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Split Geodetic Number of a Line Graph

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Abstract: A set $S \subseteq V[L(G)]$ is a split geodetic set of $L(G)$, if S is a geodetic set and $\langle V - S \rangle$ is disconnected. The split geodetic number of a line graph $L(G)$, denoted by $g_s[L(G)]$ is the minimum cardinality of a split geodetic set of $L(G)$. In this paper we obtain the split geodetic number of line graph of any graph. Also obtain many bounds on split geodetic number in terms of elements of G and covering number of G. We also investigate the relationship between split geodetic number and geodetic number.

Key Words: Label Cartesian product, distance, edge covering number, line graph, Smarandache k-split geodetic set, split geodetic number, vertex covering number.

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§1. Introduction

In this paper we follow the notations of [3]. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively. The graphs considered here have at least one component which is not complete or at least two non trivial components.

For any graph $G(V, E)$, the line graph $L(G)$ whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the maximum eccentricity is the diameter, diam G. A $u - v$ path of length d(u, v) is called a $u - v$ geodesic. We define $I[u, v]$ to the set (interval) of all vertices lying on some $u - v$ geodesic of G and for a nonempty subset S of $V(G)$, $I[S] = \bigcup_{u,v\in S} I[u, v]$.

A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by $g(G)$.

Split geodetic number of a graph was studied by in [5]. A *Smarandache k-split geodetic set S* of a graph $G = (V, E)$ is such a split geodetic set that the induced subgraph $\langle V - S \rangle$ is k-connected. Particularly, if $k = 0$, such a split geodetic set is called a *split geodetic set* S of a graph G. The split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set. Geodetic number of a

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line graph was studied by in [4]. Geodetic number of a line graph $L(G)$ of G is a set $S^{'}$ of vertices of $L(G) = H$ is called the geodetic set in H if $I(S^{'}) = V(H)$ and a geodetic set of minimum cardinality is the geodetic number of $L(G)$ and is denoted by $g[L(G)]$. Now we define split geodetic number of a line graph. A set $S^{'}$ of vertices of $L(G) = H$ is called the split geodetic set in H if the induced subgraph $V(H) - S^{'}$ is disconnected and a split geodetic set of minimum cardinality is the split geodetic number of $L(G)$ and is denoted by $g_s[L(G)]$.

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbors is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G.

For terminologies and notations not mentioned here, we follow references [2] and [3].

§2. Preliminary Notes

We need results following for proving results in this paper.

Theorem 2.1([1]) *Every geodetic set of a graph contains its extreme vertices.*

Theorem 2.2([5]) *For cycle C_n of order $n > 3$,*

$$g_s(C_n) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd.} \end{cases}$$

Theorem 2.3([1]) *Let G be a connected graph of order at least 3. If G contains a minimum geodetic set S with a vertex x such that every vertex of G lies on some $x - w$ geodesic in G for some $w \in S$, then $g(G) = g(G \times K_2)$.*

Proposition 2.4 *For any graph G, $g(G) \leq g_s(G)$.*

Proposition 2.5 *For any tree T of order n and number of cut vertices c_i then the number of end edges is $n - c_i$.*

§3. Main Results

Theorem 3.1 *For any tree T with k end edges and c_i be the number of cut vertices, having more than three internal vertices, $g_s[L(T)] = n - c_i + 1$.*

Proof Let S be the set of all extreme vertices of a line graph $L(T)$ of a tree T. Let v_i be a cut vertex in $V - S$ and $S^{'} = S \cup \{v_i\}$. By Theorem 2.1, $g_s[L(T)] \geq |S^{'}|$. On the other hand, for an internal vertex v of $L(T)$, there exists x, y of $L(T)$ such that v lies on a unique $x - y$ geodesic in $L(T)$. The corresponding end edges of T are the extreme vertices of $L(T)$ and the induced subgraph $V - S^{'}$ is disconnected. Thus $g_s[L(T)] \leq |S^{'}|$. Also, every split geodetic set S_1 of $L(T)$ must contain $S^{'}$ which is the unique minimum split geodetic set. Thus $|S^{'}| = |S_1| = k + 1$. By Proposition 2.5, $|S_1| = n - c_i + 1$. Hence, $g_s[L(T)] = n - c_i + 1$. □

Corollary 3.2 *For any path P_n, $n \geq 6$, $g_s[L(P_n)] = 3$.*

Proof Clearly, the set of two end vertices of a path P_n is its unique geodetic set. From Theorem 3.1, the results follows. □

Proposition 3.3 *Line graph of a cycle is again a cycle of same order.*

Theorem 3.4 *For cycle C_n of order $n > 3$,*

$$g_s[L(C_n)] = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd.} \end{cases}$$

Proof The result follows from Proposition 3.3 and Theorem 2.2. □

Theorem 3.5 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$,*

$$g_s[L(W_n)] = \begin{cases} \dfrac{n}{2} + 2 & \text{if n is even} \\ \dfrac{n+1}{2} + 2 & \text{if n is odd.} \end{cases}$$

Proof Let $W_n = K_1 + C_{n-1}(n \geq 6)$ and let $V(W_n) = \{x, v_1, v_2, \cdots, v_{n-1}\}$, where $deg(x) = n-1 > 3$ and $deg(v_i) = 3$ for each $i \in \{1, 2, \cdots, n-1\}$. Now $U = \{u_1, u_2, \cdots, u_j\}$ are the vertices of $L(W_n)$ formed from edges of C_{n-1}, i.e., $U \subseteq V[L(W_n)]$ and $Y = \{y_1, y_2, \cdots, y_j\}$ are the vertices of $L(W_n)$ formed from internal edges of W_n. Thus, $Y \subseteq V[L(W_n)]$. We consider the following cases.

Case 1. n is even.

Let $H \subseteq U$. Now $S = H \cup \{y_j\}$ forms a minimum geodetic set of $L(W_n)$. Let $P = \{p_1, p_2, \cdots, p_i\}$ be the vertices of $V[L(W_n)] - S$. Clearly, $S \cup \{p_l, p_k\}$ forms a minimum split geodetic set of $L(W_n)$ and $|S \cup \{p_l, p_k\}| = \dfrac{n}{2} + 2$. Therefore, $g_s[L(W_n)] = \dfrac{n}{2} + 2$.

Case 2. n is odd.

Let $H \subseteq U$, now $S = H \cup \{y_j, y_{j-1}\}$ forms a minimum geodetic set of $L(W_n)$. Let $P = \{p_1, p_2, \cdots, p_i\}$ be the vertices of $V[L(W_n)] - S$. Now $S \cup \{p_l, p_k\}$ forms a minimum split geodetic set of $L(W_n)$. Clearly, $|S \cup \{p_l, p_k\}| = \dfrac{n+1}{2} + 2$. Therefore, $g_s[L(W_n)] = \dfrac{n+1}{2} + 2$. □

As an immediate consequence of the above theorem we have the following.

Corollary 3.6 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$,*

$$g_s[L(W_n)] = \begin{cases} \dfrac{\Delta + \delta}{2} & \text{if n is even} \\ \dfrac{\Delta + \delta + 1}{2} & \text{if n is odd.} \end{cases}$$

Proof Minimum degree(δ) of $L(W_n)$ is equal to 4 and maximum degree(Δ) of $L(W_n)$ is equal to n. i,e number of vertices in W_n.

Case 1. n is even.

We have known from Case 1 of Theorem 3.5 that

$$\begin{aligned} g_s[L(W_n)] &= \frac{n}{2}+2 \\ g_s[L(W_n)] &= \frac{n+4}{2} \\ g_s[L(W_n)] &= \frac{\Delta+\delta+1}{2} \end{aligned}$$

Case 2. n is odd.

We have known from Case 2 of Theorem 3.5 that

$$\begin{aligned} g_s[L(W_n)] &= \frac{n+1}{2}+2 \\ g_s[L(W_n)] &= \frac{n+4+1}{2} \\ g_s[L(W_n)] &= \frac{\Delta+\delta+1}{2}. \end{aligned}$$

□

Theorem 3.7 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$, $g_s[L(W_n)] + g[L(W_n)] \leq m$.*

Proof Let $U = \{u_1, u_2, \cdots, u_j\} \subseteq V[L(W_n)]$ be the set of vertices formed from edges of C_{n-1} and $Y = \{y_1, y_2, \cdots, y_j\} \subseteq V[L(W_n)$ be the set of vertices formed from internal edges of W_n. Consider $S = H \cup \{y_j\}$, where $H \subset U$ forms a minimum geodetic set of $L(W_n)$. Furthermore, if $P = \{p_1, p_2, \cdots, p_i\}$ is the set of vertices of $V[L(W_n)] - S$, then $S^{'} = S \cup \{p_l, p_m\}$ forms a minimum split geodetic set of $L(W_n)$. Notice that $V[L(G)] = E(G) = m$. It follows that $|S^{'}| \cup |S| \leq m$. Thus, $g_s[L(W_n)] + g[L(W_n)] \leq m$.□

Theorem 3.8 *For a tree T with more than three internal vertices, $g_s[L(T)] \geq m - \alpha_1 + 1$, where α_1 is the edge covering number.*

Proof Suppose $S = \{e_1, e_2, \cdots, e_k\}$ to be the set of all end edges in T. Then $S \cup J$, where $J \subseteq E(T) - S$ is the minimal set of edges which covers all the vertices of T such that $|S \cup J| = \alpha_1(T)$. Without loss of generality, let $I = \{u_1, u_2, \cdots, u_n\} \subseteq V[L(T)]$ be the set of vertices in $L(T)$ formed by the end edges in T. Suppose $H = \{u_1, u_2, \cdots, u_i\} \subseteq V[L(T)] - I$. Then $I \cup \{u_i\}$ forms a minimum split geodetic set of $L(T)$, where each $u_i \in H$ with $deg \geq 2$. Clearly, it follows that $|I \cup \{u_i\}| \geq |E(T)| - |S \cup J| + 1$. Therefore, $g_s[L(T)] \geq m - \alpha_1(T) + 1$. □

Theorem 3.9 *If every non end vertex of a tree T with more than three internal vertex is adjacent to at least one end vertex, then $g_s[L(T)] \geq n - k$, where k is the number of end vertices in T.*

Proof Let $S^{'} = \{v_1, v_2, \cdots, v_k\}$ be the set of all end vertices in T with $|S^{'}| = k$. Without loss of generality, let every end edge of T be the extreme vertices of $L(T)$. Suppose $L(T)$ does not contain any end vertex. Then $S = I \cup \{u_j\}$, where $I = \{u_1, u_2, \cdots, u_i\} \subseteq V[L(T)]$ and $u_j \in V[L(T)] - I$ with $deg \geq 2$ forms a minimum split geodetic set of $L(T)$. Furthermore, if $L(T)$ contain at least one end vertex v_i, then the set $S \cup \{v_i\}$ forms a minimum split geodetic set of $L(T)$. Therefore, we obtain $|S \cup \{v_i\}| \geq n - |S^{'}|$. Clearly it follows that $g_s[L(T)] \geq n - k$. □

Theorem 3.10 *For any connected graph G of order n, $g_s(G) + g_s[L(G)] \leq 2n$.*

Proof Let $S = \{v_1, v_2, \cdots, v_n\} \subseteq V(G)$ be the minimum split geodetic set of G. Now without loss of generality, if $F = \{u_1, u_2, \cdots, u_k\}$ is the set of all end vertices in $L(G)$, then $F \cup H$, where $H \subseteq V[L(G)] - F$ forms a minimum split geodetic set of $L(G)$. Since each vertex in $L(G)$ corresponds to two adjacent vertices of G, it follows that $|S| \cup |F \cup H| \leq 2n$. Therefore $g_s(G) + g_s[L(G)] \leq 2n$. □

Theorem 3.11 *Let G be a connected graph of order n with diameter $d > 4$. Then $g_s[L(G)] \leq n-d+2$.*

Proof Let u and v be vertices of $L(G)$ for which $d(u,v) = d$ and let $u = v_0, v_1, \cdots, v_d = v$ be the $u-v$ path of length d. Now let $S = V[L(G)] - \{v_1, v_2, \cdots, v_{d-1}\}$. Then $I(S) = V[L(G)]$, $V[L(G)] - (S \cup \{v_2\})$ is disconnected and $g_s[L(G)] \leq |S| = n - d + 2$. □

Theorem 3.12 *For any integers $r, s \geq 2$, $g_s[L(K_{r,s})] \leq rs$.*

Proof Notice that the diameter of $L(K_{r,s})$ is 2 and the number of vertices in $L(K_{r,s})$ is rs. By Theorem 3.11, $g_s[L(G)] \leq n-d+2$. Now we have $g_s[L(K_{r,s})] \leq rs-2+2$. Therefore, $g_s[L(K_{r,s})] \leq rs$. □

Theorem 3.13 *For any integer $n \geq 4$, $g_s[L(K_n)] \leq \dfrac{n(n-1)}{2}$.*

Proof Let $n \geq 4$ be the vertices of the given graph K_n with diameter d. Since diameter of $L(K_n)$ is 2 and the number of vertices in $L(K_n)$ is $\dfrac{n(n-1)}{2}$. By Theorem 3.11, $g_s[L(G)] \leq n - d + 2$. We have

$$g_s[L(K_n)] \leq \frac{n(n-1)}{2} - 2 + 2 \Rightarrow g_s[L(K_n)] \leq \frac{n(n-1)}{2}.$$ □

Theorem 3.14 *For any cycle C_n with $n \equiv 0(\text{mod}2)$, $g_s[L(C_n)] = \dfrac{n}{\alpha_0(C_n)}$, where α_0 is the vertex covering number.*

Proof Let $n > 3$ be number of vertices which is even and let α_0 be the vertex covering number of C_n. By Theorem 3.4, $g_s[L(C_n)] = 2$. Also, for even cycle, the vertex covering number $\alpha_0(C_n) = \dfrac{n}{2}$. Hence $g_s[L(C_n)] = 2 = \dfrac{n}{n/2} = \dfrac{n}{\alpha_0(C_n)}$. □

Theorem 3.15 *For any cycle C_n with $n \equiv 1(\text{mod}2)$, $g_s[L(C_n)] = \dfrac{n+1}{\alpha_0(C_n)} + 1$, where α_0 is the vertex covering number.*

Proof Let $n > 3$ be the number of vertices which is odd and let α_0 be the vertex covering number of C_n. By Theorem 3.4, $g_s[L(C_n)] = 3$. Also, for odd cycle, vertex covering number $\alpha_0(C_n) = \dfrac{n+1}{2}$. Hence $g_s[L(C_n)] = 2 + 1 = \dfrac{n+1}{\alpha_0(C_n)} + 1$. □

§4. Adding an End Edge

For an edge $e = (u, v)$ of a graph G with $deg(u) = 1$ and $deg(v) > 1$, we call e an end-edge and u an end-vertex.

Theorem 4.1 *Let G' be the graph obtained by adding k end edges $\{(u, v_1), (u, v_2), \cdots, (u, v_k)\}$ to a cycle $C_n = G$ of order $n > 3$, with $u \in G$ and $\{v_1, v_2, \cdots, v_k\} \notin G$. Then $g_s[L(G')] = k + 2$.*

Proof Let $\{e_1, e_2, \cdots, e_n, e_1\}$ be edges on a cycle of order n and let G' be the graph obtained from $G = C_n$ by adding end edges (u, v_i), $i = 1, 2, \cdots, k$ such that $u \in G$ but $v_i \notin G$.

Case 1. n is even.

By definition, $L(G^{'})$ has $\langle K_{k+2}\rangle$ as an induced subgraph. Also the edges (u, v_i), $i = 1, 2, \cdots, k$ becomes vertices of $L(G^{'})$ and it belongs to some geodetic set of $L(G^{'})$. Hence $\{e_1, e_2, \cdots, e_k, e_l, e_m\}$ are the vertices of $L(G^{'})$, where e_l, e_m are the edges incident on the antipodal vertex of u in $G^{'}$ and these vertices belongs to some geodetic set of $L(G^{'})$. $L(G^{'}) = C_n \cup K_{k+2}$. Let$S = \{e_1, e_2, \cdots, e_k, e_l, e_m\}$ be the geodetic set. Suppose $P = \{e_1, e_2, \cdots, e_k\}$ is the set of vertices of $L(G^{'})$ such that $|P| < |S|$. Then, P is not a geodetic set of $L(G^{'})$. Clearly, S is the minimum geodetic set. Since $V - S$ is disconnected S is the minimum split geodetic set. Therefore $g_s[L(C_{2n})] = k + 2$.

Case 2. n is odd.

By definition, $L(G^{'})$ has $\langle K_{k+2}\rangle$ as an induced subgraph, also the edges $(u, v_i) = \{e_1, e_2, \cdots, e_k\}$ becomes vertices of $L(G^{'})$. Let $e_l = (a, b) \in G$ such that $d(u, a) = d(u, b)$ in the graph $L(G^{'})$. and let $S = \{e_1, e_2, \cdots, e_k, e_l\}$ be the geodetic set. Now $S^{'} = S \cup \{e_m\}$ is a split geodetic set, where e_m is the vertex from $V - S$ with $deg \geq 2$. It is clear that $S^{'}$ is the minimum split geodetic set. Therefore $g_s[L(C_{2n+1})] = k + 2$. □

Theorem 4.2 *Let $G^{'}$ be the graph obtained by adding end edge (u_i, v_j), $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, k$ to each vertex of $G = C_n$ of order $n > 3$ such that $u_i \in G$, $v_j \notin G$. Then $g_s[L(G^{'})] = k + 2$.*

Proof Let $\{e_1, e_2, \cdots, e_n, e_1\}$ be edges on a cycle $G = C_n$ and let $G^{'}$ be the graph obtained by adding end edge (u_i, v_j), $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, k$ to each vertex of G such that $u_i \in G$ but $v_j \notin G$. Clearly, k be the number of end vertices of $G^{'}$. By definition, $L(G^{'})$ have n copies of K_3 as an induced subgraph. The edges $(u_i, v_j) = e_j$ for all j becomes k vertices of $L(G^{'})$ and those lies on geodetic set of $L(G^{'})$. They form the extreme vertices of $L(G^{'})$. By Theorem 2.1 $S = \{e_1, e_2, \cdots, e_k\}$ forms a geodetic set. Now consider any two vertices $\{e_l, e_m\} \in V - S$ which are not adjacent. $S^{'} = \{e_1, e_2, \cdots, e_k, e_l, e_m\}$ forms a split geodetic set of $L(G^{'})$. Suppose $P = \{e_1, e_2, \cdots, e_k, e_l\}$ is the set of vertices of $L(G^{'})$ such that $|P| < |S^{'}|$. Then, $V - P$ is connected. Hence it is clear that $S^{'}$ is the minimum split geodetic set of $L(G^{'})$. There fore $g_s[L(G^{'})] = k + 2$. □

§5. Cartesian Product

The Cartesian product of the graphs H_1 and H_2, written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_1)$.

Theorem 5.1 *For any path P_n of order n,*

$$g_s[L(K_2 \times P_n)] = \begin{cases} 2 & \text{for } n = 2 \\ 3 & \text{for } n = 3 \\ 4 & \text{for } n > 3. \end{cases}$$

Proof Let $K_2 \times P_n$ be formed from two copies of G_1 and G_2 of P_n. Now, $L(K_2 \times P_n)$ formed from two copies of $G_1^{'}, G_2^{'}$ of $L(P_n)$. And let $U = \{u_1, u_2, \cdots, u_{n-1}\} \in V(G_1^{'})$, $W = \{w_1, w_2, \cdots, w_{n-1}\} \in V(G_2^{'})$. We have the following cases.

Case 1. If $n = 2$, by the definition $L(K_2 \times P_2) = K_2 \times P_2$. By Theorem 2.3,

$$g_s[L(K_2 \times P_2)] = g[L(K_2 \times P_2)] = g(P_2) = 2.$$

Case 2. If $n = 3$, $L(K_2 \times P_3)$ is formed from two copies of P_2. Clearly, $g_s[L(K_2 \times P_3)] = 3$.

Case 3. If $n > 3$, let S be the split geodetic set of $L(K_2 \times P_n)$. We claim that S contains two elements (end vertices) from each set $\{u_1, u_{n-1}, w_1, w_{n-1}\}$ and $V - S$ is disconnected. Since $I(S) = V[L(K_2 \times P_n)]$, it follows that $g_s[L(K_2 \times P_n)] \leq 4$. It remains to show that if $S^{'}$ is a three element subset of $V[L(K_2 \times P_n)]$, then $I(S^{'}) \neq V[L(K_2 \times P_n)]$. First assume that $S^{'}$ is a subset U or W, say the former. Then $I(S^{'}) = S^{'} \cup W \neq V$. Therefore, we may take that $S^{'} \cap U = \{u_i, u_j\}$ and $S^{'} \cap W = \{w_k\}$. Then

$$I(S^{'}) = \{u_i, u_j\} \cup W \neq V[L(K_2 \times P_n)].$$

□

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