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Semientire Equitable Dominating Graphs

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Abstract: The semientire equitable dominating graph $SE_qD(G)$ of a graph $G=(V,E)$ is the graph with vertex set $V \cup S$, where S is the collection of all minimal equitable dominating sets of G and with two vertices $u, v \in V \cup S$ adjacent if $u, v \in D$, where D is the minimal equitable dominating set or $u \in V(G)$ and $v = D$ is a minimal equitable dominating set of G containing u. In this paper, some necessary and sufficient conditions are given for $SE_qD(G)$ to be connected and Eulerian. Finally, some bounds on domination number of $SE_qD(G)$ are obtained in terms of vertices and edges of G.

Key Words: Dominating set, equitable dominating set, semientire equitable dominating graph.

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§1. Introduction

All graphs considered here are finite, undirected with no loops and multiple edges. As usual $p = |V(G)|$ and $q = |E(G)|$ denote the number of vertices and edges of a graph $G = (V, E)$ respectively. For any graph theoretic terminology and notations we refer to Harary [3] and for more details about parameters of domination number, we refer [4] and [6].

A set D of vertices in a graph G is called a *dominating* set of G if every vertex in $V - D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all minimal dominating sets of G. (See Ore [7]).

A subset D of V is called an *equitable dominating set* if for every $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such dominating sets is denoted by $\gamma^e(G)$ and called the *equitable domination number* of G [8].

In this paper, we use this idea to introduce a new graph valued function in the field of domination theory in graphs.

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§2. Semientire Equitable Dominating Graph

Definition 1 *Let $G = (V, E)$ be a graph. Let S be the collection of all minimal equitable dominating sets of G. The semientire equitable dominating graph $SE_qD(G)$ of a graph G is the graph with vertex set $V \cup S$ and two vertices $u, v \in V \cup S$ adjacent if $u, v \in D$, where D is a minimal equitable dominating set or $u \in V(G)$ and $v = D$ is a minimal equitable dominating set containing u.*

In Fig.1, a graph G and its semientire equitable dominating graph $SE_qD(G)$ are shown. Here $D_1 = \{1, 4, 5\}$, $D_2 = \{2, 4, 5\}$ and $D_3 = \{3, 4, 5\}$ are minimal equitable dominating sets of G.

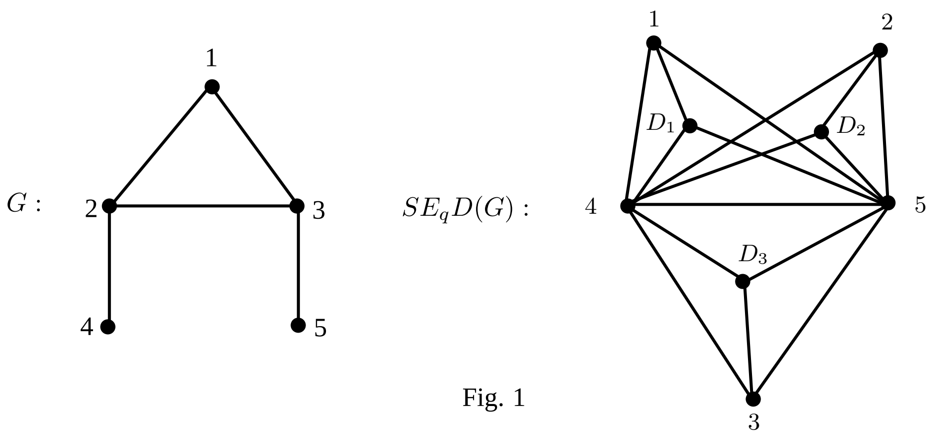

Fig. 1

§3. Results

Observation 1 In any graph G, the degree of a vertex D in $SE_qD(G)$ is the cardinality of minimal equitable dominating set D of G.

The following will be useful in the proof of our results.

Theorem A([2]) *Let G be a graph. If D is a maximal equitable independent set of G, then D is also a minimal equitable dominating set of G.*

Theorem 3.1 *For any nontrivial connected graph G, $\overline{G} \subseteq SE_qD(G)$.*

Proof Let u and v be any two adjacent vertices in $\overline{G}$ but which are not adjacent in G, then we can extend the set {u, v} into maximal equitable independent set D in G which is also a minimal equitable dominating set that is u and v are adjacent vertices in $SE_qD(G)$. Hence $\overline{G} \subseteq SE_qD(G)$. □

A subset D of V is called an *equitable independent set*, if for any $u \in D$, $v \notin N(u)$, for all $v \in D - \{u\}$. If a vertex $u \in V(G)$ be such that $|deg(u) - deg(v)| \geq 2$ for all $v \in N(u)$ then u is in each equitable dominating set. Such vertices are called *equitable isolates*.

First we obtain a necessary and sufficient condition on a graph G such that the semientire equitable dominating graph $SE_qD(G)$ is connected.

Theorem 3.2 *For any nontrivial connected graph G, the semientire equitable dominating graph $SE_qD(G)$ is connected if and only if $\Delta(G) \leq p-1$ and $\gamma^e(G) \geq 2$.*

Proof Let $\Delta(G) \leq p-1$ and u, v be any two vertices in G. Then we have the following cases.

Case 1 If u and v are not adjacent in G, then by Theorem 3.1, u is adjacent to v in $SE_qD(G)$.

Case 2 If u and v are adjacent in G and there is a vertex w in G which is not adjacent to both u and v, then u and v are joined by a path uwv in $SE_qD(G)$.

Case 3 Let u and v are adjacent in G and w is another vertex in G which is adjacent to both u and v, then there exist two maximal equitable independent sets D_1 and D_2 are minimal equitable dominating sets in G. Hence u and v connected through w in $SE_qD(G)$. From the above cases, we get $SE_qD(G)$ is connected.

Suppose $\gamma^e(G) = 1$. Then every vertex of G has $\Delta(G) = p-1$ and forms a minimal equitable dominating set except one vertex which is adjacent to all the other vertices in G. Therefore by definition, the semientire equitable dominating graph is disconnected, a contradiction. Hence $\gamma^e(G) \geq 2$.

Conversely, suppose $SE_qD(G)$ is connected. On the contrary $\gamma^e(G) = 1$. If G is a graph having $\Delta(G) \leq p-1$ with no equitable isolated vertices, then every vertex of G forms a minimal equitable dominating set D of G. This implies $SE_qD(G)$ is disconnected, a contradiction. Hence $\gamma^e(G) \geq 2$. □

Let k and $k+1$ be any two positive integers, $1 \leq k \leq k+1$. A graph G is said to be $(k, k+1)$ *bi-regular graph*, if its vertices have degree either k or $k+1$.

Theorem 3.3 *For any unicyclic graph G without isolated vertices, then $SE_qD(G)$ is a $(p+2, p-2)$ bi-regular graph.*

Proof Let G be a unicyclic graph of order p and contain no isolated vertices. Then from the definition of semientire equitable dominating graph, every vertex of $SE_qD(G)$ has the degree either $p+2$ or $p-2$. Hence $SE_qD(G)$ is a $(p+2, p-2)$ bi-regular graph. □

Remark 1 If T is a tree of order p, then $SE_qD(T)$ is a p-regular graph.

Proposition 3.1 *The semientire equitable dominating graph $SE_qD(G)$ is pK_2 if and only if $G = K_p$; $p \geq 2$.*

Proof Suppose $G = K_p; p \geq 2$. Then clearly each vertex of G will form a minimal equitable dominating set. Hence $SE_qD(G) = pK_2$.

Conversely, suppose $SE_qD(G) = pK_2$ and $G \neq K_p$. Then there exists at least one minimal equitable dominating set D containing two vertices of G. Then by the definition of semientire equitable dominating graph, D will form C_3 in $SE_qD(G)$. Hence $G = K_p; p \geq 2$. □

Theorem 3.4 *Let the semientire equitable dominating graph $SE_qD(G)$ is a graph with $2p$ vertices and p edges if and only if $G = K_p; p \geq 2$.*

Proof Suppose $G = K_p; p \geq 2$. Then by definition of $SE_qD(G)$, it is clear that $SE_qD(G)$ is a graph with $2p$ vertices and q edges.

Conversely, suppose $SE_qD(G)$ is a $(2p, p)$ graph. Then the graph pK_2 is the only graph with $2p$ vertices and p edges. Then by Proposition 3.1, $G = K_p; p \geq 2$. □

Corollary 1 *If $G = K_{1,n}$; $n \geq 3$, then $SE_qD(G) = K_{n+2}$.*

Theorem 3.5 *If G is a connected graph with $\Delta(G) < p - 1$, then $diam(SE_qD(G)) \leq 2$, where $diam(G)$ is the diameter of a graph G.*

Proof Let G be a nontrivial connected graph and by Theorem 3.2, $SE_qD(G)$ is connected. Let $u, v \in V(SE_qD(G))$ be any two arbitrary vertices. We consider the following cases.

Case 1 Suppose $u, v \in V(G)$, u and v are nonadjacent vertices in G.
Then $d_{SE_qD(G)}(u, v) = 1$. If u and v are adjacent in G and there is no minimal equitable dominating set containing both u and v. Then there exists another vertex w in $V(G)$, which is not adjacent to both u and v. Let D_1 and D_2 be any two equitable dominating sets containing u, w and v, w respectively. Hence u and v are connected in $SE_qD(G)$ by a path uwv. Thus $d_{SE_qD(G)}(u, v) \leq 2$.

Case 2 Suppose $u \in V(G)$ and $v \notin V(G)$. Then $v = D$ is a minimal equitable dominating set of G. If $u \in D$ then $d_{SE_qD(G)}(u, v) = 1$. If $u \notin D$, then there exist a vertex $w \in D$ which is adjacent to both u and v. Hence $d_{SE_qD(G)}(u, v) = d(u, w) + d(w, v) = 2$.

Case 3 Suppose $u, v \in V(G)$. Then $u = D$ and $v = D'$ are two minimal equitable dominating sets of G. If D and D' are disjoint, then every vertex in $w \in D$ is adjacent to some vertex $z \in D'$ and vice versa. This implies that

$$d_{SE_qD(G)}(u, v) = d(u, w) + d(w, z) + d(z, v) = 3.$$

If D and D' have a vertex in common, then $d_{SE_qD(G)}(u, v) = d(u, w) + d(w, v) = 2$. Thus from all these cases the result follows. □

The *equitable dominating graph* $E_qD(G)$ of a graph $G = (V, E)$ is the graph with vertex set $V \cup D$, where D is the set of all minimal equitable dominating sets of G and with two adjacent vertices $u, v \in V \cup D$ if $u \in V$ and v is a minimal equitable dominating set of G containing u.

Proposition 3.2([1]) *The equitable dominating graph $E_qD(G)$ is pK_2 if and only if $G = K_p; p \geq 2$.*

Theorem 3.6 *The equitable dominating graph is isomorphic to the semientire equitable dominating graph if and only if G is a nontrivial complete graph.*

Proof Let G be a nontrivial complete graph K_p. Then from Proposition 3.2, $E_qD(G) = pK_2$, and we have Proposition 3.1, Hence $E_qD(G) = SE_qD(G) = pK_2$.

Conversely, suppose $E_qD(G) = SE_qD(G)$, Propositions 3.1 and 3.2, G must be complete graph. Hence $G = K_p; p \geq 2$. □

We need the following theorem for the proof of our next results.

Theorem B([3]) *A connected graph G is eulerian if and only if every vertex of G has even degree.*

Next, we prove the necessary and sufficient condition for $SE_qD(G)$ to be Eulerian.

Theorem 3.7 *For any graph G with no isolated vertices, $SE_qD(G)$ is Eulerian if and only if the cardinality of each minimal equitable dominating set is even.*

Proof Let $\Delta(G) \leq p-1$ and $\gamma^e(G) \geq 2$, by Theorem 3.2, $SE_qD(G)$ is connected. Suppose $SE_qD(G)$ is Eulerian. On the contrary, every minimal equitable dominating set contains odd number of vertices and by observation 1, hence $SE_qD(G)$ has a vertex of odd degree, therefore by Theorem B, $SE_qD(G)$ is not Eulerian. Hence the cardinality of minimal equitable dominating set is even.

Conversely, suppose the cardinality of minimal equitable dominating set is even. Then degree of each vertex in $SE_qD(G)$ is even. Therefore by Theorem B, $SE_qD(G)$ is Eulerian. □

$4. Domination in $SE_qD(G)$

We first calculate the domination number of $SE_qD(G)$ of some standard class of graphs.

Theorem 4.1 *Let G be a graph without isolated vertices. Then,*

1. if $G = K_p; p \geq 2$, then $\gamma(SE_qD(K_p) = p$.
2. if $G = K_{1,p}; p \geq 1$, then $\gamma(SE_qD(K_{1,p}) = 1$.
3. if $G = P_p$, $p \geq 2$, then $\gamma(SE_qD(P_p) = 2$.
4. if $G = C_p; p \geq 4$, then $\gamma(SE_qD(C_p) = 3$.
5. if $G = W_p; p \geq 5$, then $\gamma(SE_qD(W_p) = 1$.

Theorem 4.2 *Let G be any graph of order p and $S = \{S_1, S_2, S_3, \cdots S_n\}$ be the minimal equitable dominating set of G, then $\gamma(SE_qD(G)) \leq \gamma(\overline{G}) + |S|$.*

Proof Let G be a connected graph. Let $D = \{v_1, v_2, v_3, \cdots v_i\}$; $1 \leq i \leq p$ be the set of all minimal equitable dominating sets of $\overline{G}$. By the definition of $SE_qD(G)$, each S_i; $1 \leq i \leq p$ is independent in $SE_qD(G)$. Hence $D' = D \cup S$ will form a dominating set in $SE_qD(G)$. Therefore $\gamma(SE_qD(G)) \leq |D'| = |D \cup S| = \gamma(\overline{G}) + |S|$. □

Further, we get the Nordhaus-Gaddum type result for semientire equitable dominating graph.

Theorem 4.3 *Let G be a graph such that both G and $\overline{G}$ are connected of order $p \geq 2$. Then*

1. $\gamma(SE_qD(G)) + \gamma(SE_qD(\overline{G})) \leq p$.
2. $\gamma(SE_qD(G)).\gamma(SE_qD(\overline{G})) \leq 2p$.

Further, the equality holds good if and only if $G = P_4$.

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