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Total Near Equitable Domination in Graphs

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Abstract: Let $G = (V, E)$ be a graph, $D \subseteq V$ and u be any vertex in D. Then the out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$. A subset $D \subseteq V(G)$ is called a near equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. A near equitable dominating set D is said to be a total near equitable dominating set (tned-set) if every vertex $w \in V$ is adjacent to an element of D. The minimum cardinality of tned-set of G is called the total near equitable domination number of G and is denoted by $\gamma_{tne}(G)$. The maximum order of a partition of V into tned-sets is called the total near equitable domatic number of G and is denoted by $d_{tne}(G)$. In this paper we initiate a study of these parameters.

Key Words: Equitable domination number, near equitable domination number, near equitable domatic number, total near equitable domination Number, total near equitable domatic number, Smarandachely k-dominator coloring.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [2].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Let G be a graph without isolated vertices. For an integer $k \geqslant 1$, a Smarandachely k-dominator coloring of G is a proper coloring of G with the extra property that every vertex in G properly dominates a k-color classes. Particularly, a subset S of V is called a dominating set if $N[S] = V$, i.e., a Smarandachely 1-dominator set. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ ($\Gamma(G)$). An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [5]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [6]. Various types of domination have been defined and

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studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [5]. E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi [3] introduced the concept of total domination in graphs. A dominating set D of a graph G is a total dominating set if every vertex of V is adjacent to some vertex of D. The cardinality of a smallest total dominating set in a graph G is called the total domination number of G and is denoted by $\gamma_t(G)$.

A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

Equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly.

Let $D \subseteq V(G)$ and u be any vertex in D. The out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$. D is called near equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_{ne} and is called the near equitable domination number of G. A partition $P = \{V_1, V_2, \cdots, V_l\}$ of a vertex set $V(G)$ of a graph is called near equitable domatic partition of G if V_i is near equitable dominating set for every $1 \leq i \leq l$. The near equitable domatic number of G is the maximum cardinality of near equitable domatic partition of G and denoted by $d_{ne}(G)$ [7].

For a near equitable dominating set D of G it is natural to look at how total D behaves. For example, for the cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_2, v_3, v_4\}$ are near equitable dominating sets, S_1 is not total and S_2 is total.

In this paper, we introduce the concept of a total near equitable domination to initiate a study of a total near equitable domination number and a total near equitable domatic number.

We need the following to prove main results.

Definition 1.1([7]) *Let $G = (V, E)$ be a graph and D be a near equitable dominating set of G. Then $u \in D$ is a near equitable pendant vertex if $od_D(u) = 1$. A set D is called a near equitable pendant set if every vertex in D is an equitable pendant vertex.*

Theorem 1.2([7]) *Let T be a wounded spider obtained from the star $K_{1,n-1}$, $n \geq 5$ by subdividing m edges exactly once. Then*

$$\gamma_{ne}(T) = \begin{cases} n, & \text{if } m = n-1\ ; \\ n-1, & \text{if } m = n-2; \\ n-2, & \text{if } m \leq n-3. \end{cases}$$

§2. Total Near Equitable Domination in Graphs

A near equitable dominating set D of a graph G is said to be a *total near equitable dominating set* (tned-set) if every vertex $w \in V$ is adjacent to an element of D. The minimum of the cardinality of tned-set of G is called a total near equitable domination number and is denoted by $\gamma_{tne}(G)$. A subset D of V is a minimal tned-set if no proper subset of D is a tned-set.

We note that this parameter is only defined for graphs without isolated vertices and, since each total near equitable dominating set is a near equitable dominating set, we have $\gamma_{ne}(G) \leq \gamma_{tne}(G)$. Since each total near equitable dominating set is a total dominating set, we have $\gamma_t(G) \leq \gamma_{tne}(G)$. The bound is sharp for rK_2, $r \geq 1$. In fact $\gamma_{tne}(G) = \gamma_t(G) = |V|$, for $G = rK_2$, it is easy to see however, that rK_2, $r \geq 1$ is the only graph with this property. Furthermore, the difference $\gamma_{tne}(G) - \gamma_t(G)$ can be arbitrarily large in a graph G. It can be easily checked that $\gamma_t(K_{1,r}) = 2$, while $\gamma_{tne}(K_{1,r}) = n - 2$.

We now proceed to compute $\gamma_{tne}(G)$ for some standard graphs.

1. For any path P_n, $n \geq 4$,

$$\gamma_{tne}(P_n) = \gamma_t(P_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \equiv 2 \ (mod\ 4); \\ \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

   where $\lceil x \rceil$ is a least integer not less than x.

2. For any cycle C_n, $n \geq 4$,

$$\gamma_{tne}(C_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \equiv 2 \ (mod\ 4); \\ \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

   .

3. For the complete graph K_n, $n \geq 4$ $\gamma_{tne}(K_n) = \gamma_{ne}(K_n) = \lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ is a greatest integer not exceeding x.

4. For the double star $S_{n,m}$,

$$\gamma_{tne}(S_{n,m}) = \gamma_{ne}(S_{n,m}) = \begin{cases} 2, & \text{if } n, m \leq 2 \ ; \\ n + m - 2, & \text{if } n, m \geq 2 \text{ and } n \text{ or } m \geq 3. \end{cases}$$

5. For the complete bipartite graph $K_{n,m}$ with $2 < m \leq n$, we have

$$\gamma_{tne}(K_{n,m}) = \gamma_{ne}(K_{n,m}) = \begin{cases} m - 1, & \text{if } n = m \text{ and } m \geq 3; \\ m, & \text{if } n - m = 1; \\ n - 1, & \text{if } n - m \geq 2. \end{cases}$$

6. For the wheel W_n on n vertices,

$$\gamma_{tne}(W_n) = \gamma_{ne}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1.$$

Theorem 2.1 *Let G be a graph and D be a minimum tned- set of G containing t near equitable pendant vertices. Then $\gamma_{tne}(G) \geq \frac{n+t}{3}$.*

Proof Let D be any minimum tned- set of G containing t near equitable pendant vertices . Then $2|D| - t \geq |V - D|$. It follows that, $3|D| - t \geq n$. Hence $\gamma_{_{tne}}(G) \geq \dfrac{n+t}{3}$. □

Theorem 2.2 *Let T be a wounded spider obtained from the star $K_{1,n-1}$, $n \geq 5$ by subdividing m edges exactly once. Then*

$$\gamma_{tne}(T) = \gamma_{ne}(T) = \begin{cases} n, & \text{if } m = n-1\ ; \\ n-1, & \text{if } m = n-2; \\ n-2, & \text{if } m \leq n-3. \end{cases}$$

Proof Proof follows from Theorem 1.2. □

Theorem 2.3 *Let T be a tree of order n, $n \geq 4$ in which every non-pendant vertex is either a support or adjacent to a support and every non- pendant vertex which is support is adjacent to at least two pendant vertices. Then $\gamma_{tne}(T) = \gamma_{ne}(T)$.*

Proof Let D denote set of all non-pendant vertices and all pendant vertices except two for each support of T. Clearly, D is a γ_{ne}-set. Since any support vertex adjacent to at least two pendant vertices, it follows that $\langle D \rangle$ contains no isolate vertex. Hence D is a tned-set and hence $\gamma_{tne}(T) \leq \gamma_{ne}(T)$. Since $\gamma_{ne}(T) \leq \gamma_{tne}(T)$, it follows that $\gamma_{tne}(T) = \gamma_{ne}(T)$. □

Theorem 2.4 *Let G be a connected graph of order n, $n \geq 4$. Then,*

$$\gamma_{tne}(G) \leq n - 2.$$

Proof It is enough to show that for any minimum total near equitable dominating set D of G, $|V - D| \geq 2$. Since G is a connected graph of order n, $n \geq 4$, it follows that $\delta(G) \geq 1$. Suppose $v \in V - D$ and adjacent to $u \in D$. Since $od_{_{V-D}}(v) \geq 1$, then $od_{_D}(u) \geq 2$. □

The star graph $G \cong K_{1,n}$ is an example of a connected graph for which $\gamma_{tne}(G) = 2n - (\Delta(G) + 3)$. The following theorem shows that, this is the best possible upper bound for $\gamma_{tne}(G)$.

Theorem 2.5 *If G is connected of order n, $n \geq 4$, then,*

$$\gamma_{tne}(G) \leq 2n - (\Delta(G) + 3).$$

Proof Let G be a connected graph of order n, $n \geq 4$, then by Theorem 2.4, $\gamma_{tne}(G) \leq n - 2 = 2n - (n - 1 + 3) \leq 2n - (\Delta(G) + 3)$. □

Theorem 2.6 *If G is a graph of order n, $n \geq 4$ and $\Delta(G) \leq n - 2$ such that both G and $\overline{G}$ connected, then*

$$\gamma_{tne}(G) + \gamma_{tne}(\overline{G}) \leq 3n - 6.$$

Proof Let G be a connected graph and $\Delta(G) \leq n - 2$. By Theorem 2.4, $\gamma_{tne}(G) \leq 2n - (\Delta(G)+4) \leq 2n-(\delta(G)+4)$. Since $\overline{G}$ is a connected, by Theorem 2.5, $\gamma_{tne}(\overline{G}) \leq 2n-(\Delta(\overline{G})+3)$, it follows that

$$\begin{aligned}\gamma_{tne}(G)+\gamma_{tne}(\overline{G}) &\leq 2n-(\delta(G)+4)+2n-(\Delta(\overline{G})+3)\\ &= 4n-(\delta(G)+\Delta(\overline{G}))-7\\ &= 3n-6.\end{aligned}$$ □

The bound is sharp for C_4.

Theorem 2.7 *Let G be a graph such that both G and $\overline{G}$ connected. Then,*

$$\gamma_{tne}(G)+\gamma_{tne}(\overline{G}) \leq 2n-4.$$

Proof Since both G and $\overline{G}$ are a connected, it follows by Theorem 2.4 that, $\gamma_{tne}(G)+\gamma_{tne}(\overline{G}) \leq 2n-4$. □

The bound is sharp for P_4. We now proceed to obtain a characterization of minimal tned-sets.

Theorem 2.8 *A tned- set D of a graph G is a minimal tned- set if and only if one of the following holds:*

(i) D is a minimal near equitable dominating set;
(ii) There exist $x, y \in D$ such that $N(y) \cap N(D-\{x\}) = \phi$.

Proof Suppose that D is a minimal tned-set of G. Then for any $u \in D$, $D-\{u\}$ is not tned-set. If D is a minimal near equitable dominating set, then we are done. If not, then there exists a vertex $x \in D$ such that $D-\{x\}$ is a near equitable dominating set, but not a tned- set. Therefore there exists a vertex $y \in D-\{x\}$ such that y is an isolated vertex in $\langle(D-\{x\})\rangle$. Hence $N\{y\} \cap N(D-\{x\}) = \phi$.

Conversely, let D be a tned- set and (i) holds. Suppose D is not a minimal tned-set. Then for every $u \in D$, $D-\{u\}$ is a tned- set. So, D is not a minimal near equitable dominating set, a contradiction. Next, suppose that D is a tned- set and (ii) holds. Then there exist $x, y \in D$ such that $N(y) \cap N(D-\{x\}) = \phi$.
Suppose to the contrary, D is not a minimal tned- set. Then for every $u \in D$, $D-\{u\}$ is a tned- set. So, $\langle(D-\{u\})\rangle$ does not contain any isolated vertex. Therefore for every $x, y \in D$, $N(y) \cap N(D-\{x\}) \neq \phi$, a contradiction. □

Theorem 2.9 *For any positive integer m, there exists a graph G such that $\gamma_{tne}(G) - \left\lfloor \frac{n}{\Delta+1} \right\rfloor = m$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.*

Proof For $m=1$, let $G=K_{3,3}$. Then, $\gamma_{tne}(G) - \left\lfloor \frac{n}{\Delta+1} \right\rfloor = 2-1 = 1$.
For $m=2$, let $G=K_{2,4}$. Then, $\gamma_{tne}(G) - \left\lfloor \frac{n}{\Delta+1} \right\rfloor = 3-1 = 2$.
For $m \geq 3$, let $G=S_{r,s}$, where $r+s=m+3$, $s \geq r+3$, $r \geq 2$, $\gamma_{tne}(G) = r+s-2 = m+1$,

$$\left\lfloor \frac{n}{\Delta+1} \right\rfloor = \left\lfloor \frac{r+s+2}{s+2} \right\rfloor = 1$$

and

$$\gamma_{_{tne}}(G) - \lfloor \frac{n}{\Delta + 1} \rfloor = r + s - 3 = m. \qquad \Box$$

§3. Total Near Equitable Domatic Number

The maximum order of a partition of the vertex set V of a graph G into dominating sets is called the domatic number of G and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [9]. In this section we present few basic results on the total near equitable domatic number of a graph.

Let G be a graph without isolated vertices. A total near equitable domatic partition (tne-domatic partition) of G is a partition $\{V_1, V_2, \cdots, V_k\}$ of $V(G)$ in which each V_i is a tned-set of G. The maximum order of a tne-domatic partition of G is called the *total near equitable domatic number* (tne-domatic number) of G and is denoted by $d_{tne}(G)$.

We now proceed to compute $d_{tne}(G)$ for some standard graphs.

1. For any complete graph K_n, $n \geq 4$, $d_{tne}(K_n) = d_{ne}(K_n) = 2$.

2. For any $n \geq 1$, $d_{tne}(C_{4n}) = 2$.

3. For any star $K_{1,n}$, $n \geq 3$, $d_{tne}(K_{1,n}) = d_{ne}(K_{1,n}) = 1$.

4. For the wheel W_n on n vertices, then $d_{tne}(W_n) = d_{ne}(W_n) = 1$.

5. For the complete bipartite graph $K_{n,m}$, with $2 < m \leq n$

$$d_{tne}(K_{n,m}) = d_{ne}(K_{n,m}) = \begin{cases} 2, & \text{if } |n-m| \leq 2 \text{ ;} \\ 1, & \text{if } |n-m| \geq 3, \, n, m \geq 2. \end{cases}$$

It is obvious that any total near equitable domatic partition of a graph G is also a total domatic partition and any total domatic partition is also a domatic partition, thus we obtain the obvious bound $d_{tne}(G) \leq d_t(G) \leq d(G)$.

Remark 3.1 Let $v \in V(G)$ and $deg(v) = \delta$. Since any tned-set of G must contain either v or a neighbour of v and $d_{tne}(G) \leq d_t(G)$, it follows that $d_{tne}(G) \leq \delta$.

Definition 3.2 *A graph G is called tne-domatically full if $d_{tne}(G) = \delta$.*

For example, a star $K_{1,n}$ is tne-domatically full.

Remark 3.3 Since every member of any tne-domatic partition of a graph G on n vertices has at least $\gamma_{tne}(G)$ vertices, it follows that $d_{tne}(G) \leq \dfrac{n}{\gamma_{tne}(G)}$. This inequality can be strict for rK_2, $r \geq 1$.

Theorem 3.4 *Let G be a graph of order n, $n \geq 4$ with $\Delta(G) \leq 2$ such that both G and $\overline{G}$ are connected. Then $d_{tne}(\overline{G}) \leq 2$.*

proof Since $\Delta(G) \leq 2$, it follows that for any $v \in \overline{G}$, $deg(v) \geq n-3$. Hence $\gamma_{tne}(\overline{G}) \leq \lceil \frac{n}{2} \rceil$. Thus by Remark 3.3, $d_{tne}(G) \leq 2$. □

The bound is sharp for P_n, $n \geq 6$.

Theorem 3.5 *Let G be a graph of order n, $n \geq 4$ with $\Delta(G) \leq 2$ such that both G and $\overline{G}$ are connected. Then $\gamma_{tne}(G) + d_{tne}(\overline{G}) \leq n$.*

Proof Proof follows by Theorem 2.4 and Theorem 3.4. □

theorem 3.6 *For any graph G, $\gamma_{tne}(G) + d_{tne}(G) \leq 2n - 3$.*

proof By Theorem 2.5,

$$\gamma_{tne}(G) \leq 2n - (\Delta(G) + 3) \leq 2n - (\delta(G) + 3) \leq 2n - (d_{tne}(G) + 3).$$

Therefor, $\gamma_{tne}(G) + d_{tne}(G) \leq 2n - 3$. □

The bound is sharp for $2K_2$.

theorem 3.7 *For any graph G, $\gamma_{tne}(G) + d_{tne}(G) \leq n + \delta - 2$.*

Proof Since $d_{tne}(G) \leq d_t(G) \leq \delta(G)$, by Theorem 2.4,

$$\gamma_{tne}(G) + d_{tne}(G) \leq n + \delta - 2.$$ □

The bound is sharp for $K_{1,n}$.

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