subcase 1.3 $n\equiv 2(mod3)$

Define $f:V(G)\to\{0,1,2\}$ as follows:

$$f(v_i)=\begin{cases}0, & 1\leqslant i\leqslant \frac{n+1}{3}\\ 1, & \frac{n+1}{3}+1\leqslant i\leqslant \frac{2n+2}{3}\\ 2, & \frac{2n+2}{3}+1\leqslant i\leqslant n\end{cases}$$

$$f(w_{ij})=\begin{cases}0, & 1\leqslant i\leqslant \frac{n+1}{3}-1 & 1\leqslant j\leqslant m\\ 0, & i=\frac{n+1}{3} & 1\leqslant j\leqslant \frac{2m}{3}\\ 1, & i=\frac{n+1}{3} & \frac{2m}{3}+1\leqslant j\leqslant m\\ 1, & \frac{n+4}{3}\leqslant i\leqslant \frac{2n+2}{3}-1 & 1\leqslant j\leqslant m\\ 1, & i=\frac{2n+2}{3} & 1\leqslant j\leqslant \frac{m}{3}\\ 2, & i=\frac{2n+2}{3} & \frac{m}{3}+1\leqslant j\leqslant m\\ 2, & \frac{2n+2}{3}+1\leqslant i\leqslant n & 1\leqslant j\leqslant m\end{cases}$$

We get the induced edge labelling $f^*:E(G)\longrightarrow\{0,1,2\}$ as follows:

$$f^*(v_iv_{i+1})=\begin{cases}0, & 1\leqslant i\leqslant \frac{n+1}{3}-1\\ 1, & \frac{n+1}{3}\leqslant i\leqslant \frac{2n+2}{3}-1\\ 2, & \frac{2n+2}{3}\leqslant i\leqslant n-1\end{cases}$$

$$f^*(v_iw_{ij})=\begin{cases}0, & 1\leqslant i\leqslant \frac{n-2}{3} & 1\leqslant j\leqslant m\\ 0, & i=\frac{n-2}{3}+1 & 1\leqslant j\leqslant \frac{2m}{3}\\ 1, & i=\frac{n-2}{3}+1 & \frac{2m}{3}+1\leqslant j\leqslant m\\ 1, & \frac{n+4}{3}\leqslant i\leqslant \frac{2n+2}{3}-1 & 1\leqslant j\leqslant m\\ 1, & i=\frac{2n+2}{3} & 1\leqslant j\leqslant \frac{m}{3}\\ 2, & i=\frac{2n+2}{3} & \frac{m}{3}+1\leqslant j\leqslant m\\ 2, & \frac{2n+2}{3}+1\leqslant i\leqslant n & 1\leqslant j\leqslant m\end{cases}$$

Then,

$$v_f(0)=\frac{n+nm+1}{3},\ v_f(1)=\frac{n+nm+1}{3},\ v_f(2)=\frac{n+nm-2}{3},$$
$$e_f^*(0)=\frac{mn+n-2}{3},\ e_f^*(1)=\frac{mn+n+1}{3},\ e_f^*(2)=\frac{mn+n-2}{3}.$$

Thus,

$$|v_f(i)-v_f(j)|\leqslant 1\ \text{ and }\ |e_f^*(i)-e_f^*(j)|\leqslant 1\ \forall\, i,j\in\{0,1,2\}.$$

Hence f is a mean cordial labelling.

Case 2. $m \equiv 1(mod3)$

Subcase 2.1 $n \equiv 0(mod3)$

Define $f : V(G) \to \{0,1,2\}$ as follows:

$$f(v_i)=\begin{cases}0, & 1 \le i \le \frac{n}{3}\\ 1, & \frac{n}{3}+1 \leqslant i \leqslant \frac{2n}{3}\\ 2, & \frac{2n}{3}+1 \leqslant i \leqslant n\end{cases}$$

$$f(w_{ij})=\begin{cases}0, & 1 \leqslant i \leqslant \frac{n}{3} & 1 \leqslant j \leqslant m\\ 1, & \frac{n}{3}+1 \leqslant i \leqslant \frac{2n}{3} & 1 \leqslant j \leqslant m\\ 2, & \frac{2n}{3}+1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m\end{cases}$$

We then know the induced edge labelling $f^* : E(G) \to \{0,1,2\}$ as follows:

$$f^*(v_iv_{i+1})=\begin{cases}0, & 1 \leqslant i \leqslant \frac{n}{3}-1\\ 1, & \frac{n}{3} \leqslant i \leqslant \frac{2n}{3}-1\\ 2, & \frac{2n}{3} \leqslant i \leqslant n-1\end{cases}$$

$$f^*(v_iw_{ij})=\begin{cases}0, & 1 \leqslant i \leqslant \frac{n}{3} & 1 \leqslant j \leqslant m\\ 1, & \frac{n}{3}+1 \leqslant i \leqslant \frac{2n}{3} & 1 \leqslant j \leqslant m\\ 2, & \frac{2n}{3}+1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m\end{cases}$$

Then,

$$v_f(0)=\frac{nm+n}{3},\quad v_f(1)=\frac{nm+n}{3},\quad v_f(2)=\frac{nm+n}{3},$$
$$e^*_f(0)=\frac{mn+n-3}{3},\quad e^*_f(1)=\frac{mn+n}{3},\quad e^*_f(2)=\frac{mn+n}{3}.$$

Thus,

$$|v_f(i)-v_f(j)| \leqslant 1 \text{ and } |e^*_f(i)-e^*_f(j)| \leqslant 1 \qquad \forall \quad i,j \in \{0,1,2\}.$$

Hence f is a mean cordial labelling.

Subcase 2.2 $n \equiv 1(mod3)$

Define $f : V(G) \to \{0,1,2\}$ as follows:

$$f(v_i)=\begin{cases}0, & 1 \leqslant i \leqslant \frac{n+2}{3}\\ 1, & \frac{n+2}{3}+1 \leqslant i \leqslant \frac{2n+1}{3}\\ 2, & \frac{2n+1}{3}+1 \leqslant i \leqslant n\end{cases}$$

$$f(w_{ij})=\begin{cases}0, & 1 \leqslant i \leqslant \frac{n+2}{3}-1 & 1 \leqslant j \leqslant m\\ 0, & i=\frac{n+2}{3} & 1 \leqslant j \leqslant \frac{m-1}{3}\\ 1, & i=\frac{n+2}{3} & \frac{m-1}{3}+1 \leqslant j \leqslant m\\ 1, & \frac{n+5}{3} \leqslant i \leqslant \frac{2n+1}{3} & 1 \leqslant j \leqslant m\\ 1, & i=\frac{2n+1}{3} & 1 \leqslant j \leqslant \frac{2m+1}{3}\\ 2, & i=\frac{2n+1}{3} & \frac{2m+1}{3}+1 \leqslant j \leqslant m\\ 2, & \frac{2n+4}{3} \leqslant i \leqslant n & 1 \leqslant j \leqslant m\end{cases}$$

The induced edge labelling$f^* : E(G) \longrightarrow \{0,1,2\}$ is known as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} - 1 \\ 1, & \frac{n+2}{3} \leqslant i \leqslant \frac{2n+1}{3} - 1 \\ 2, & \frac{2n+1}{3} \leqslant i \leqslant n-1 \end{cases}$$

$$f^*(v_iw_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} - 1 & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+2}{3} & 1 \leqslant j \leqslant \frac{m-1}{3} \\ 1, & i = \frac{n+2}{3} & \frac{m-1}{3} + 1 \leqslant j \leqslant m \\ 1, & \frac{n+5}{3} \leqslant i \leqslant \frac{2n-2}{3} & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+1}{3} & 1 \leqslant j \leqslant \frac{2m+1}{3} \\ 2, & i = \frac{2n+1}{3} & \frac{2m+1}{3} + 1 \leqslant j \leqslant m \\ 2, & \frac{2n+1}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

Then,

$$v_f(0) = \frac{n+nm+1}{3},\ v_f(1) = \frac{n+nm+1}{3},\ v_f(2) = \frac{n+nm-2}{3},$$
$$e_f^*(0) = \frac{mn+n-2}{3},\ e_f^*(1) = \frac{mn+n+1}{3},\ e_f^*(2) = \frac{mn+n-2}{3}.$$

Thus,

$$|v_f(i) - v_f(j)| \leqslant 1 \text{ and } |e_f^*(i) - e_f^*(j)| \leqslant 1 \quad \forall\, i,j \in \{0,1,2\}.$$

Hence f is a mean cordial labelling.

Subcase 2.3 $n \equiv 2(mod 3)$

Define $f : V(G) \to \{0,1,2\}$ as follows:

$$f(v_i) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+1}{3} \\ 1, & \frac{n+1}{3} + 1 \leqslant i \leqslant \frac{2n+2}{3} \\ 2, & \frac{2n+2}{3} + 1 \leqslant i \leqslant n \end{cases}$$

$$f(w_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+1}{3} - 1 & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+1}{3} & 1 \leqslant j \leqslant \frac{2m+1}{3} \\ 1, & i = \frac{n+1}{3} & \frac{2m+1}{3} + 1 \leqslant j \leqslant m \\ 1, & \frac{n+4}{3} \leqslant i \leqslant \frac{2n+2}{3} - 1 & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+2}{3} & 1 \leqslant j \leqslant \frac{m-1}{3} \\ 2, & i = \frac{2n+2}{3} & \frac{m-1}{3} + 1 \leqslant j \leqslant m \\ 2, & \frac{2n+2}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

The induced edge labelling $f^* : E(G) \longrightarrow \{0,1,2\}$ is known as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+1}{3} - 1 \\ 1, & \frac{n+1}{3} \leqslant i \leqslant \frac{2n+2}{3} - 1 \\ 2, & \frac{2n+2}{3} \leqslant i \leqslant n-1 \end{cases}$$

$$f^*(v_iw_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+1}{3} - 1 & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+1}{3} & 1 \leqslant j \leqslant \frac{2m+1}{3} \\ 1, & i = \frac{n+1}{3} & \frac{2m+1}{3} + 1 \leqslant j \leqslant m \\ 1, & \frac{n+1}{3} + 1 \leqslant i \leqslant \frac{2n+2}{3} - 1 & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+2}{3} & 1 \leqslant j \leqslant \frac{m-1}{3} \\ 2, & i = \frac{2n+2}{3} & \frac{m-1}{3} + 1 \leqslant j \leqslant m \\ 2, & \frac{2n+2}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

Then,

$$v_f(0) = \frac{n + nm + 2}{3},\ v_f(1) = \frac{n + nm - 1}{3},\ v_f(2) = \frac{n + nm - 1}{3},$$
$$e_f^*(0) = \frac{mn + n - 1}{3},\ e_f^*(1) = \frac{mn + n - 1}{3},\ e_f^*(2) = \frac{mn + n - 1}{3}.$$

Thus,

$$|v_f(i) - v_f(j)| \leqslant 1 \text{ and } |e_f^*(i) - e_f^*(j)| \leqslant 1 \ \forall\, i, j \in \{0, 1, 2\}.$$

Hence f is a mean cordial labelling.

Case 3. $m \equiv 2(mod3)$

Subcase 3.1 $n \equiv 0(mod3)$

Define $f : V(G) \to \{0, 1, 2\}$ as follows:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{3} \\ 1, & \frac{n}{3} + 1 \leqslant i \leqslant \frac{2n}{3} \\ 2, & \frac{2n}{3} + 1 \leqslant i \leqslant n \end{cases}$$

$$f(w_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n}{3} & 1 \leqslant j \leqslant m \\ 1, & \frac{n}{3} + 1 \leqslant i \leqslant \frac{2n}{3} & 1 \leqslant j \leqslant m \\ 2, & \frac{2n}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

We get the induced edge labelling $f^* : E(G) \to \{0, 1, 2\}$ as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n}{3} - 1 \\ 1, & \frac{n}{3} \leqslant i \leqslant \frac{2n}{3} - 1 \\ 2, & \frac{2n}{3} \leqslant i \leqslant n - 1 \end{cases}$$

$$f^*(v_iw_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n}{3} & 1 \leqslant j \leqslant m \\ 1, & \frac{n}{3} + 1 \leqslant i \leqslant \frac{2n}{3} & 1 \leqslant j \leqslant m \\ 2, & \frac{2n}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

Then,

$$v_f(0) = \frac{nm + n}{3}, \quad v_f(1) = \frac{nm + n}{3}, \quad v_f(2) = \frac{nm + n}{3},$$
$$e_f^*(0) = \frac{mn + n - 3}{3}, \quad e_f^*(1) = \frac{mn + n}{3}, \quad e_f^*(2) = \frac{mn + n}{3}.$$

Thus,

$$|v_f(i) - v_f(j)| \leqslant 1 \text{ and } |e_f^*(i) - e_f^*(j)| \leqslant 1 \quad \forall \quad i, j \in \{0, 1, 2\}.$$

Hence f is a mean cordial labelling.

Subcase 3.2 $n \equiv 1(mod3)$

Define $f : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$f(v_i) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} \\ 1, & \frac{n+2}{3} + 1 \leqslant i \leqslant \frac{2n+1}{3} \\ 2, & \frac{2n+1}{3} + 1 \leqslant i \leqslant n \end{cases}$$

$$f(w_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} - 1 & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+2}{3} & 1 \leqslant j \leqslant \frac{m-2}{3} \\ 1, & i = \frac{n+2}{3} & \frac{m-2}{3} + 1 \leqslant j \leqslant m \\ 1, & \frac{n+2}{3} + 1 \leqslant i \leqslant \frac{2n+1}{3} - 1 & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+1}{3} & 1 \leqslant j \leqslant \frac{2m-1}{3} \\ 2, & i = \frac{2n+1}{3} & \frac{2m-1}{3} + 1 \leqslant j \leqslant m \\ 2, & \frac{2n+1}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

The induced edge labelling$f^* : E(G) \longrightarrow \{0, 1, 2\}$ is calculated as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} - 1 \\ 1, & \frac{n+2}{3} \leqslant i \leqslant \frac{2n+1}{3} - 1 \\ 2, & \frac{2n+1}{3} \leqslant i \leqslant n - 1 \end{cases}$$

$$f^*(v_i w_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+2}{3} - 1 & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+2}{3} & 1 \leqslant j \leqslant \frac{m-2}{3} \\ 1, & i = \frac{n+2}{3} & \frac{m-2}{3} + 1 \leqslant j \leqslant m \\ 1, & \frac{n+2}{3} + 1 \leqslant i \leqslant \frac{2n+1}{3} - 1 & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+1}{3} & 1 \leqslant j \leqslant \frac{2m-1}{3} \\ 2, & i = \frac{2n+1}{3} & \frac{2m-1}{3} + 1 \leqslant j \leqslant m \\ 2, & \frac{2n+1}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

Then,

$$v_f(0) = \frac{n + nm}{3},\ v_f(1) = \frac{n + nm}{3},\ v_f(2) = \frac{n + nm}{3},$$
$$e^*_f(0) = \frac{mn + n - 3}{3},\ e^*_f(1) = \frac{mn + n}{3},\ e^*_f(2) = \frac{mn + n}{3}.$$

Thus,

$$|v_f(i) - v_f(j)| \leqslant 1 \ \text{ and } \ |e^*_f(i) - e^*_f(j)| \leqslant 1 \ \forall\, i, j \in \{0, 1, 2\}.$$

Hence f is a mean cordial labelling.

Subcase 3.3 $n \equiv 2(mod3)$

Define $f : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$f(v_i) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+1}{3} \\ 1, & \frac{n+1}{3} + 1 \leqslant i \leqslant \frac{2n+2}{3} \\ 2, & \frac{2n+2}{3} + 1 \leqslant i \leqslant n \end{cases}$$

$$f(w_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+1}{3} - 1 \quad & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+1}{3} & 1 \leqslant j \leqslant \frac{2m-1}{3} \\ 1, & i = \frac{n+1}{3} & \frac{2m-1}{3} + 1 \leqslant j \leqslant m \\ 1, & \frac{n+1}{3} + 1 \leqslant i \leqslant \frac{2n+2}{3} - 1 & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+2}{3} & 1 \leqslant j \leqslant \frac{m-2}{3} \\ 2, & i = \frac{2n+2}{3} & \frac{m-2}{3} + 1 \leqslant j \leqslant m \\ 2, & \frac{2n+2}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

The induced edge labelling $f^* : E(G) \longrightarrow \{0, 1, 2\}$ is found as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+1}{3} - 1 \\ 1, & \frac{n+1}{3} \leqslant i \leqslant \frac{2n+2}{3} - 1 \\ 2, & \frac{2n+2}{3} \leqslant i \leqslant n - 1 \end{cases}$$

$$f^*(v_i w_{ij}) = \begin{cases} 0, & 1 \leqslant i \leqslant \frac{n+1}{3} - 1 \quad & 1 \leqslant j \leqslant m \\ 0, & i = \frac{n+1}{3} & 1 \leqslant j \leqslant \frac{2m-1}{3} \\ 1, & i = \frac{n+1}{3} & \frac{2m-1}{3} + 1 \leqslant j \leqslant m \\ 1, & \frac{n+1}{3} + 1 \leqslant i \leqslant \frac{2n+2}{3} - 1 & 1 \leqslant j \leqslant m \\ 1, & i = \frac{2n+2}{3} & 1 \leqslant j \leqslant \frac{m-2}{3} \\ 2, & i = \frac{2n+2}{3} & \frac{m-2}{3} + 1 \leqslant j \leqslant m \\ 2, & \frac{2n+2}{3} + 1 \leqslant i \leqslant n & 1 \leqslant j \leqslant m \end{cases}$$

Then,

$$v_f(0) = \frac{n + nm}{3},\ v_f(1) = \frac{n + nm}{3},\ v_f(2) = \frac{n + nm}{3},$$
$$e_f^*(0) = \frac{mn + n - 3}{3},\ e_f^*(1) = \frac{mn + n}{3},\ e_f^*(2) = \frac{mn + n}{3}.$$

Thus,

$$|v_f(i) - v_f(j)| \leqslant 1 \ \text{ and } \ |e_f^*(i) - e_f^*(j)| \leqslant 1 \ \forall\, i, j \in \{0, 1, 2\},$$

Hence f is a mean cordial labelling. Thus, from all these above cases we conclude that the path union of n copies of star $K_{1,m}$ is a mean cordial graph. □

Illustration 2.5 A mean cordial labelling of four copies of star $K_{1,6}$ is shown in Figure 4.

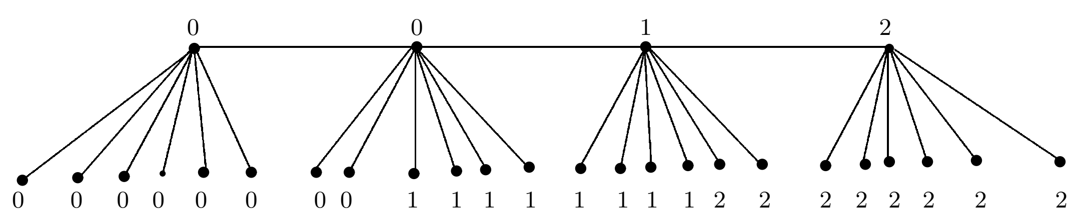

Figure 4

Illustration 2.6 A mean cordial labelling of two copies of star $K_{1,6}$ is shown in Figure 5.

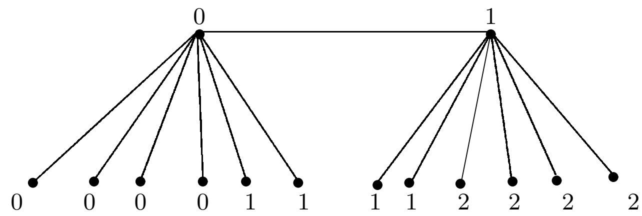

Figure 5

Illustration 2.7 A mean cordial labelling of six copies of star $K_{1,3}$ is shown in Figure 6.

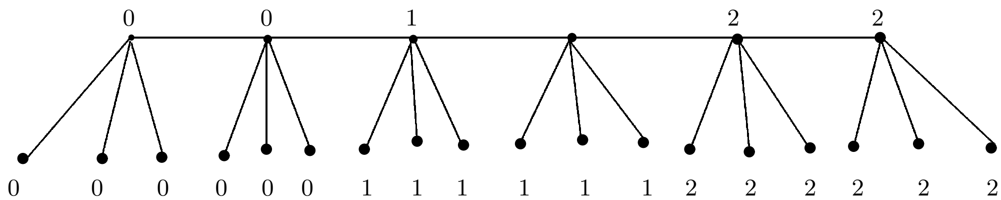

Figure 6

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Some New Families of Odd Graceful Graphs

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Abstract: A labeling or numbering of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a labeling depending on the vertex labels $f(u)$ and $f(v)$. In this paper we study some new families of odd graceful graphs.

Key Words: Labeling, odd-even graceful graph, tree.

AMS(2010): 05C78.

§1. Introduction

Unless mentioned otherwise, a graph in this paper shall mean a simple finite graph without isolated vertices. For all terminology and notations in Graph Theory, we follow [1], and all terminology regarding to labeling, we follow [2] and [3].

Gnanajothi [3] introduced the concept of odd graceful graphs as an extension of graceful graphs. A graph $G = (V, E)$ with p vertices and q edges is said to admit an *odd graceful labeling* if $f : V(G) \to \{0, 1, 2, \cdots, 2q - 1\}$ is injective and the induced function $f^* : E(G) \to \{1, 3, 5, \cdots, 2q - 1\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is bijective. The graph which admits odd graceful labeling is called an odd graceful graph. In the present paper, we investigate some new families of odd graceful graphs generated from various graph operations on the given graph.

§2. Main Results

Definition 2.1 *Let G_n^* be a graph with vertex set $V = \{a_i, b_i / i = 1, 2, \cdots, n\}$ and $E = \{a_i a_{i+1}, b_i b_{i+1}, a_i b_{i+1}, b_i a_{i+1} / i = 1, 2, \cdots, n - 1\}$.*

Definition 2.2 *D_n^* be a graph with $V = \{a_{ij} / i = 1, 2, \cdots, n; j = 1, 2, 3, 4\}$ and $E = \{a_{i,1} a_{i+1,1} / i = 1, 2, \cdots, n - 1\} \cup \{a_{i,3} a_{i+1,3} / i = 1, 2, \cdots, n - 1\} \cup \{a_{i,1} a_{i,2}; a_{i,2} a_{i,3}; a_{i,3} a_{i,4}; a_{i,4} a_{i,1} / i = 1, 2, \cdots n\}$.*

Theorem 2.1 *Let G_n^* be a graph with vertex $V = \{a_i, b_i / i = 1, 2, \cdots, n\}$ and $E = \{a_i a_{i+1}, b_i b_{i+1}, a_i b_{i+1}, b_i a_{i+1} / i = 1, 2, \cdots, n - 1\}$. Then G is odd graceful.*

Proof Let G_n^* be a graph with vertex set $V = \{a_i, b_i / i = 1, 2, \cdots, n\}$ and $E = \{a_i a_{i+1}, b_i b_{i+1}, a_i b_{i+1}, b_i a_{i+1} / i = 1, 2, \cdots, n - 1\}$. Note that, it has $2n$ vertices and $4n - 4$ edges.

[1]Received July 25, 2015, Accepted August 31, 2016.

Define a function $f : V(G_n^*) \to \{0, 1, 2, \cdots, 8n-9 = 2(4n-4)-1\}$ such that

$$\begin{aligned}
f(a_1) &= 0\\
f(a_{2i-1}) &= 2(4i-3); 2 \leqslant i \leqslant \frac{n}{2} \text{ if } n \text{ is even, or } 2 \leqslant i \leqslant \frac{n+1}{2} \text{ if } n \text{ is odd}\\
f(a_{2i}) &= 8n-5-8i; 1 \leqslant i \leqslant \frac{n}{2} \text{ if } n \text{ is even, or } 1 \leqslant i \leqslant \frac{n-1}{2} \text{ if } n \text{ is odd}\\
f(b_1) &= 2\\
f(b_{2i-1}) &= 8(i-1); 2 \leqslant i \leqslant \frac{n}{2} \text{ if } n \text{ is even, or } 2 \leqslant i \leqslant \frac{n+1}{2} \text{ if } n \text{ is odd}\\
f(b_{2i}) &= 8n-1-8i; 1 \leqslant i \leqslant \frac{n}{2} \text{ if } n \text{ is even, or } 1 \leqslant i \leqslant \frac{n-1}{2} \text{ if } n \text{ is odd}
\end{aligned}$$

It is easy to show that f is injective. Also $\max\limits_{v \in V(G_n^*)} f(v) = 8n-9$. Thus, $f(v) \in \{0, 1, 2, \cdots, 8n-9\}$, for all $v \in V(G_n^*)$.

Now, it can be easily verified that all the edge values are in the interval $[1, 8n-9]$. Thus, f is an odd graceful numbering. Hence, the graph G_n^* is odd graceful. □

An odd graceful labelling of the graph G_8^* is displayed in Figure 2.1.

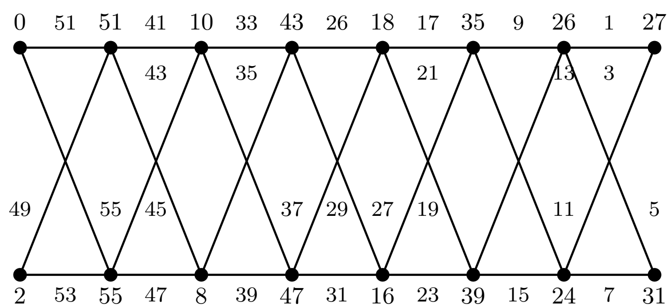

Figure 2.1

Theorem 2.2 *Let D_n^* be a graph with $V = \{a_{ij}/i = 1, 2, \cdots, n, j = 1, 2, 3, 4\}$ and $E = \{a_{i,1}a_{i+1,1}/i = 1, 2, \cdots, n-1\} \cup \{a_{i,3}a_{i+1,3}/i = 1, 2, \cdots, n-1\} \cup \{a_{i,1}a_{i,2}; a_{i,2}a_{i,3}; a_{i,3}a_{i,4}; a_{i,4}a_{i,1},\ i = 1, 2, \cdots n\}$. Then D_n^* is odd graceful for any n.*

Proof Let $\{a_{ij}/i = 1, 2, \cdots, n; j = 1, 2, 3, 4\}$ be the set of vertices of D_n^*. Note that, D_n^* has $4n$ vertices and $6n-2$ edges.

Define a function $f : V(D_n^*) \to \{0, 1, 2, \cdots, 12n-5 = 2(6n-2)-1\}$ such that

$$f(a_{i,1}) = \begin{cases} 12i-4 & \text{if } i \text{ is even}\\ 12(n-i)+7 & \text{if } n \text{ is odd}\end{cases}$$

$$f(a_{i,2}) = \begin{cases} 12(n-i)-3) & \text{if } i \text{ is even}\\ 12(i-1) & \text{if } i \text{ is odd}\end{cases}$$

$$f(a_{i,3}) = \begin{cases} 12i-6 & \text{if } i \text{ is even}\\ 12(n-i)+3 & \text{if } i \text{ is odd}\end{cases}$$

$$f(a_{i,4}) = \begin{cases} 12(n-i)+1 & \text{if } i \text{ is even} \\ 12i-10 & \text{if } i \text{ is odd} \end{cases}$$

for $i = 1, 2, \cdots n$. Let f^* be the edge labelling induced by f such that $f^*(uv) = |f(u) - f(v)|$. Now we split the edge set of D_n^* into three disjoint sum.

Let $P = \{f^*(a_{ij}, a_{i,j+1})/i = 1, 2, \cdots, n; j = 1, 2, 3\} \cup \{f^*(a_{i,4}a_{i1})/i = 1, 2, \cdots, n\}$. Then the values of edges under P is $\{1, 3, 5, 7; 13, 15, 17, 19; 25, 27, 29, 31; \cdots; 12n-11, 12n-9, 12n-7, 12n-5\}$.

Let $Q = \{f^*(a_{i1}, a_{i+1,1})/i = 1, 2, \cdots, n-1\}$. Then the corresponding edge values are $\{11, 23, 35, \cdots, 12n-13\}$.

Let $R = \{f^*(a_{i,3}a_{i+1,3})/i = 1, 2, \cdots, n-1\}$. Then the corresponding edge values are equal to $\{9, 21, 31, \cdots, 12n-15\}$.

Next, consider $P \cup Q \cup R$.

$$P \cup Q \cup R = \{1, 3, 5, 7, \ldots, 12n-5 = 2q-1\}$$

That is, the edge values of D_n^* are $\{1, 3, 5, 7, \ldots, 12n-5 = 2q-1\}$. It can be easily shown that f is injective on $V(D_n^*)$ and it is obvious that f is an odd graceful numbering. Hence, the graph D_n^* is odd graceful. □

Figure 2.2 gives an odd graceful labelling of the graph D_6^*.

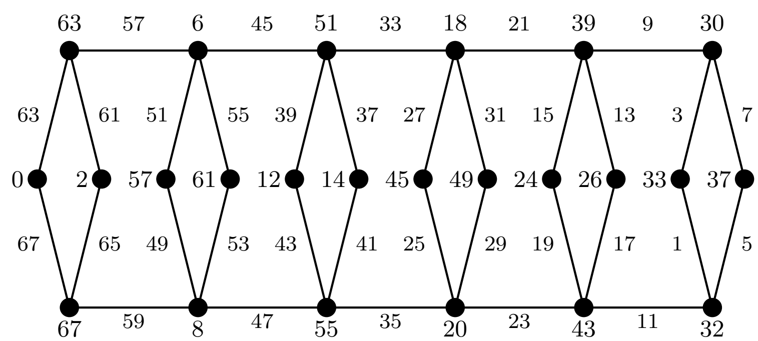

Figure 2.2

Observation 2.1 Let K_2 be a complete graph on two vertices. Take $2n$ copies of K_2. Keep n copies of K_2 in one set and n copies in the second set. Let $u_{i,1}, u_{i,2}$ for $i = 1, 2, \ldots, n$ to the vertices of one set and $u'_{i,1}, u'_{i,2}$ for $i = 1, 2, \cdots, n$ be the vertices in the second set. Now join $u'_{i,2}$ to $u'_{i+1,1}$ and $u'_{i,1}$ to $u_{i+1,1}$. The resultant graph is denoted as $G(K_2)$ and it has $4n$ vertices and $6n-2$ edges.

The graph $G(K_2)$ obtained by 3 copies of K_2 is shown in Figure 2.3.

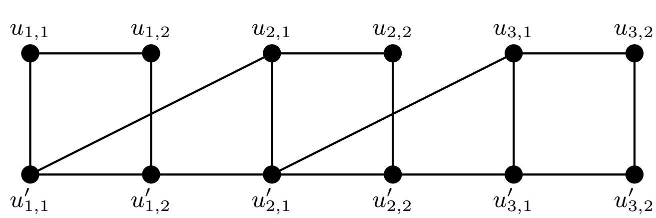

Figure 2.3

Theorem 2.3 *The graph $G(K_2)$ is odd graceful.*

Proof Let $u_{i,1}, u_{i,2}, u'_{i,1}, u'_{i,2}$ for $i = 1, 2, \cdots, n$ be the set of vertices of $G(K_2)$. Define a function $f : V(G) \to \{0, 1, 2, \cdots, 12n-5 = [2(6n-2)]-1\}$ such that

$$
\begin{aligned}
f(u_{1,1}) &= 0\\
f(u_{i+1,1}) &= 8i-4, \text{for } i = 1, 2, \cdots, n-1\\
f(u_{1,2}) &= 12n-5\\
f(u_{2i-1,2}) &= 12n-17-8(i-2);\ 2 \leqslant i \leqslant \frac{n}{2} \text{ if } n \text{ is even}\\
& \qquad\qquad\qquad\qquad\qquad 2 \leqslant i \leqslant \frac{n+1}{2} \text{ if } n \text{ is odd}\\
f(u_{2i,2}) &= 12n-15-8(i-1), 1 \leqslant i \leqslant \frac{n}{2} \quad \text{if } n \text{ is even}\\
& \qquad\qquad\qquad\qquad\qquad 1 \leqslant i \leqslant \frac{n-1}{2} \text{ if } n \text{ is odd}\\
f(u'_{1,1}) &= 12n-9\\
f(u'_{2i-1,1}) &= 12n-19-8(i-2);\ 2 \leqslant i \leqslant \frac{n}{2} \text{ if } n \text{ is even}\\
& \qquad\qquad\qquad\qquad\qquad 2 \leqslant i \leqslant \frac{n+1}{2} \text{ if } n \text{ is odd}\\
f(u'_{2i,1}) &= 12n-13-8(i-1), 1 \leqslant i \leqslant \frac{n}{2} \text{ if } n \text{ is even}\\
& \qquad\qquad\qquad\qquad\qquad 1 \leqslant i \leqslant \frac{n-1}{2} \text{ if } n \text{ is odd}\\
f(u'_{1,2}) &= 2\\
f(u'_{i+1,2}) &= 8i; 1 \leqslant i \leqslant n-1
\end{aligned}
$$

Easily, it can be verified that f is injective. Also, $\max f(v) = 12n-5, v \in V(K_2)$. Thus, $f(v) \in \{0, 1, 2, \cdots, 12n-5\}$. Finally, it can be easily proved that, the values of the edges are in the interval $[1, 12n-5]$. Thus, f is an odd graceful labeling . Hence, the graph $G(K_2)$ is odd graceful. □

Figure 2.4 gives an odd graceful labeling on graph $G(K_2)$ obtained by 6 copies of K_2.

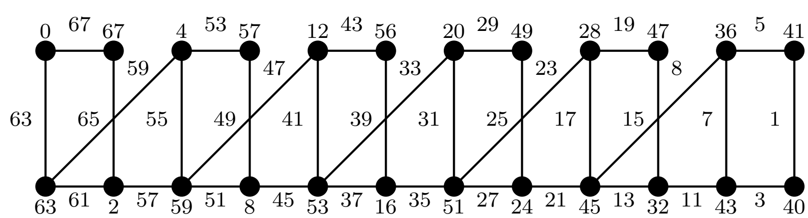

Figure 2.4

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Scientific conclusions are the gold with limited amount; while scientific means is the magic that can be utilized to produce endless amount of gold.

By Cai Yuanpei, a Chinese educator.

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