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Vertex-to-Edge-set Distance Neighborhood Pattern Matrices

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Abstract: The vertex to edge set (VTES) distance $d_1(u,e)$ from a vertex $u \in V(G)$ to an edge $e \in E(G)$ is the number of edges on $(u-e)$ path. For each $u \in V(G)$ define $N_j^M[u] = \{e \in M \subseteq V(G) : d(u,e) = j$, where $0 \leq j \leq d_1(G)$ $\}$ and a non-negative integer matrix $D_1^M(G) = (|N_j^M[u]|)$ of order $V(G) \times ((d_1(G)+1)$ called the VTES-M-distance neighborhood pattern (M-dnp) matrix of G. If $f_M : u \longmapsto f_M(u)$ is an injective function, where $f_M(u) = \{j : N_j^M[u] \neq \emptyset\}$, then the set M is a VTES-distance pattern distinguishing (M-dpd) set of G and G is a VTES-dpd-graph. This paper is a study of VTES M-dnp-matrices of a VTES-dpd-graph.

Key Words: Distance (in Graph), vertex-to-edge-set distance-pattern distinguishing sets, VTES-distance neighborhood pattern matrix.

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§1. Introduction

For all terminology which are not defined in this paper, we refer the reader to F.Harary [6]. Unless mentioned otherwise, all the graphs considered in this paper are nontrivial, simple, finite and connected.

Distance between two elements (vertex to vertex, vertex to edge, edge to vertex, and edge to edge) in graphs is already defined in the literature (refer [9]), but here we are using vertex to edge-distance. For subsets $S, T \subseteq V(G)$, and any vertex v, let $d(v,S) = min\{d(v,u) : u \in S\}$ and $d(S,T) = min\{d(x,y) : x \in S, y \in T\}$. In particular, if $f = xy$ is an edge in G, then the vertex to edge distance between v and f is given by $d(v,f) = min\{d(v,x), d(v,y)\}$ [9].

A study of these sets is expected to be useful in a number of areas of application such as facility location [5] and design of indices of "quantitative structure activity relationships" (QSAR) in chemistry ([2], [8]).

Definition 1.1([9]) *For any vertex v in a connected graph G, the vertex-to-edge eccentricity $\epsilon(v)$ of v is $\epsilon(v) = max\{d(v,e) : e \in E(G)\}$. The vertex-to-edge diameter $d_1(G) = max\{\epsilon(v)\}$ and the vertex-to-edge radius $r_1(G) = min\{\epsilon(v)\}$. A vertex v for which $\epsilon(v)$ is minimum is called a vertex-to-edge central vertex of G and the set of all vertex-to-edge central vertices of G is the vertex-to-edge center $C_1(G)$ of G. Any edge e for which $\epsilon(v) = d(v,e)$ called an eccentric edge of v.*

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The vertex-to-vertex eccentricities and the vertex-to-edge eccentricities of the vertices of graphs G and H in Fig.1 are given in the Table 1.1 and Table 1.2, respectively.

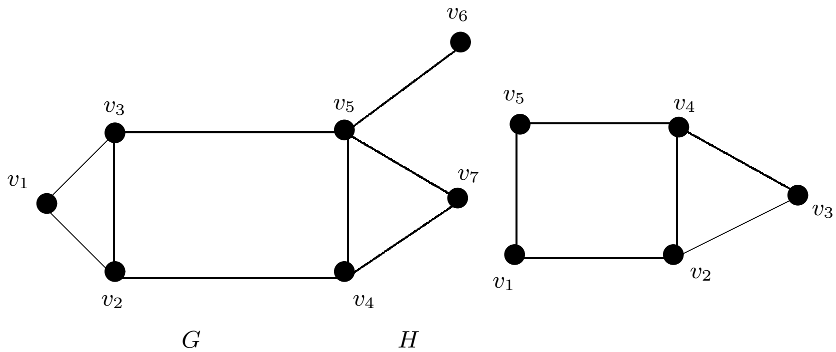

Fig 1

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
$e(v)$	3	3	2	2	2	3	3
$\epsilon(v)$	2	2	2	2	2	3	2

Table 1.1

v	v_1	v_2	v_3	v_4	v_5
$e(v)$	2	2	2	2	2
$\epsilon(v)$	2	1	2	1	2

Table 1.2

Definition 1.2 *Let $G = (V, E)$ be a given connected simple (p,q)-graph, $M \subseteq E(G)$ and for each $u \in V(G)$, let $f_M(u) = \{d(u, e) : e \in M\}$ be the distance-pattern of u with respect to M. If f_M is injective then the set M is a distance-pattern distinguishing set (or, a "VTES-dpd-set" in short) of G and G is a VTES-dpd-graph. If $f_M(u) - \{0\}$ is independent of the choice of u in G then M is an open distance-pattern uniform (or, VTES-odpu) set of G and G is called an VTES-odpu-graph. The minimum cardinality of a VTES-dpd-set (VTES-odpu-set) in G, if it exists, is the VTES-dpd-number(VTES-odpu-number) of G and it is denoted by $\rho(G)$.*

For an arbitrarily fixed vertex u in G and for any nonnegative integer j, we let $N_j[u] = \{e \in E(G) : d(u, e) = j\}$. Clearly, $|N_0[u]| = \{deg(u)\}$, $\forall u \in V(G)$ and $N_j[u] = V(G) - V(\xi_u)$ whenever j exceeds the vertex-to-edge eccentricity $\epsilon(u)$ of u in the component ξ_u to which u belongs. Thus, if G is connected then, $N_j[u] = \phi$ if and only if $j > \epsilon(u)$. If G is a connected graph then the vectors $\bar{u} = (|N_0[u]|, |N_1[u]|, |N_2[u]|, \cdots, |N_{\epsilon(u)}[u]|)$ associated with $u \in V(G)$ can be arranged as a $p \times (d_{1G} + 1)$ matrix D_{1G} whose entries are nonnegative integers given by

$$\begin{pmatrix} |N_0[v_1]| & |N_1[v_1]| & |N_2[v_1]| & ... & |N_{\epsilon(v_1)}[v_1]| & 0 & 0 & 0 \\ |N_0[v_2]| & |N_1[v_2]| & |N_2[v_2]| & ... & ... & |N_{\epsilon(v_2)}[v_2]| & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ |N_0[v_p]| & |N_1[v_p]| & |N_2[v_p]| & ... & ... & ... & ... & |N_{\epsilon(v_p)}[v_p]| \end{pmatrix}$$

where d_{1G} denotes the vertex-toedge diameter of G; we call D_{1G} *VTES-distance neighborhood pattern matrix (or, VTES-dnp-matrix)* of G. For a VTES-dnp-matrix the following observations are immediate.

Observation 1.3 Entries in the first column of D_{1G} are nonzero entries.

Observation 1.34 In each row of D_{1G}, entry zero will be after some nonzero entries. Zero entries may or may not be present in rows.

Observation 1.5 The entries in the first column of D_{1G} correspond to the degrees of the corresponding vertices in G.

Proposition 1.6 *For each $u \in V(G)$ of a connected graph G, $\{N_j[u] : N_j[u] \neq \phi,\ 0 \leq j \leq d_{1G}\}$ gives a partition of $E(G)$.*

Proof If possible, let $e \in N_j[u] \cap N_k[u]$, for some $e \in E(G)$ and $u \in V(G)$. Then $d(u,e) = j$ and $d(u,e) = k$, and hence $j = k$. Therefore,$N_j[u] \cap N_k[u] = \phi$ for any (j,k) with $j \neq k$. Now, clearly, $\bigcup_{j=o}^{d_{1G}} N_j[u] \subseteq E(G)$. Also, for any $e \in E(G)$, since G is connected, $d(u,e) = k$, for some $k \in \{0,1,2,\cdots,d_{1G}\}$. That is, $e \in N_k[u]$ for some $k \in \{0,1,2,\cdots,d_{1G}\}$ which implies $E(G) \subseteq \bigcup_{j=o}^{d_{1G}} N_j[u]$. Hence $\bigcup_{j=o}^{d_{1G}} N_j[u] = E(G)$. □

Corollary 1.7 *Each row of the VTES-dnp-matrix D_{1G} of a graph G is the partition of $|E(G)|$. Hence, sum of the entries in each row of the VTES-dnp-matrix D_{1G} of a graph G is equal to the number of edges of G.*

§2. M-VTES-Distance Neighborhood Pattern Matrix of a Graph

Given an arbitrary nonempty subset $M \subseteq E(G)$ of G and for each $u \in V(G)$, define $N_j^M[u] = \{e \in M : d(u,e) = j\}$; clearly then $N_j^{E(G)}[u] = N_j[u]$. One can define the M-VTES-eccentricity of u as the largest integer for which $N_j^M[u] \neq \phi$ and the $p \times (d_{1G}+1)$ nonnegative integer matrix $D_{1G}^M = (|N_j^M[u]|)$ is called the M-VTES-distance neighborhood pattern (or, M-VTES-dnp) matrix of G. D_G^{*M} is obtained from D_{1G}^M by replacing each nonzero entry by 1.

B.D.Acarya ([1]) defined VTES-dnp matrix of any graph and in particular, M-VTES-dnp matrix of VTES-dpd-graph as follows:

Definition 2.1 *Let $G = (V,E)$ be a given connected simple (p,q)-graph, $M(\neq \phi) \subseteq E(G)$ and $u \in V(G)$. Then, the M-VTES-distance-pattern of u is the set $f_M(u) = \{d(u,e) : e \in M\}$. Clearly, $f_M(u) = \left\{j : N_j^M[u] \neq \phi\right\}$. Hence, in particular, if $f_M : u \longmapsto f_M(u)$ is an injective function, then the set M is a VTES-distance-pattern distinguishing set (or, a "VTES-dpd-set" is short) of G and if $f_M(u) - \{0\}$ is independent of the choice of u in G then M is a VTES-open distance-pattern uniform (or, VTES-odpu) set of G. A graph G with a VTES-dpd-set(VTES-odpu-set) is called a VTES-dpd-(VTES-odpu)-graph.*

Following are some interesting results on M-VTES-dnp matrix of a connected nontrivial graph G.

Observation 2.2 Both D_{1G}^M and D_{1G}^{*M} do not admit null rows.

Proposition 2.3 *For each $u \in V(G)$,*

$$N_0^M[u] = \begin{cases} M \cap D_u, & if \quad M \cap D_u \neq \phi; \\ \emptyset, & if \quad M \cap D_u = \phi, \end{cases}$$

where $D_u = \{e_i : 1 \leq i \leq degu$ and u is adjacent to e_i $\}$.

Therefore, the entries in the first column of D_{1G}^M are zero or an integer $k, 1 \leq k \leq degu$ and the entries in the first column of D_{1G}^{*M} are either 0 or 1.

Remark 2.4 It should be noted that Observation is not true in the case of D_{1G}^{*M}.

Remark 2.5 For a graph $G \cong C_n$,

vertex-toedge diameter $d_{1G} = \begin{cases} \frac{n-2}{2} & if \quad \text{n} \quad is \quad even \quad integer \\ \frac{n-1}{2} & if \quad \text{n} \quad is \quad odd \quad integer \end{cases}$

Remark 2.6 For a graph $G \cong P_n, n \geq 2$, the vertex-to-edge diameter $d_{1G} = n - 2$.

Lemma 2.7 is similar to Proposition 1.6.

Lemma 2.7 *For each $u \in V(G)$ of a connected graph G, $\{N_j^M[u] : N_j^M[u] \neq \emptyset, 0 \leq j \leq d_{1G}\}$ gives a partition of M.*

Proof If possible, let $e \in N_j^M[u] \cap N_k^M[u]$, for some $e \in M$ and $u \in V(G)$. Then $d(u,e) = j$ and $d(u,e) = k$, and hence $j = k$. Therefore, $N_j^M[u] \cap N_k^M[u] = \phi$ for any (j,k) with $j \neq k$. Now, clearly, $\bigcup_{j=o}^{d_{1G}} N_j^M[u] \subseteq M$. Also, for any $e \in M$, since G is connected, $d(u,e) = k$, for some $k \in \{0, 1, 2, \cdots, d_{1G}\}$. That is, $e \in N_k^M[u]$ for some $k \in \{0, 1, 2, \cdots, d_{1G}\}$ which implies $M \subseteq \bigcup_{j=o}^{d_{1G}} N_j^M[u]$. Hence $\bigcup_{j=o}^{d_{1G}} N_j^M[u] = M$. □

Corollary 2.8 *Each row of D_{1G}^M is a partition of $|M|$.*

Corollary 2.9 *Sum of the entries in each row of D_{1G}^M gives $|M|$ and sum of the entries in each row of D_{1G}^{*M} is less than or equal to $|M|$.*

§3. M-VTES-Distance Neighborhood Pattern Matrix of a VTES-dpd Graph

In this section we find out some results of D_{1G}^{*M} of a VTES-dpd-graph. From the definition of D_{1G}^{*M}, we have the following observations.

Observation 3.1 In any graph G, a nonempty $M \subseteq E(G)$ is a VTES-dpd-set if and only if no two rows of D_{1G}^{*M} are identical.

The following Theorem 3.2 shows that M is a proper subset of $E(G)$.

Theorem 3.2 *For a VTES-dpd-graph G of order p and size q, a VTES-dpd set M is such that $3 \leqq |M| \leqq q - 1$.*

Proof For lower bound, let M be a VTES-dpd set. If $M = \{e\}$, for some $e = uv \in E(G)$, then D_{1G}^{*M} contains a $2 \times (d_{1G} + 1)$ submatrix, such that the rows of submatrix represent the M-VTES-dnp of the vertices u and v in D_{1G}^{*M}. That is, entry 1 is at the first column of submatrix and the rows are as shown below,

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence D_{1G}^{*M} contains identical rows, Therefore M is not a VTES-dpd-set and hence $|M| \neq 1$.

Next, suppose $M = \{e_k, e_l\}$ for some $e_k = u_iu_j \in E(G)$ and $e_l = v_iv_j \in E(G)$. We consider the following cases.

Case 1. e_k is adjacent to e_l. Let $u_j = v_i$. Then $d_1(u_i, e_k) = d(v_j, e_l) = 0$ and $d(u_i, e_l) = d(v_j, e_k) = 1$. Then D_{1G}^{*M} contains a $2 \times (d_{1G}+1)$ submatrix, such that the rows of submatrix represent the M-VTES-dnp of the vertices u_i and v_j in D_{1G}^{*M}. That is, entry 1 is at the first and second column of submatrix and rows are as shown below.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Case 2. e_k is not adjacent to e_l. Then the rows of the $2 \times (d_{1G}+1)$ submatrix corresponding to the M-VTES-dnp of u_i and v_j in D_{1G}^{*M} are as follows.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Thus D_{1G}^{*M} contains identical rows if $|M| = 2$ and so, M is not a VTES-dpd-set. Hence the lower bound follows.

For upper bound, suppose on contrary, there exist a VTES-dpd-set M with $|M| = q$. We prove by induction on $p \geq 2$.

Suppose $p = 2$ with $|M| = q$. Then the graph $G \cong K_2$ and $|M| = 1$. By lower bound $|M| \geq 3$, a contradiction. Suppose $p = 3$ with $|M| = q$ then the graph G is either $K_{1,2}$ or K_3. If $G \cong K_{1,2}$, with $|M| = q = 2$. By lower bound $|M| \geq 3$, a contradiction. If $G \cong K_3$ with $|M| = q = 3$. Then, we have a VTES-dpd-matrix

$$D_{1G}^{*M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Clearly rows are identical hence, a contradiction. Therefore $|M| \neq q$ for $p = 2$ and $p = 3$. Suppose that $|M| \neq q$ for $p = n$. We claim that $|M| \neq q$ for $p = n + 1$. Let $V(G) = \{v_1, v_2, \cdots, v_n, v_{n+1}\}$ be the vertex set of G. One can observe that every graph has atleast one vertex-to-edge central vertex. Let $C_1(G)$ be set of vertex-to-edge central vertices. We consider the following cases.

Case 1. $|C_1(G)| = 1$. Let $v_i \in C_1(G)$, then D_{1G}^{*M} contains a $(degv_i) \times (d_{1G}+1)$ submatrix, rows of which represent the M-VTES-dnp of the vertices $v_j \in N(v_i)$; $j = 1, 2, \cdots, degv_i$ in D_{1G}^{*M} as shown below.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence D_{1G}^{*M} contains identical rows , a contradiction. Hence $|M| \neq q$ for $p = n+1$. By mathematical induction, the result follows for all p.

Case 2. $|C_1(G)| \geq 2$. Let $C_1(G) = \{v_1, v_2, \cdots, v_i\}, i \geq 2$. For every $v_i, v_j \in C_1(G)$, there exists an edge e_k such that $d(v_i, e_k) = d(v_j, e_k)$. Then D_{1G}^{*M} contains a $2 \times (d_{1G}+1)$ submatrix, the rows of which represent the M-VTES-dnp of vertices v_i and v_j in D_{1G}^{*M} as shown below.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence D_{1G}^{*M} contains identical rows, a contradiction. Thus, $|M| \leqq q-1$. □

Lemma 3.3 *If $d_{1G} \leq 2$, then G does not possess a VTES-dpd-set.*

Proof One can verify from [6] Appendix 1,Table A_1. □

Corollary 3.4 *If $G \cong K_n$, $K_n - e$ or $K_{m,n}$, then G does not possess a VTES-dpd-set.*

Theorem 3.5 *A graph $G \cong P_n$ of order n admits a VTES-dpd-set if and only if $n \geq 5$.*

Proof Suppose that $G \cong P_n$, where $P_n = (v_1, e_1, v_2, e_2, v_3, e_3, \cdots, e_{n-1}, v_n,)$. Let $M = \{e_1, e_2, e_4\}$. Then

$$D_{1G}^{*M} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Now, we can partition D_{1G}^{*M} into two submatrices say, A and B where A is a $4 \times (d_{1G}+1)$ submatrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we can find the 4×4 submatrix A_1 of A which is of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The remaining $4 \times (d_{1G} - 3)$ submatrix A_2 of A has all the entries as zero.

Also, Each i^th row, $1 \leq i \leq (n-4)$, of the submatrix B of order $(n-4) \times (d_{1G}+1)$ has entry 1 only in the $i^{th}, (i+2)^{nd}$,and $(i+3)^{rd}$ columns. None of the rows of the submatrices A and B are identical and hence no two rows of D_{1G}^{*M} are identical. Hence $\{e_1, e_2, e_4\}$ form a VTES-dpd-set. Therefore, any graph $G \cong P_n$ of order $n \geq 5$ admits a VTES-dpd-set.

Now to complete the proof we need to show that P_n is not a VTES-dpd-graph for $n \leq 4$. So, suppose that $G \cong P_n$ and $n \leq 4$. Since $n \leq 4$, $d_1(P_n) \leq 2$ for $n \leqq 4$. By Lemma 3.3, the proof follows. □

Corollary 3.6 *$G \cong P_5$ is the smallest VTES-dpd-graph with $M = \{e_1, e_2, e_4\}$*

Theorem 3.7 *A cycle $G \cong C_n$ of order n admits a VTES-dpd-set if and only if $n \geq 7$.*

Proof Let $C_n = (v_1, e_1, v_2, e_2, v_3, e_3, \cdots, e_{n-1}, v_n, e_1, v_1)$ be a cycle on n vertices. We consider the following cases.

Case 1. n is an even integer, and ≥ 8 . Let $M = \{e_1, e_2, e_4\}$. Then

$$D_{1G}^{*M} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we can partition D_{1G}^{*M} into four submatrices say, A,B,C and D where A is a $4 \times (d_{1G} + 1)$ submatrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we can find the 4×4 sub-matrix A_1 in A which is of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Here the remaining $4 \times (d_{1G} - 3)$ sub-matrix A_2 of A has all the entries as zero. The submatrix B of order $\frac{(n-6)}{2} \times (d_{1G} + 1)$ is of the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Each i^th row, $1 \le i \le \frac{(n-6)}{2}$, of the submatrix B of order $(n-6) \times (d_{1G} + 1)$ has entry 1 only in the i^{th}, $(i+2)^{nd}$,and $(i+3)^{rd}$ columns.

We also choose submatrix C of order

$$(n - 4 - \frac{(n-6)}{2} - \frac{(n-8)}{2}) \times (d_{1G} + 1)$$

of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Finally we can choose a submatrix D of order

$$\frac{(n-8)}{2} \times (d_{1G} + 1)$$

of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly we can observe that none of the rows of submatrices A,B,C and D are identical and hence no two rows of D_{1G}^{*M} are identical. Therefore, any graph $G \cong C_n$ of order $n \ge 8$ admits a VTES-dpd-set.

Case 2. n is an odd integer and ≥ 7 Let $M = \{e_1, e_2, e_4\}$. Then

$$D_{1G}^{*M} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we can partition D_{1G}^{*M} into four submatrices say, A,B,C and D, where A is a $4 \times (d_{1G} + 1)$ submatrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we can find the 4×4 submatrix A_1 of A which is of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Here the remaining $4 \times (d_{1G} - 3)$ submatrix A_2 of A has all the entries as zero. The submatrix B of order $\dfrac{(n-5)}{2} \times (d_{1G} + 1)$ is of the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Each i^th row, $1 \leq i \leq \frac{(n-5)}{2}$, of the submatrix B of order $(n-5) \times (d_{1G}+1)$ has entry 1 only in the i^{th}, $(i+2)^{nd}$,and $(i+3)^{rd}$ columns.

Also, we choose a submatrix C of order $(n - 4 - \frac{(n-5)}{2} - \frac{n-7}{2}) \times (d_{1G}+1)$ of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Finally, we can choose a $(\frac{n-7}{2}) \times (d_{1G}+1)$ submatrix D of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, one can observe that the rows of A,B,C and D of D_{1G}^{*M} are not identical. Therefore, any graph $G \cong C_n$ of order $n \geq 7$ admits a VTES-dpd-set.

Now to complete the proof we need to show that the C_n is not a VTES-dpd-graph for $n \leq 6$. So, suppose that $G \cong C_n$ and $n \leq 6$. Then $d_{1G} \leq 2$. The proof follows by Lemma 3.3. □

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