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On Cosets and Normal Subgroups

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Abstract: The paper [5] has worked on fuzzy cosets and fuzzy normal subgroups of a group, [8] has extended the idea to fuzzy middle coset. In addition to what has been done, we make a link between fuzzy coset and fuzzy middle coset and investigate some more properties of the fuzzy middle coset. [7] made attempt with some results needing adjustment. [2], [8] and [9] have shown that if $f \in F(S_n)$, the set of all fuzzy subgroups of S_n, is such that Imf has the highest order and f is constant on the conjugacy classes of S_n, then it is co-fuzzy symmetric subgroup of S_n. Then, using some results of [5], we get another result.

Key Words: Middle cosets, fuzzy normal, normal subgroups, fuzzy μ-commutativity.

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§1. Introduction

This paper seeks to contribute to the body of knowledge existing in the area of fuzzy normal subgroup without any damage to the existing one.

§2. Preliminaries

Definition 2.1 *Let X be a non-empty set. A fuzzy subset μ of the set G is a function $\mu : G \to [0, 1]$.*

Definition 2.2 *Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if*

(*i*) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$;
(*ii*) $\mu(x^{-1}) = \mu(x)$;
(*iii*) *μ is called a fuzzy normal subgroup if $\mu(xy) = \mu(yx)$ for all x and y in G.*

Definition 2.3 *Let μ be a fuzzy subset (subgroup) of X. Then, for some t in $[0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset (subgroup) of the fuzzy subset (subgroup) μ.*

Definition 2.4 *Let μ be a fuzzy subgroup of a group G. For a in G, the fuzzy left (or right) coset*

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$a\mu$ (or μa) of G determined by a and μ is defined by $(a\mu)(x) = \mu(a^{-1}x)$ (or $(\mu a)(x) = \mu(xa^{-1})$) for all x in G.

Definition 2.5 *Let μ be a fuzzy subgroup of a group G. For a and b in G, the fuzzy middle coset $a\mu b$ of G is defined by $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$ for all x in G.*

Proposition 2.6 *Let G be a group and μ a fuzzy subset of G. Then μ is a fuzzy subgroup of G if and only if G^t_μ is a level subgroup of G for every t in $[0, \mu(e)]$, where e is the identity of G.*

Theorem 2.7 *Let μ be a fuzzy normal subgroup of a group G. Let $t \in [0, 1]$ such that $t \leq \mu(e)$, where e is the identity of G. Then G^t_μ is a normal subgroup of G.*

Remark 2.8 The paper [5] have also shown that the collection $\{G^t_\mu\}$ form a chain of normal subgroups of G.

Theorem 2.9 *Let μ and λ be fuzzy subgroups of G. Then they are conjugate if for some $a \in G$ we have $\mu(a^{-1}xa) = \lambda(x)$ $\forall x \in G$.*

Theorem 2.10 *Let μ and λ be any two fuzzy subgroup of any group G. Then, μ and λ are conjugate fuzzy subgroup of G if and only if $\mu = \lambda$.*

Theorem 2.11([8]) *Let μ be a fuzzy normal subgroup of G, Then for any $g \in G$, $\mu(gxg^{-1}) = \mu(g^{-1}xg)$ for every $x \in G$.*

§3. Some Results on Fuzzy Normal and Cosets

Theorem 3.1 *Let $a^{-1}\mu a$ be a fuzzy middle coset of G for some $a \in G$. Then all such a form the normalizer $N(\mu)$ of fuzzy subgroup μ of G if and only if μ is fuzzy normal.*

Proof The paper [5] defined the normalizer of μ by $N(\mu) = \{a \in G : \mu(axa^{-1}) = \mu(x)\}$. Then, $\mu(axa^{-1}) = \mu(x) \Leftrightarrow \mu$ is fuzzy normal so that $\mu(axa^{-1}a) = \mu(xa) \Leftrightarrow \mu(ax) = \mu(xa)$.

Conversely, let μ be fuzzy normal and $a^{-1}\mu a$ a middle coset in G. Then, for all $x \in G$ and some $a \in G$,

$$(a^{-1}\mu a)(x) = \mu(axa^{-1}) = \mu(aa^{-1}x) = \mu(x).$$

This implies that

$$\mu(axa^{-1}) = \mu(x).$$

Hence,

$$\{a\} = N(\mu).$$ □

Proposition 3.2 *Let μ be a fuzzy normal subgroup of G by a and b. Then every fuzzy middle coset $a\mu b$ coincides with some left and right cosets $c\mu$ and μc respectively, where c^{-1} is the product $b^{-1}a^{-1}$.*

Proof By associativity in G and 2.2(iii), we have that

$$(a\mu b)(x) = \mu((a^{-1}x)b^{-1}) = \mu(b^{-1}(a^{-1}x))$$

$$\mu(b^{-1}a^{-1}x) = \mu(c^{-1}x) = \mu(xc^{-1}) \text{ still by 2.2(iii).}$$

Thus,

$$(a\mu b) = c\mu = \mu c. \quad \square$$

Theorem 3.3 *Let G be a group of order 2 and μ a fuzzy normal subgroup of G. Then, for some $a \in G$ and $\forall x \in G$, the middle coset $a\mu a$ coincides with fuzzy subgroup μ.*

Proof In the middle coset $a\mu b$, take $a = b$. By associativity in G, we have

$$(a\mu a)(x) = \mu((a^{-1}x)a^{-1})$$

. By 3.2,

$$\mu((a^{-1}x)a^{-1}) = \mu(a^{-2}x)$$

. Since $a^{-1} \in G$ and G is of order 2,

$$\mu(a^{-2}x) = \mu((a^{-1})^2x) = \mu(ex) = \mu(x).$$

Therefore,

$$a\mu a = \mu. \quad \square$$

Now we introduce the notion of fuzzy μ-commutativity.

Definition 3.4 *Let μ be a fuzzy subgroup of G. Two elements a and b in G are said to be fuzzy μ-commutative if $a\mu b = b\mu a$.*

Theorem 3.5 *Let μ be a fuzzy normal subgroup of G. Then any two elements a and b in G are fuzzy μ-commutative.*

Proof Notice that

$$(a\mu b)(x) = \mu(a^{-1}xb^{-1}).$$

Then, by 2.11,

$$\mu(a^{-1}xb^{-1}) = \mu(b^{-1}xa^{-1}) = (b\mu a)(x).$$

Thus,

$$a\mu b = b\mu a. \quad \square$$

In [7], it is claimed that every middle coset of a group G is a fuzzy subgroup. But here is a counter example.

Example 3.6 Let $G = (\mathbb{Z}_4, +)$ and choose $a = b = 1$ then $a^{-1} = b^{-1} = 3$.

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0 = e \\ 0.6, & \text{otherwise.} \end{cases}$$

It can be seen that μ is a fuzzy subgroup of G. But the middle coset $a\mu b$ defined by

$$(a\mu b)(x) = \begin{cases} 1, & \text{if } x = 2 \\ 0.6, & \text{otherwise.} \end{cases}$$

is such that $(a\mu b)(2) > (a\mu b)(e)$. But this is a contradiction, since, usually, if μ is a fuzzy subgroup of a group G, $\mu(e) \geq \mu(x)\ \forall x \in G$.

In the following theorem, a necessary condition for middle coset of a group to be fuzzy subgroup is given.

Theorem 3.7 *Every middle coset $a\mu b$ of a group G is a fuzzy subgroup if μ is fuzzy conjugate to some fuzzy subgroup λ of G.*

Proof Let $b = a^{-1}$ for some $a, b \in G$ and μ and λ be fuzzy conjugate subgroups of G.

$$(a\mu b)(xy^{-1}) = (a\mu a^{-1})(xy^{-1}) = \mu(a^{-1}xy^{-1}a) = \lambda(xy^{-1}) \geq min\{\lambda(x), \lambda(y)\}$$

This implies that

$$min\{\lambda(x), \lambda(y)\} = min\{\mu(a^{-1}xa), \mu(a^{-1}ya)\} = min\{(a\mu a^{-1})(x), (a\mu a^{-1})(y)\}.$$

Hence,

$$(a\mu b)(xy^{-1}) \geq min\{(a\mu b)(x), (a\mu b)(y)\}.$$ □

Remark 3.8 If $b = a^{-1}$, the middle coset $a\mu a^{-1}$ is a fuzzy subgroup since μ is self conjugate. Hence, the result of 3.7 generalizes the theorem 1.2.10 of [8].

Proposition 3.8 of [7] says that fuzzy middle cosets form normal subgroup of G. But here is a counter example.

Example 3.9 Let $G = S_3$ and $a = (123),\ b = (12),\ x = (12),\ y^{-1} = (123)$. Also, define the fuzzy group μ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = e \\ 0.5, & \text{if } x = (123), (132) \\ 0.3, & \text{otherwise.} \end{cases}$$

Then, $(a\mu b)(xy^{-1}) = 0.3$ and $(a\mu b)(y^{-1}x) = 1$. Thus,

$$(a\mu b)(xy^{-1}) \neq (a\mu b)(y^{-1}x),$$

which implies that $a\mu b$ is not fuzzy normal.

We now give a characterization for $a\mu b$ to be fuzzy normal. It is noteworthy that [8] has shown that $a\mu a^{-1}$ and μ are conjugates.

Theorem 3.10 *A fuzzy middle coset $a\mu b$ is fuzzy normal if and only if $b = a^{-1}$ and μ is fuzzy normal.*

Proof Let μ be fuzzy normal. By definition,

$$(a\mu b)(xy) = \mu(a^{-1}xyb^{-1}).$$

By 2.9 and 2.10, if we take $b = a^{-1}$, $a\mu b$ and μ are conjugate so that

$$\mu(a^{-1}xyb^{-1}) = (a\mu b)(xy) = \mu(xy).$$

Since, μ is fuzzy normal,

$$\mu(xy) = \mu(yx) = \mu(a^{-1}yxb^{-1}) = (a\mu b)(yx).$$

Thus,

$$(a\mu b)(xy) = (a\mu b)(yx).$$

Conversely, assume that $a\mu b$ is fuzzy normal. Then,

$$(a\mu b)(xy) = (a\mu b)(yx).$$

It follows from 2.10 that

$$(a\mu b)(xy) = \mu(a^{-1}xyb^{-1}) = \mu(xy) \Leftrightarrow b = a^{-1}$$

and

$$(a\mu b)(yx) = \mu(a^{-1}yxb^{-1}) = \mu(yx) \Leftrightarrow b = a^{-1}.$$

This implies that

$$\mu(xy) = (a\mu b)(xy) = (a\mu b)(yx) = \mu(yx) \Leftrightarrow b = a^{-1}.$$

Hence, μ is fuzzy normal and $b = a^{-1}$. □

Proposition 3.11 *Let $\mu \in F(S_n)$ be co-fuzzy symmetric subgroup of S_n. Then μ is fuzzy normal.*

Proof Since μ is co-fuzzy, it is constant on $x^{-1}\Pi x$, the conjugate class of S_n containing Π. Hence $\mu(x^{-1}\Pi x) = \mu(\Pi)$ for $\forall \Pi \in C(\Pi)$ and some $x \in S_n$. By 2.2(iii), μ is fuzzy normal. □

Theorem 3.12 *Every symmetric group with co-fuzzy symmetric subgroup μ is such that the level subgroups μ_t are normal subgroups of the symmetric group so that for $t \in [0, 1]$ and $t < \mu(e)$, the collection $\{\mu_t\}$ is a chain of normal subgroups of S_n.*

Proof Let $\mu \in F(S_n)$ be such that μ is co-fuzzy. Then it is fuzzy normal by 3.11. Then every level subgroup μ_t (which is a subgroup of S_n) of μ is a normal subgroup by 2.7. Then, by 2.8, the collection $\{\mu_t\}$ is a chain of normal subgroups of S_n. □

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