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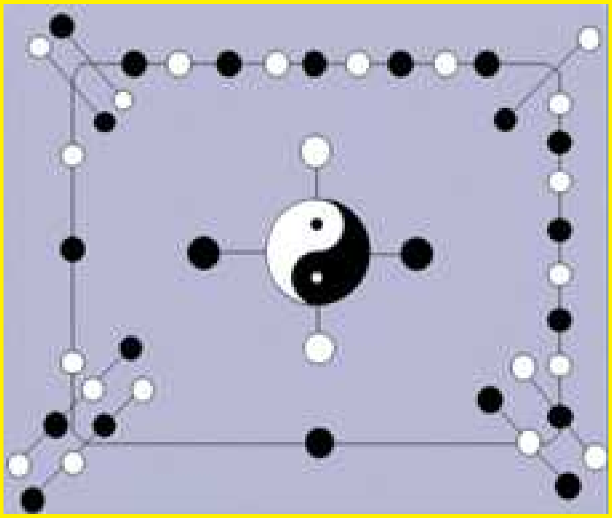

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Famous Words:

Too much experience is a dangerous thing.

By Ooscar Wilde, A British dramatist.

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Mathematics on Non-Mathematics

— A Combinatorial Contribution

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Abstract: A classical system of mathematics is homogenous without contradictions. But it is a little ambiguous for modern mathematics, for instance, the Smarandache geometry. Let $\mathscr{F}$ be a family of things such as those of particles or organizations. Then, *how to hold its global behaviors or true face*? Generally, $\mathscr{F}$ is not a mathematical system in usual unless a set, i.e., a system with contradictions. There are no mathematical subfields applicable. Indeed, the trend of mathematical developing in 20th century shows that a mathematical system is more concise, its conclusion is more extended, but farther to the true face for its abandoned more characters of things. This effect implies an important step should be taken for mathematical development, i.e., turn the way to extending non-mathematics in classical to mathematics, which also be provided with the philosophy. All of us know *there always exists a universal connection between things in* $\mathscr{F}$. Thus there is an underlying structure, i.e., a vertex-edge labeled graph G for things in $\mathscr{F}$. Such a labeled graph G is invariant accompanied with $\mathscr{F}$. The main purpose of this paper is to survey how to extend classical mathematical non-systems, such as those of algebraic systems with contradictions, algebraic or differential equations with contradictions, geometries with contradictions, and generally, classical mathematics systems with contradictions to mathematics by the underlying structure G. All of these discussions show that a non-mathematics in classical is in fact a mathematics underlying a topological structure G, i.e., mathematical combinatorics, and contribute more to physics and other sciences.

Key Words: Non-mathematics, topological graph, Smarandache system, non-solvable equation, CC conjecture, mathematical combinatorics.

AMS(2010): 03A10,05C15,20A05, 34A26,35A01,51A05,51D20,53A35

§1. Introduction

A thing is complex, and hybrid with other things sometimes. That is why it is difficult to know the true face of all things, included in "Name named is not the eternal Name; the unnamable is the eternally real and naming the origin of all things", the first chapter of *TAO TEH KING* [9], a well-known Chinese book written by an ideologist, *Lao Zi* of China. In fact, all of things with

[1]Received February 16, 2014, Accepted August 8, 2014.

universal laws acknowledged come from the six organs of mankind. Thus, the words "*existence*" and "non-existence" are knowledged by human, which maybe not implies the true existence or not in the universe. Thus the existence or not for a thing is *invariant*, independent on human knowledge.

The boundedness of human beings brings about a unilateral knowledge for things in the world. Such as those shown in a famous proverb "the blind men with an elephant". In this proverb, there are six blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, such as those shown in Fig.1 following. Each of them insisted on his own and not accepted others. They then entered into an endless argument.

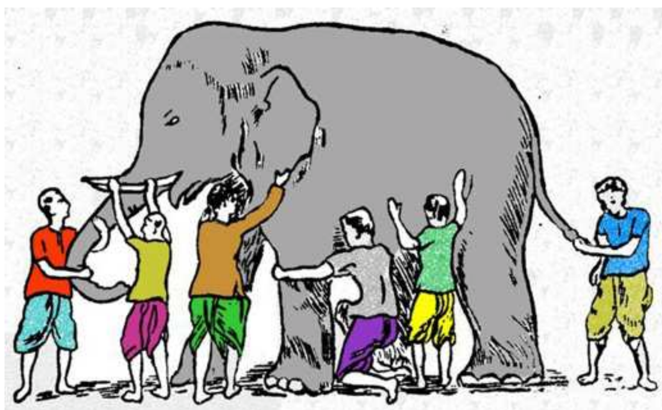


Fig.1

All of you are right! A wise man explains to them: *why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said*. Thus, the best result on an elephant for these blind men is

$$\begin{aligned}\text{An elephant} &= \{4\text{ pillars}\}\bigcup\{1\text{ rope}\}\bigcup\{1\text{ tree branch}\}\\ &\bigcup \{2\text{ hand fans}\}\bigcup\{1\text{ wall}\}\bigcup\{1\text{ solid pipe}\}\end{aligned}$$

What is the meaning of this proverb for understanding things in the world? It lies in that the situation of human beings knowing things in the world is analogous to these blind men. Usually, a thing T is identified with its known characters (or name) at one time, and this process is advanced gradually by ours. For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be its known and $\nu_i, i \geq 1$ unknown characters at time t. Then, the thing T is understood by

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right)\bigcup\left(\bigcup_{k\geq 1}\{\nu_k\}\right) \qquad (1.1)$$

in logic and with an approximation $T^\circ = \bigcup\limits_{i=1}^{n}\{\mu_i\}$ for T at time t. This also answered why difficult for human beings knowing a thing really.

Generally, let Σ be a finite or infinite set. A *rule* or a *law* on a set Σ is a mapping $\underbrace{\Sigma \times \Sigma \cdots \times \Sigma}_{n} \rightarrow \Sigma$ for some integers n. Then, a *mathematical system* is a pair $(\Sigma; \mathcal{R})$, where $\mathcal{R}$ consists those of rules on Σ by logic providing all these resultants are still in Σ.

Definition 1.1([28]-[30]) *Let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be m mathematical system, different two by two. A Smarandache multi-system $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m} \Sigma_i$ with rules $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$ on $\widetilde{\Sigma}$, denoted by $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$.*

Consequently, the thing T is nothing else but a Smarandache multi-system (1.1). However, these characters $\nu_k, k \geq 1$ are unknown for one at time t. Thus, $T \approx T^\circ$ is only an approximation for its true face and it will never be ended in this way for knowing T, i.e., "Name named is not the eternal Name", as Lao Zi said.

But one's life is limited by its nature. It is nearly impossible to find all characters $\nu_k,\ k \geq 1$ identifying with thing T. Thus one can only understands a thing T relatively, namely find invariant characters $\mathcal{I}$ on $\nu_k,\ k \geq 1$ independent on artificial frame of references. In fact, this notion is consistent with *Erlangen Programme* on developing geometry by Klein [10]: *given a manifold and a group of transformations of the same, to investigate the configurations belonging to the manifold with regard to such properties as are not altered by the transformations of the group*, also the fountainhead of *General Relativity* of Einstein [2]: *any equation describing the law of physics should have the same form in all reference frame*, which means that a universal law does not moves with the volition of human beings. Thus, an applicable mathematical theory for a thing T should be an *invariant theory* acting on by all automorphisms of the artificial frame of reference for thing T.

All of us have known that things are inherently related, not isolated in philosophy, which implies that these is an underlying structure in characters $\mu_i,\ 1 \leq i \leq n$ for a thing T, namely, an inherited topological graph G. Such a graph G should be independent on the volition of human beings. Generally, a labeled graph G for a Smarandache multi-space is introduced following.

Definition 1.2([21]) *For any integer $m \geq 1$, let $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ be a Smarandache multi-system consisting of m mathematical systems $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$. An inherited topological structure $G[\widetilde{S}]$ of $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ is a topological vertex-edge labeled graph defined following:*

$$V(G[\widetilde{S}]) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$
$$E(G[\widetilde{S}]) = \{(\Sigma_i, \Sigma_j) | \Sigma_i \bigcap \Sigma_j \neq \emptyset,\ 1 \leq i \neq j \leq m\} \text{ with labeling}$$
$$L:\ \Sigma_i \rightarrow L(\Sigma_i) = \Sigma_i \quad \text{and} \quad L:\ (\Sigma_i, \Sigma_j) \rightarrow L(\Sigma_i, \Sigma_j) = \Sigma_i \bigcap \Sigma_j$$

for integers $1 \leq i \neq j \leq m$.

However, classical combinatorics paid attentions mainly on techniques for catering the need of other sciences, particularly, the computer science and children games by artificially giving up individual characters on each system $(\Sigma, \mathcal{R})$. For applying more it to other branch sciences initiatively, a good idea is pullback these individual characters on combinatorial objects again, ignored by the classical combinatorics, and back to the true face of things, i.e., an interesting conjecture on mathematics following:

Conjecture 1.3(CC Conjecture, [15],[19]) *A mathematics can be reconstructed from or turned into combinatorization.*

Certainly, this conjecture is true in philosophy. So it is in fact a combinatorial notion on developing mathematical sciences. Thus:

(1) *One can combine different branches into a new theory and this process ended until it has been done for all mathematical sciences, for instance, topological groups and Lie groups.*

(2) *One can selects finite combinatorial rulers and axioms to reconstruct or make generalizations for classical mathematics, for instance, complexes and surfaces.*

From its formulated, the CC conjecture brings about a new way for developing mathematics , and it has affected on mathematics more and more. For example, it contributed to groups, rings and modules ([11]-[14]), topology ([23]-[24]), geometry ([16]) and theoretical physics ([17]-[18]), particularly, these 3 monographs [19]-[21] motivated by this notion.

A *mathematical non-system* is such a system with contradictions. Formally, let $\mathscr{R}$ be mathematical rules on a set Σ. A pair $(\Sigma;\mathscr{R})$ is non-mathematics if there is at least one ruler $R \in \mathscr{R}$ validated and invalided on Σ simultaneously. Notice that a multi-system defined in Definition 1.1 is in fact a system with contradictions in the classical view, but it is cooperated with logic by Definition 1.2. Thus, it lights up the hope of transferring a system with contradictions to mathematics, consistent with logic by combinatorial notion.

The main purpose of this paper is to show how to transfer a mathematical non-system, such as those of non-algebra, non-group, non-ring, non-solvable algebraic equations, non-solvable ordinary differential equations, non-solvable partial differential equations and non-Euclidean geometry, mixed geometry, differential non- Euclidean geometry, $\cdots$, etc. classical mathematics systems with contradictions to mathematics underlying a topological structure G, i.e., mathematical combinatorics. All of these discussions show that *a mathematical non-system is a mathematical system inherited a non-trivial topological graph, respect to that of the classical underlying a trivial K_1 or K_2.* Applications of these non-mathematic systems to theoretical physics, such as those of gravitational field, infectious disease control, circulating economical field can be also found in this paper.

All terminologies and notations in this paper are standard. For those not mentioned here, we follow [1] and [19] for algebraic systems, [5] and [6] for algebraic invariant theory, [3] and [32] for differential equations, [4], [8] and [21] for topology and topological graphs and [20], [28]-[31] for Smarandache systems.

§2. Algebraic Systems

Notice that the graph constructed in Definition 1.2 is in fact on sets $\Sigma_i,\ 1 \leq i \leq m$ with relations on their intersections. Such combinatorial invariants are suitable for algebraic systems. All operations $\circ :\ \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ on a set $\mathscr{A}$ considered in this section are closed and single valued, i.e., $a \circ b$ is uniquely determined in $\mathscr{A}$, and it is said to be *Abelian* if $a \circ b = b \circ a$ for

$\forall a, b \in \mathscr{A}$.

2.1 Non-Algebraic Systems

An algebraic system is a pair $(\mathscr{A}; \mathcal{R})$ holds with $a \circ b \in \mathscr{A}$ for $\forall a, b \in \mathscr{A}$ and $\circ \in \mathcal{R}$. A *non-algebraic system* $\neg(\mathscr{A}; \mathcal{R})$ on an algebraic system $(\mathscr{A}; \mathcal{R})$ is

AS^{-1}: *there maybe exist an operation $\circ \in \mathcal{R}$, elements $a, b \in \mathscr{A}$ with $a \circ b$ undetermined.*

Similarly to classical algebra, an isomorphism on $\neg(\mathscr{A}; \mathcal{R})$ is such a mapping on $\mathscr{A}$ that for $\forall \circ \in \mathcal{R}$,

$$h(a \circ b) = h(a) \circ h(b)$$

holds for $\forall a, b \in \mathscr{A}$ providing $a \circ b$ is defined in $\neg(\mathscr{A}; \mathcal{R})$ and $h(a) = h(b)$ if and only if $a = b$. Not loss of generality, let $\circ \in \mathcal{R}$ be a chosen operation. Then, there exist closed subsets $\mathscr{C}_i, \ i \geq 1$ of $\mathscr{A}$. For instance,

$$\langle a \rangle^{\circ} = \{a, a \circ a, a \circ a \circ a, \cdots, \underbrace{a \circ a \circ \cdots \circ a}_{k}, \cdots\}$$

is a closed subset of $\mathscr{A}$ for $\forall a \in \mathscr{A}$. Thus, there exists a decomposition $\mathscr{A}_1^{\circ}, \mathscr{A}_2^{\circ}, \cdots, \mathscr{A}_n^{\circ}$ of $\mathscr{A}$ such that $a \circ b \in \mathscr{A}_i^{\circ}$ for $\forall a, b \in \mathscr{A}_i^{\circ}$ for integers $1 \leq i \leq n$.

Define a topological graph $G[\neg(\mathscr{A}; \circ)]$ following:

$$V(G[\neg(\mathscr{A}; \circ)]) = \{\mathscr{A}_1^{\circ}, \mathscr{A}_2^{\circ}, \cdots, \mathscr{A}_n^{\circ}\};$$
$$E(G[\neg(\mathscr{A}; \circ)]) = \{(\mathscr{A}_i^{\circ}, \mathscr{A}_j^{\circ}) \text{ if } \mathscr{A}_i^{\circ} \bigcap \mathscr{A}_j^{\circ} \neq \emptyset, 1 \leq i, \neq j \leq n\}$$

with labels

$$L: \ \mathscr{A}_i^{\circ} \in V(G[\neg(\mathscr{A}; \circ)]) \to L(\mathscr{A}_i^{\circ}) = \mathscr{A}_i^{\circ},$$
$$L: \ (\mathscr{A}_i^{\circ}, \mathscr{A}_j^{\circ}) \in E(G[\neg(\mathscr{A}; \circ)]) \to \mathscr{A}_i^{\circ} \bigcap \mathscr{A}_j^{\circ} \text{ for integers } 1 \leq i \neq j \leq n.$$

For example, let $\mathscr{A}_1^{\circ} = \{a, b, c\}$, $\mathscr{A}_2^{\circ} = \{a, d, f\}$, $\mathscr{A}_3^{\circ} = \{c, d, e\}$, $\mathscr{A}_4^{\circ} = \{a, e, f\}$ and $\mathscr{A}_5^{\circ} = \{d, e, f\}$. Calculation shows that $\mathscr{A}_1^{\circ} \bigcap \mathscr{A}_2^{\circ} = \{a\}$, $\mathscr{A}_1^{\circ} \bigcap \mathscr{A}_3^{\circ} = \{c\}$, $\mathscr{A}_1^{\circ} \bigcap \mathscr{A}_4^{\circ} = \{a\}$, $\mathscr{A}_1^{\circ} \bigcap \mathscr{A}_5^{\circ} = \emptyset$, $\mathscr{A}_2^{\circ} \bigcap \mathscr{A}_3^{\circ} = \{d\}$, $\mathscr{A}_2^{\circ} \bigcap \mathscr{A}_4^{\circ} = \{a\}$, $\mathscr{A}_2^{\circ} \bigcap \mathscr{A}_5^{\circ} = \{d, f\}$,$\mathscr{A}_3^{\circ} \bigcap \mathscr{A}_4^{\circ} = \{e\}$, $\mathscr{A}_3^{\circ} \bigcap \mathscr{A}_5^{\circ} = \{d, e\}$ and $\mathscr{A}_4^{\circ} \bigcap \mathscr{A}_5^{\circ} = \{e, f\}$. Then, the labeled graph $G[\neg(\mathscr{A}; \circ)]$ is shown in Fig.2.

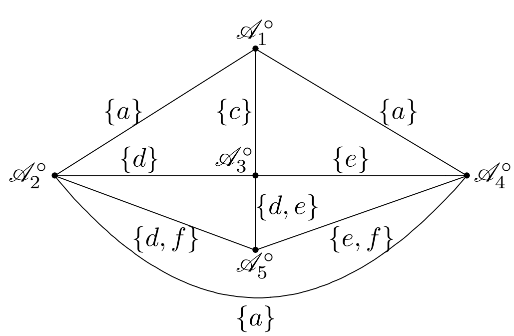


Fig.2

Let $h: \mathscr{A} \rightarrow \mathscr{A}$ be an isomorphism on $\neg(\mathscr{A};\circ)$. Then $\forall a, b \in \mathscr{A}_i^\circ)$, $h(a)\circ h(b) = h(a\circ b) \in h(A_i^\circ)$ and $h(A_i^\circ)\bigcap h(A_j^\circ) = h(A_i^\circ \bigcap A_j^\circ) = \emptyset$ if and only if $A_i^\circ \bigcap A_j^\circ = \emptyset$ for integers $1 \leq i \neq j \leq n$. Whence, if $G^h[\neg(\mathscr{A};\circ)]$ defined by

$$V(G^h[\neg(\mathscr{A};\circ)]) = \{h(\mathscr{A}_1^\circ), h(\mathscr{A}_2^\circ), \cdots, h(\mathscr{A}_n^\circ)\};$$
$$E(G^h[\neg(\mathscr{A};\circ)]) = \{(h(\mathscr{A}_i^\circ), h(\mathscr{A}_j^\circ)) \text{ if } h(\mathscr{A}_i^\circ)\bigcap h(\mathscr{A}_j^\circ) \neq \emptyset, 1 \leq i, \neq j \leq n\}$$

with labels

$$L^h:\ h(\mathscr{A}_i^\circ) \in V(G^h[\neg(\mathscr{A};\circ)]) \rightarrow L(h(\mathscr{A}_i^\circ)) = h(\mathscr{A}_i^\circ),$$
$$L^h:\ (h(\mathscr{A}_i^\circ), h(\mathscr{A}_j^\circ)) \in E(G^h[\neg(\mathscr{A};\circ)]) \rightarrow h(\mathscr{A}_i^\circ)\bigcap h(\mathscr{A}_j^\circ)$$

for integers $1 \leq i \neq j \leq n$. Thus $h: \mathscr{A} \rightarrow \mathscr{A}$ induces an isomorphism of graph $h^*: G[\neg(\mathscr{A};\circ)] \rightarrow G^h[\neg(\mathscr{A};\circ)]$. We therefore get the following result.

Theorem 2.1 *A non-algebraic system $\neg(\mathscr{A};\circ)$ in type AS^{-1} inherits an invariant $G[\neg(\mathscr{A};\circ)]$ of labeled graph.*

Let

$$G[\neg(\mathscr{A};\mathcal{R})] = \bigcup_{\circ\in\mathcal{R}} G[\neg(\mathscr{A};\circ)]$$

be a topological graph on $\neg(\mathscr{A};\mathcal{R})$. Theorem 2.1 naturally leads to the conclusion for non-algebraic system $\neg(\mathscr{A};\mathcal{R})$ following.

Theorem 2.2 *A non-algebraic system $\neg(\mathscr{A};\mathcal{R})$ in type AS^{-1} inherits an invariant $G[\neg(\mathscr{A};\mathcal{R})]$ of topological graph.*

Similarly, we can also discuss *algebraic non-associative systems*, *algebraic non-Abelian systems* and find inherited invariants $G[\neg(\mathscr{A};\circ)]$ of graphs. Usually, we adopt different notations for operations in $\mathcal{R}$, which consists of a multi-system $(\mathscr{A};\mathcal{R})$. For example, $\mathcal{R} = \{+,\cdot\}$ in an algebraic field $(R;+,\cdot)$. If we view the operation $+$ is the same as $\cdot$, throw out $0\cdot a$, $a\cdot 0$ and $1+a$, $a+1$ for $\forall a \in R$ in R, then $(R;+,\cdot)$ comes to be a non-algebraic system $(R;\cdot)$ with topological graph $G[R;\cdot]$ shown in Fig.3.

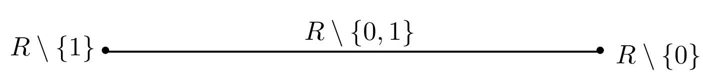


Fig.3

2.2 Non-Groups

A group is an associative system $(\mathscr{G};\circ)$ holds with identity and inverse elements for all elements in $\mathscr{G}$. Thus, for $a, b, c \in \mathscr{G}$, $(a\circ b)\circ c = a\circ(b\circ c)$, $\exists 1_{\mathscr{G}} \in \mathscr{G}$ such that $1_{\mathscr{G}}\circ a = a\circ 1_{\mathscr{G}} = a$ and for $\forall a \in \mathscr{G}$, $\exists a^{-1} \in \mathscr{A}\mathscr{G}$ such that $a\circ a^{-1} = 1_{\mathscr{G}}$. A *non-group* $\neg(\mathscr{G};\circ)$ on a group $(\mathscr{G};\circ)$ is an algebraic system in 3 types following:

$\mathbf{AG}_1^{-1}$: *there maybe exist a_1, b_1, c_1 and $a_2, b_2, c_2 \in \mathscr{G}$ such that $(a_1 \circ b_1) \circ c_1 = a_1 \circ (b_1 \circ c_1)$ but $(a_2 \circ b_2) \circ c_2 \neq a_2 \circ (b_2 \circ c_2)$, also holds with identity $1_{\mathscr{G}}$ and inverse element a^{-1} for all elements in $a \in \mathscr{G}$.*

$\mathbf{AG}_2^{-1}$: *there maybe exist distinct $1_{\mathscr{G}}, 1'_{\mathscr{G}} \in \mathscr{G}$ such that $a_1 \circ 1_{\mathscr{G}} = 1_{\mathscr{G}} \circ a_1 = a_1$ and $a_2 \circ 1'_{\mathscr{G}} = 1'_{\mathscr{G}} \circ a_2 = a_2$ for $a_1 \neq a_2 \in \mathscr{G}$, also holds with associative and inverse elements a^{-1} on $1_{\mathscr{G}}$ and $1'_{\mathscr{G}}$ for $\forall a \in \mathscr{G}$.*

$\mathbf{AG}_3^{-1}$: *there maybe exist distinct inverse elements $a^{-1}, \dot{a}^{-1}$ for $a \in \mathscr{G}$, also holds with associative and identity elements.*

Notice that $(a \circ a) \circ a = a \circ (a \circ a)$ always holds with $a \in \mathscr{G}$ in an algebraic system. Thus there exists a decomposition $\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_n$ of $\mathscr{G}$ such that $(\mathscr{G}_i; \circ)$ is a group for integers $1 \leq i \leq n$ for Type AG_1^{-1}.

Type AG_2^{-1} is true only if $1_{\mathscr{G}} \circ 1'_{\mathscr{G}} \neq 1_{\mathscr{G}}$ and $\neq 1'_{\mathscr{G}}$. Thus $1_{\mathscr{G}}$ and $1'_{\mathscr{G}}$ are local, not a global identity on $\mathscr{G}$. Define

$$\mathscr{G}(1_{\mathscr{G}}) = \{a \in \mathscr{G} \text{ if } a \circ 1_{\mathscr{G}} = 1_{\mathscr{G}} \circ a = a\}.$$

Then $\mathscr{G}(1_{\mathscr{G}}) \neq \mathscr{G}(1'_{\mathscr{G}})$ if $1_{\mathscr{G}} \neq 1'_{\mathscr{G}}$. Denoted by $I(\mathscr{G})$ the set of all local identities on $\mathscr{G}$. Then $\mathscr{G}(1_{\mathscr{G}}),\ 1_{\mathscr{G}} \in I(\mathscr{G})$ is a decomposition of $\mathscr{G}$ such that $(\mathscr{G}(1_{\mathscr{G}}); \circ)$ is a group for $\forall 1_{\mathscr{G}} \in I(\mathscr{G})$.

Type AG_3^{-1} is true only if there are distinct local identities $1_{\mathscr{G}}$ on $\mathscr{G}$. Denoted by $I(\mathscr{G})$ the set of all local identities on $\mathscr{G}$. We can similarly find a decomposition of $\mathscr{G}$ with group $(\mathscr{G}(1_{\mathscr{G}}); \circ)$ holds for $\forall 1_{\mathscr{G}} \in I(\mathscr{G})$ in this type.

Thus, for a non-group $\neg(\mathscr{G}; \circ)$ of AG_1^{-1}-AG_3^{-1}, we can always find groups $(\mathscr{G}_1; \circ), (\mathscr{G}_2; \circ), \cdots$, $(\mathscr{G}_n; \circ)$ for an integer $n \geq 1$ with $\mathscr{G} = \bigcup\limits_{i=1}^{n} \mathscr{G}_i$. Particularly, if $(\mathscr{G}; \circ)$ is itself a group, then such a decomposition is clearly exists by its subgroups.

Define a topological graph $G[\neg(\mathscr{G}; \circ)]$ following:

$$\begin{aligned} &V(G[\neg(\mathscr{G}; \circ)]) = \{\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_n\}; \\ &E(G[\neg(\mathscr{G}; \circ)]) = \{(\mathscr{G}_i, \mathscr{G}_j) \text{ if } \mathscr{G}_i \bigcap \mathscr{G}_j \neq \emptyset, 1 \leq i, \neq j \leq n\} \end{aligned}$$

with labels

$$\begin{aligned} &L: \ \mathscr{G}_i \in V(G[\neg(\mathscr{G}; \circ)]) \to L(\mathscr{G}_i) = \mathscr{G}_i, \\ &L: \ (\mathscr{G}_i, \mathscr{G}_j) \in E(G[\neg(\mathscr{G}; \circ)]) \to \mathscr{G}_i \bigcap \mathscr{G}_j \text{ for integers } 1 \leq i \neq j \leq n. \end{aligned}$$

For example, let $\mathscr{G}_1 = \langle \alpha, \beta \rangle$, $\mathscr{G}_2 = \langle \alpha, \gamma, \theta \rangle$, $\mathscr{G}_3 = \langle \beta, \gamma \rangle$, $\mathscr{G}_4 = \langle \beta, \delta, \theta \rangle$ be 4 free Abelian groups with $\alpha \neq \beta \neq \gamma \neq \delta \neq \theta$. Calculation shows that $\mathscr{G}_1 \bigcap \mathscr{G}_2 = \langle \alpha \rangle$, $\mathscr{G}_2 \bigcap \mathscr{G}_3 = \langle \gamma \rangle$, $\mathscr{G}_3 \bigcap \mathscr{G}_4 = \langle \delta \rangle$, $\mathscr{G}_1 \bigcap \mathscr{G}_4 = \langle \beta \rangle$ and $\mathscr{G}_2 \bigcap \mathscr{G}_4 = \langle \theta \rangle$. Then, the topological graph $G[\neg(\mathscr{G}; \circ)]$ is shown in Fig.4.

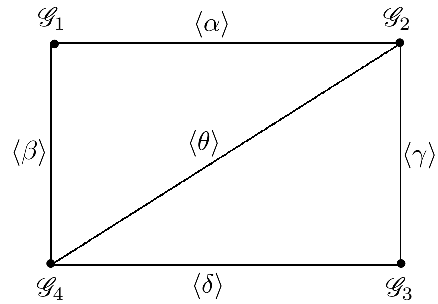


Fig.4

For an isomorphism $g: \mathscr{G} \rightarrow \mathscr{G}$ on $\neg(\mathscr{G};\circ)$, it naturally induces a 1-1 mapping $g^*: V(G[\neg(\mathscr{G};\circ)]) \rightarrow V(G[\neg(\mathscr{G};\circ)])$ such that each $g^*(\mathscr{G}_i)$ is also a group and $g^*(\mathscr{G}_i) \bigcap g^*(\mathscr{G}_j) \neq \emptyset$ if and only if $\mathscr{G}_i \bigcap \mathscr{G}_j \neq \emptyset$ for integers $1 \leq i \neq j \leq n$. Thus g induced an isomorphism g^* of graph from $G[\neg(\mathscr{G};\circ)]$ to $g^*(G[\neg(\mathscr{G};\circ)])$, which implies a conclusion following.

Theorem 2.3 *A non-group $\neg(\mathscr{G};\circ)$ in type AG_1^{-1}-AG_3^{-1} inherits an invariant $G[\neg(\mathscr{G};\circ)]$ of topological graph.*

Similarly, we can discuss more non-groups with some special properties, such as those of *non-Abelian group*, *non-solvable group*, *non-nilpotent group* and find inherited invariants $G[\neg(\mathscr{G};\circ)]$. Notice that([19]) any group $\mathscr{G}$ can be decomposed into disjoint classes $C(H_1)$, $C(H_2), \cdots, C(H_s)$ of conjugate subgroups, particularly, disjoint classes $Z(a_1)$, $Z(a_2), \cdots, Z(a_l)$ of centralizers with $|C(H_i)| = |\mathscr{G} : N_{\mathscr{G}}(H_i)|$, $|Z(a_j)| = |\mathscr{G} : Z_{\mathscr{G}}(a_j)|$, $1 \leq i \leq s$, $1 \leq j \leq l$ and $|C(H_1)| + |C(H_2)| + \cdots + |C(H_s)| = |\mathscr{G}|$, $|Z(a_1)| + |Z(a_2)| + \cdots + |Z(a_l)| = |\mathscr{G}|$, where $N_{\mathscr{G}}(H)$, $Z(a)$ denote respectively the normalizer of subgroup H and centralizer of element a in group $\mathscr{G}$. This fact enables one furthermore to construct topological structures of non-groups with special classes of groups following:

Replace a vertex $\mathscr{G}_i$ by s_i (or l_i) isolated vertices labeled with $C(H_1)$, $C(H_2), \cdots, C(H_{s_i})$ (or $Z(a_1)$, $Z(a_2), \cdots, Z(a_{l_i})$) in $G[\neg(\mathscr{G};\circ)]$ and denoted the resultant by $\widehat{G}[\neg(\mathscr{G};\circ)]$.

We then get results following on non-groups with special topological structures by Theorem 2.3.

Theorem 2.4 *A non-group $\neg(\mathscr{G};\circ)$ in type AG_1^{-1}-AG_3^{-1} inherits an invariant $\widehat{G}[\neg(\mathscr{G};\circ)]$ of topological graph labeled with conjugate classes of subgroups on its vertices.*

Theorem 2.5 *A non-group $\neg(\mathscr{G};\circ)$ in type AG_1^{-1}-AG_3^{-1} inherits an invariant $\widehat{G}[\neg(\mathscr{G};\circ)]$ of topological graph labeled with Abelian subgroups, particularly, with centralizers of elements in $\mathscr{G}$ on its vertices.*

Particularly, for a group the following is a readily conclusion of Theorems 2.4 and 2.5.

Corollary 2.6 *A group $(\mathscr{G};\circ)$ inherits an invariant $\widehat{G}[\mathscr{G};\circ]$ of topological graph labeled with conjugate classes of subgroups (or centralizers) on its vertices, with $E(\widehat{G}[\mathscr{G};\circ]) = \emptyset$*

2.3 Non-Rings

A ring is an associative algebraic system $(R;+,\circ)$ on 2 binary operations "+", "$\circ$", hold with an Abelian group $(R;+)$ and for $\forall x,y,z\in R$, $x\circ(y+z)=x\circ y+x\circ z$ and $(x+y)\circ z=x\circ z+y\circ z$. Denote the identity by 0_+, the inverse of a by $-a$ in $(R;+)$. A *non-ring* $\neg(R;+,\circ)$ on a ring $(R;+,\circ)$ is an algebraic system on operations "+", "$\circ$" in 5 types following:

$\mathbf{AR}_1^{-1}$: there maybe exist $a,b\in R$ such that $a+b\neq b+a$, but hold with the associative in $(R;\circ)$ and a group $(R;+)$;

$\mathbf{AR}_2^{-1}$: there maybe exist a_1,b_1,c_1 and $a_2,b_2,c_2\in R$ such that $(a_1\circ b_1)\circ c_1=a_1\circ(b_1\circ c_1)$, $(a_2\circ b_2)\circ c_2\neq a_2\circ(b_2\circ c_2)$, but holds with an Abelian group $(R;+)$.

$\mathbf{AR}_3^{-1}$: there maybe exist a_1,b_1,c_1 and $a_2,b_2,c_2\in R$ such that $(a_1+b_1)+c_1=a_1+(b_1+c_1)$, $(a_2+b_2)+c_2\neq a_2+(b_2+c_2)$, but holds with $(a\circ b)\circ c=a\circ(b\circ c)$, identity 0_+ and $-a$ in $(R;+)$ for $\forall a,b,c\in R$.

$\mathbf{AR}_4^{-1}$: there maybe exist distinct $0_+,0'_+\in R$ such that $a+0_+=0_++a=a$ and $b+0'_+=0'_++b=b$ for $a\neq b\in R$, but holds with the associative in $(R;+)$, $(R;\circ)$ and inverse elements $-a$ on 0_+, $0'_+$ in $(R;+)$ for $\forall a\in R$.

$\mathbf{AR}_5^{-1}$: there maybe exist distinct inverse elements $-a,-\dot{a}$ for $a\in R$ in $(R;+)$, but holds with the associative in $(R;+)$, $(R;\circ)$ and identity elements in $(R;+)$.

Notice that $(a+a)+a=a+(a+a)$, $a+a=a+a$ and $a\circ a=a\circ$ always hold in non-ring $\neg(R;+,\circ)$. Whence, for Types AR_1^{-1} and AR_2^{-1}, there exists a decomposition $R_1,R_2,\cdots,R_n$ of R such that $a+b=b+a$ and $(a\circ b)\circ c=a\circ(b\circ c)$ if $a,b,c\in R_i$, i.e., each $(R_i;+,\circ)$ is a ring for integers $1\leq i\leq n$. A similar discussion for Types AG_1^{-1}-AG_3^{-1} in Section 2.2 also shows such a decomposition $(R_i;+,\circ)$, $1\leq i\leq n$ of subrings exists for Types $3-5$. Define a topological graph $G[\neg(R;+,\circ)]$ by

$$V(G[\neg(R;+,\circ)])=\{R_1,R_2,\cdots,R_n\};$$
$$E(G[\neg(R;+,\circ)])=\{(R_i,R_j)\text{ if }R_i\bigcap R_j\neq\emptyset,1\leq i,\neq j\leq n\}$$

with labels

$$L:\ R_i\in V(G[\neg(R;+,\circ)])\to L(R_i)=R_i,$$
$$L:\ (R_i,R_j)\in E(G[\neg(R;+,\circ)])\to R_i\bigcap R_j\text{ for integers }1\leq i\neq j\leq n.$$

Then, such a topological graph $G[\neg(R;+,\circ)]$ is also an invariant under isomorphic actions on $\neg(R;+,\circ)$. Thus,

Theorem 2.7 *A non-ring $\neg(R;+,\circ)$ in types AR_1^{-1}-AR_5^{-1} inherits an invariant $G[\neg(R;+,\circ)]$ of topological graph.*

Furthermore, we can consider *non-associative ring*, *non-integral domain*, *non-division ring*, *skew non-field* or *non-field*, $\cdots$, etc. and find inherited invariants $G[\neg(R;+,\circ)]$ of graphs. For example, a *non-field* $\neg(F;+,\circ)$ on a field $(F;+,\circ)$ is an algebraic system on operations "+", "$\circ$" in 8 types following:

$\mathbf{AF}_1^{-1}$: there maybe exist a_1, b_1, c_1 and $a_2, b_2, c_2 \in F$ such that $(a_1 \circ b_1) \circ c_1 = a_1 \circ (b_1 \circ c_1)$, $(a_2 \circ b_2) \circ c_2 \neq a_2 \circ (b_2 \circ c_2)$, but holds with an Abelian group $(F; +)$, identity $1_\circ$, a^{-1} for $a \in F$ in $(F; \circ)$.

$\mathbf{AF}_2^{-1}$: there maybe exist a_1, b_1, c_1 and $a_2, b_2, c_2 \in F$ such that $(a_1+b_1)+c_1 = a_1+(b_1+c_1)$, $(a_2 + b_2) + c_2 \neq a_2 + (b_2 + c_2)$, but holds with an Abelian group $(F; \circ)$, identity 1_+, $-a$ for $a \in F$ in $(F; +)$.

$\mathbf{AF}_3^{-1}$: there maybe exist $a, b \in F$ such that $a \circ b \neq b \circ a$, but hold with an Abelian group $(F; +)$, a group $(F; \circ)$;

$\mathbf{AF}_4^{-1}$: there maybe exist $a, b \in F$ such that $a + b \neq b + a$, but hold with a group $(F; +)$, an Abelian group $(F; \circ)$;

$\mathbf{AF}_5^{-1}$: there maybe exist distinct $0_+, 0'_+ \in F$ such that $a + 0_+ = 0_+ + a = a$ and $b + 0'_+ = 0'_+ + b = b$ for $a \neq b \in F$, but holds with the associative, inverse elements $-a$ on 0_+, $0'_+$ in $(F; +)$ for $\forall a \in F$, an Abelian group $(F; \circ)$;

$\mathbf{AF}_6^{-1}$: there maybe exist distinct $1_\circ, 1'_\circ \in F$ such that $a \circ 1_\circ = 1_\circ \circ a = a$ and $b \circ 1'_\circ = 1'_\circ \circ b = b$ for $a \neq b \in F$, but holds with the associative, inverse elements a^{-1} on $1_\circ$, $1'_\circ$ in $(F; \circ)$ for $\forall a \in F$, an Abelian group $(F; +)$;

$\mathbf{AF}_7^{-1}$: there maybe exist distinct inverse elements $-a, -\dot{a}$ for $a \in F$ in $(F; +)$, but holds with the associative, identity elements in $(F; +)$, an Abelian group $(F; \circ)$.

$\mathbf{AF}_8^{-1}$: there maybe exist distinct inverse elements $a^{-1}, \dot{a}^{-1}$ for $a \in F$ in $(F; \circ)$, but holds with the associative, identity elements in $(F; \circ)$, an Abelian group $(F; +)$.

Similarly, we can show that there exists a decomposition $(F_i; +, \circ)$, $1 \leq i \leq n$ of fields for non-fields $\neg(F; +, \circ)$ in Types AF_1^{-1}-AF_8^{-1} and find an invariant $G[\neg(F; +, \circ)]$ of graph.

Theorem 2.8 *A non-ring $\neg(F; +, \circ)$ in types AF_1^{-1}-AF_8^{-1} inherits an invariant $G[\neg(F; +, \circ)]$ of topological graph.*

2.4 Algebraic Combinatorics

All of previous discussions with results in Sections 2.1-2.3 lead to a conclusion alluded in philosophy that *a non-algebraic system $\neg(\mathscr{A}; \mathcal{R})$ constraint with property can be decomposed into algebraic systems with the same constraints, and inherits an invariant $G[\neg(\mathscr{A}; \mathcal{R})]$ of topological graph labeled with those of algebraic systems*, i.e., algebraic combinatorics, which is in accordance with the notion for developing geometry that of Klein's. Thus, a more applicable approach for developing algebra is including non-algebra to algebra by consider various non-algebraic systems constraint with property, but such a process will never be ended if we do not firstly determine all algebraic systems. Even though, a more feasible approach is by its inverse, i.e., algebraic G-systems following:

Definition 2.9 *Let $(\mathscr{A}_1; \mathcal{R}_1), (\mathscr{A}_2; \mathcal{R}_2), \cdots, (\mathscr{A}_n; \mathcal{R}_n)$ be algebraic systems. An algebraic G-system is a topological graph G with labeling $L: v \in V(G) \to L(v) \in \{\mathscr{A}_1, \mathscr{A}_2, \cdots, \mathscr{A}_n\}$ and $L: (u, v) \in E(G) \to L(u) \bigcap L(v)$ with $L(u) \bigcap L(v) \neq \emptyset$, denoted by $G[\mathscr{A}, \mathcal{R}]$, where $\mathscr{A} = \bigcup\limits_{i=1}^{n} \mathscr{A}_i$*

and $\mathcal{R} = \bigcup_{i=1}^{n} \mathcal{R}_i$.

Clearly, if $G[\mathscr{A}, \mathcal{R}]$ is prescribed, these algebraic systems $(\mathscr{A}_1; \mathcal{R}_1), (\mathscr{A}_2; \mathcal{R}_2), \cdots, (\mathscr{A}_n; \mathcal{R}_n)$ with intersections are determined.

Problem 2.10 *Characterize algebraic G-systems $G[\mathscr{A}, \mathcal{R}]$, such as those of G-groups, G-rings, integral G-domain, skew G-fields, G-fields, $\cdots$, etc., or their combination $G - \{groups, rings\}$, $G - \{groups, integral\ domains\}$, $G - \{groups, fields\}$, $G - \{rings, fields\}$ $\cdots$. Particularly, characterize these G-algebraic systems for complete graphs $G = K_2, K_3, K_4$, path P_3, P_4 or circuit C_4 of order ≤ 4.*

In this perspective, classical algebraic systems are nothing else but mostly algebraic K_1-systems, also a few algebraic K_2-systems. For example, a field $(F; +, \cdot)$ is in fact a K_2-group prescribed by Fig.3.

§3. Algebraic Equations

All equations discussed in this paper are independent, maybe contain one or several unknowns, not an impossible equality in algebra, for instance $2^{x+y+z} = 0$.

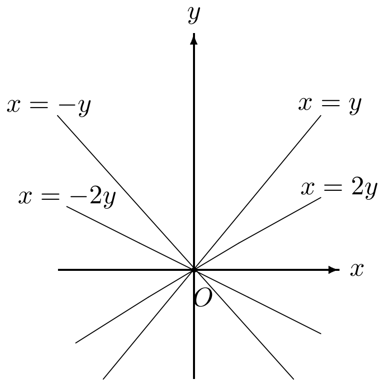
3.1 Geometry on Non-Solvable Equations

Let (LES_4^1), (LES_4^2) be two systems of linear equations following:

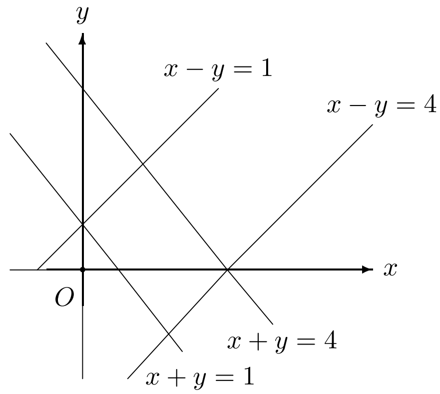
$$(LES_4^1) \begin{cases} x = y \\ x = -y \\ x = 2y \\ x = -2y \end{cases} \qquad (LES_4^2) \begin{cases} x + y = 1 \\ x + y = 4 \\ x - y = 1 \\ x - y = 4 \end{cases}$$

Clearly, the system (LES_4^1) is solvable with $x = 0, y = 0$ but (LES_4^2) is non-solvable because $x + y = 1$ is contradicts to that of $x + y = 4$ and so for $x - y = 1$ to $x - y = 4$. Even so, *is the system (LES_4^2) meaningless in the world*? Similarly, *is only the solution $x = 0,\ y = 0$ of system (LES_4^1) important to one*? Certainly NOT! This view can be readily come into being by all figures on $\mathbb{R}^2$ of these equations shown in Fig.5. Thus, if we denote by

$$\left\{ \begin{array}{l} L_1 = \{(x,y) \in \mathbb{R}^2 | x = y\} \\ L_2 = \{(x,y) \in \mathbb{R}^2 | x = -y\} \\ L_3 = \{(x,y) \in \mathbb{R}^2 | x = 2y\} \\ L_4 = \{(x,y) \in \mathbb{R}^2 | x = -2y\} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} L_1' = \{(x,y) \in \mathbb{R}^2 | x + y = 1\} \\ L_2' = \{(x,y) \in \mathbb{R}^2 | x + y = 4\} \\ L_3' = \{(x,y) \in \mathbb{R}^2 | x - y = 1\} \\ L_4' = \{(x,y) \in \mathbb{R}^2 | x - y = 4\} \end{array} \right\},$$



(LES_4^1)



(LES_4^2)

Fig.5

the global behavior of (LES_4^1), (LES_4^2) are lines $L1 - L_4$, lines $L_1' - L'4$ on $\mathbb{R}^2$ and

$$L_1 \bigcap L_2 \bigcap L_3 \bigcap L_4 = \{(0,0)\} \ \text{ but } \ L_1' \bigcap L_2' \bigcap L_3' \bigcap L_4' = \emptyset.$$

Generally, let

$$(ES_m) \quad \begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

be a system of algebraic equations in Euclidean space $\mathbb{R}^n$ for integers $m, n \geq 1$ with non-empty point set $S_{f_i} \subset \mathbb{R}^n$ such that $f_i(x_1, x_2, \cdots, x_n) = 0$ for $(x_1, x_2, \cdots, x_n) \in S_{f_i}$, $1 \leq i \leq m$. Clearly, the system (ES_m) is non-solvable or not dependent on

$$\bigcap_{i=1}^{m} S_{f_i} = \emptyset \ \text{ or } \ \neq \emptyset.$$

Conversely, let $\mathscr{G}$ be a geometrical space consisting of m parts $\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_m$ in $\mathbb{R}^n$, where, each $\mathscr{G}_i$ is determined by a system of algebraic equations

$$\begin{cases} f_1^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{m_i}^{[i]}(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

Then, the system of equations

$$\left.\begin{array}{l} f_1^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{m_i}^{[i]}(x_1, x_2, \cdots, x_n) = 0 \end{array}\right\} \ 1 \leq i \leq m$$

is non-solvable or not dependent on

$$\bigcap_{i=1}^{m} \mathscr{G}_i = \emptyset \quad \text{or} \quad \neq \emptyset.$$

Thus we obtain the following result.

Theorem 3.1 *The geometrical figure of equation system (ES_m) is a space $\mathscr{G}$ consisting of m parts $\mathscr{G}_i$ determined by equation $f_i(x_1, x_2, \cdots, x_n) = 0,\ 1 \leq i \leq m$ in (ES_m), and is non-solvable if $\bigcap\limits_{i=1}^{m} \mathscr{G}_i = \emptyset$. Conversely, if a geometrical space $\mathscr{G}$ consisting of m parts, $\mathscr{G}_1, \mathscr{G}_2, \cdots, \mathscr{G}_m$, each of them is determined by a system of algebraic equations in $\mathbb{R}^n$, then all of these equations consist a system (ES_m), which is non-solvable or not dependent on $\bigcap\limits_{i=1}^{m} \mathscr{G}_i = \emptyset$ or not.*

For example, let G be a planar graph with vertices v_1, v_2, v_3, v_4 and edges $v_1v_2, v_1v_3, v_2v_3, v_3v_4$, v_4v_1, shown in Fig.6.

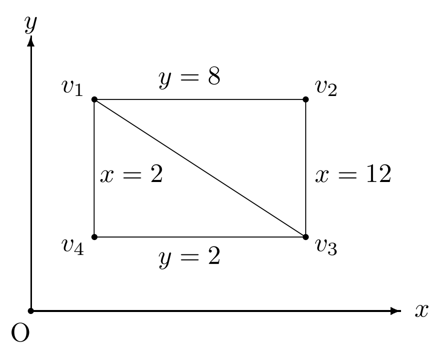


Fig.6

Then, a non-solvable system of equations with figure G on $\mathbb{R}^2$ consists of

$$(LE_5)\begin{cases} x = 2 \\ y = 8 \\ x = 12 \\ y = 2 \\ 3x + 5y = 46. \end{cases}$$

Thus G is an underlying graph of non-solvable system (LE_5).

Definition 3.2 *Let (ES_{m_i}) be a solvable system of m_i equations*

$$\begin{cases} f_1^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_n) = 0 \\ \dots\dots\dots\dots\dots\dots\dots \\ f_{m_i}^{[i]}(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

with a solution space $S_{f[i]}$ in $\mathbb{R}^n$ for integers $1 \leq i \leq m$. A topological graph $G[ES_m]$ is defined by

$$V(G[ES_m]) = \{S_{f[i]},\ 1 \leq i \leq m\};$$
$$E(G[ES_m]) = \{(S_{f[i]}, S_{f[j]}) \text{ if } S_{f[i]} \bigcap S_{f[j]} \neq \emptyset,\ 1 \leq i \neq j \leq m\}$$

with labels

$$L:\ S_{f[i]} \in V(G[ES_m])) \to L(S_{f[i]}) = S_{f[i]},$$
$$L:\ (S_{f[i]}, S_{f[j]}) \in E(G[ES_m])) \to S_{f[i]} \bigcap S_{f[j]} \ for\ integers\ 1 \leq i \neq j \leq m.$$

Applying Theorem 3.1, a conclusion following can be readily obtained.

Theorem 3.3 *A system (ES_m) consisting of equations in (ES_{m_i}), $1 \leq i \leq m$ is solvable if and only if $G[ES_m] \simeq K_m$ with $\emptyset \neq S \subset \bigcap\limits_{i=1}^{m} S_{f[i]}$. Otherwise, non-solvable, i.e., $G[ES_m] \not\simeq K_m$, or $G[ES_m] \simeq K_m$ but $\bigcap\limits_{i=1}^{m} S_{f[i]} = \emptyset$.*

Let $T:\ (x_1, x_2, \cdots, x_n) \to (x_1', x_2', \cdots, x_n')$ be linear transformation determined by an invertible matrix $[a_{ij}]_{n\times n}$, i.e., $x_i' = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$, $1 \leq i \leq n$ and let $T(S_{f[k]}) = S'_{f[k]}$ for integers $1 \leq k \leq m$. Clearly, $T:\ \{S_{f[i]},\ 1 \leq i \leq m\} \to \{S'_{f[i]},\ 1 \leq i \leq m\}$ and

$$S'_{f[i]} \bigcap S'_{f[j]} \neq \emptyset \ \text{ if and only if } \ S_{f[i]} \bigcap S_{f[j]} \neq \emptyset$$

for integers $1 \leq i \neq j \leq m$. Consequently, if $T:\ (ES_m) \leftarrow ('ES_m)$, then $G[ES_m] \simeq G['ES_m]$. Thus T induces an isomorphism T^* of graph from $G[ES_m]$ to $G['ES_m]$, which implies the following result:

Theorem 3.4 *A system (ES_m) of equations $f_i(\overline{x}) = 0, 1 \leq i \leq m$ inherits an invariant $G[ES_m]$ under the action of invertible linear transformations on $\mathbb{R}^n$.*

Theorem 3.4 enables one to introduce a definition following for algebraic system (ES_m) of equations, which expands the scope of algebraic equations.

Definition 3.5 *If $G[ES_m]$ is the topological graph of system (ES_m) consisting of equations in (ES_{m_i}) for integers $1 \leq i \leq m$, introduced in Definition 3.2, then $G[ES_m]$ is called a G-solution of system (ES_m).*

Thus, for developing the theory of algebraic equations, a central problem in front of one should be:

Problem 3.6 *For an equation system (ES_m), determine its G-solution $G[ES_m]$.*

For example, the solvable system (ES_m) in classical algebra is nothing else but a K_m-solution with $\bigcap\limits_{i=1}^{m} S_{f[i]} \neq \emptyset$, as claimed in Theorem 3.3. The readers are refereed to references [22] or [26] for more results on non-solvable equations.

3.2 Homogenous Equations

A system (ES_m) is homogenous if each of its equations $f_i(x_0, x_1, \cdots, x_n)$, $1 \leq i \leq m$ is homogenous, i.e.,

$$f_i(\lambda x_0, \lambda x_1, \cdots, \lambda x_n) = \lambda^d f_i(x_0, x_1, \cdots, x_n)$$

for a constant λ, denoted by (hES_m). For such a system, there are always existing a K_m-solution with $\{x_i = 0,\ 0 \leq i \leq n\} \subset \bigcap\limits_{i=1}^{m} S_{f[i]}$ and each $f_i(x_0, x_1, \cdots, x_n) = 0$ passes through $O = (\underbrace{0, 0, \cdots, 0}_{n+1})$ in $\mathbb{R}^n$. Clearly, an invertible linear transformation T action on such a K_m-solution is also a K_m-solution.

However, there are meaningless for such a K_m-solution in projective space $\mathbb{P}^n$ because $O \notin \mathbb{P}^n$. Thus, new invariants for such systems under projective transformations $(x_0', x_1', \cdots, x_n') = [a_{ij}]_{(n+1)\times(n+1)} (x_0, x_1, \cdots, x_n)$ should be found, where $[a_{ij}]_{(n+1)\times(n+1)}$ is invertible. In $\mathbb{R}^2$, two lines $P(x,y), Q(x,y)$ are *parallel* if they are not intersect. But in $\mathbb{P}^2$, this parallelism will never appears because the Bézout's theorem claims that any two curves $P(x,y,z)$, $Q(x,y,z)$ of degrees m, n without common components intersect precisely in mn points. However, denoted by $I(P,Q)$ the set of intersections of homogenous polynomials $P(\overline{x})$ with $Q(\overline{x})$ with $\overline{x} = (x_0, x_1, \cdots, x_n)$. The parallelism in $\mathbb{R}^n$ can be extended to $\mathbb{P}^n$ following, which enables one to find invariants on systems homogenous equations.

Definition 3.7 *Let $P(\overline{x}), Q(\overline{x})$ be two complex homogenous polynomials of degree d with $\overline{x} = (x_0, x_1, \cdots, x_n)$. They are said to be parallel, denoted by $P \parallel Q$ if $d \geq 1$ and there are constants $a, b, \cdots, c$ (not all zero) such that for $\forall \overline{x} \in I(P,Q)$, $ax_0 + bx_1 + \cdots + cx_n = 0$, i.e., all intersections of $P(\overline{x})$ with $Q(\overline{x})$ appear at a hyperplane on $\mathbb{P}^n\mathbf{C}$, or $d = 1$ with all intersections at the infinite $x_n = 0$. Otherwise, $P(\overline{x})$ are not parallel to $Q(\overline{x})$, denoted by $P \nparallel Q$.*

Definition 3.8 *Let $P_1(\overline{x}) = 0, P_2(\overline{x}) = 0, \cdots, P_m(\overline{x}) = 0$ be homogenous equations in (hES_m). Define a topological graph $G[hES_m]$ in $\mathbb{P}^n$ by*

$$V(G[hES_m]) = \{P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})\};$$
$$E(G[hES_m]) = \{(P_i(\overline{x}), P_j(\overline{x})) | P_i \nparallel P_j,\ 1 \leq i, j \leq m\}$$

with a labeling

$$L:\ P_i(\overline{x}) \to P_i(\overline{x}), \quad (P_i(\overline{x}), P_j(\overline{x})) \to I(P_i, P_j),\ \textit{where}\ 1 \leq i \neq j \leq m.$$

For any system (hES_m) of homogenous equations, $G[hES_m]$ is an indeed invariant under the action of invertible linear transformations T on $\mathbb{P}^n$. By definition in [6], a *covariant* $C(a_{\overline{k}}, \overline{x})$ on homogenous polynomials $P(\overline{x})$ is a polynomial function of coefficients $a_{\overline{k}}$ and variables $\overline{x}$. We furthermore find a topological invariant on covariants following.

Theorem 3.9 *Let (hES_m) be a system consisting of covariants $C_i(a_{\overline{k}}, \overline{x})$ on homogenous polynomials $P_i(\overline{x})$ for integers $1 \leq i \leq m$. Then, the graph $G[hES_m]$ is a covariant under the action of invertible linear transformations T, i.e., for $\forall C_i(a_{\overline{k}}, \overline{x}) \in (ES_m)$, there is $C_{i'}(a_{\overline{k}}, \overline{x}) \in (ES_m)$ with*

$$C_{i'}(a_{\overline{k}}', \overline{x}') = \Delta^p C_i(a_{\overline{k}}, \overline{x})$$

holds for integers $1 \leq i \leq m$*, where* p *is a constant and* Δ *is the determinant of* T*.*

Proof Let $G^T[hES_m]$ be the topological graph on transformed system $T(hES_m)$ defined in Definition 3.8. We show that the invertible linear transformation T naturally induces an isomorphism between graphs $G[hES_m]$ and $G^T[hES_m]$. In fact, T naturally induces a mapping $T^*: G[hES_m] \to G^T[hES_m]$ on $\mathbb{P}^n$. Clearly, $T^*: V(G[hES_m]) \to V(G^T[hES_m])$ is $1-1$, also onto by definition. In projective space $\mathbb{P}^n$, a line is transferred to a line by an invertible linear transformation. Therefore, $C_u^T \parallel C_v^T$ in $T(ES_m)$ if and only if $C_u \parallel C_v$ in (hES_m), which implies that $(C_u^T, C_v^T) \in E(G^T[ES_m])$ if and only if $(C_u, C_v) \in E(G[hES_m])$. Thus, $G[hES_m] \simeq G^T[hES_m]$ with an isomorphism T^* of graph.

Notice that $I\left(C_u^T, C_v^T\right) = T\left(I(C_u, C_v)\right)$ for $\forall (C_u, C_v) \in E(G[hES_m])$. Consequently, the induced mapping

$$T^*: V(G[hES_m]) \to V(G^T[hES_m]), \qquad E(G[hES_m]) \to E(G^T[hES_m])$$

is commutative with that of labeling L, i.e., $T^* \circ L = L \circ T^*$. Thus, T^* is an isomorphism from topological graph $G[hES_m]$ to $G^T[hES_m]$. □

Particularly, let $p = 0$, i.e., (ES_m) consisting of homogenous polynomials $P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})$ in Theorem 3.9. Then we get a result on systems of homogenous equations following.

Corollary 3.10 *A system* (hES_m) *of homogenous equations* $f_i(\overline{x}) = 0, 1 \leq i \leq m$ *inherits an invariant* $G[hES_m]$ *under the action of invertible linear transformations on* $\mathbb{P}^n$*.*

Thus, for homogenous equation systems (hES_m), the *G-solution* in Problem 3.6 should be substituted by $G[hES_m]$*-solution.*

§4. Differential Equations

4.1 Non-Solvable Ordinary Differential Equations

For integers $m,\ n \geq 1$, let

$$\dot{X} = F_i(X),\ 1 \leq i \leq m \qquad (DES_m^1)$$

be a differential equation system with continuous $F_i: \mathbf{R}^n \to \mathbf{R}^n$, $\dot{X} = \dfrac{dX}{dt}$ such that $F_i(\overline{0}) = \overline{0}$, particularly, let

$$\dot{X} = A_1X, \cdots, \dot{X} = A_kX, \cdots, \dot{X} = A_mX \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order with

$$\dot{X} = (\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n)^t = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt}\right)$$

and

$$\begin{cases} x^{(n)} + a_{11}^{[0]}x^{(n-1)} + \cdots + a_{1n}^{[0]}x = 0 \\ x^{(n)} + a_{21}^{[0]}x^{(n-1)} + \cdots + a_{2n}^{[0]}x = 0 \\ \cdots\cdots\cdots\cdots \\ x^{(n)} + a_{m1}^{[0]}x^{(n-1)} + \cdots + a_{mn}^{[0]}x = 0 \end{cases} \qquad (LDE_m^n)$$

a linear differential equation system of order n with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix},$$

where, $x^{(n)} = \dfrac{d^n x}{dt^n}$, all $a_{ij}^{[k]}$, $0 \leq k \leq m,\ 1 \leq i, j \leq n$ are numbers. Such a system (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) are called *non-solvable* if there are no function $X(t)$ (or $x(t)$) hold with (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) unless constants. For example, the following differential equation system

$$(LDE_6^2)\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

is a non-solvable system.

According to theory of ordinary differential equations ([32]), any linear differential equation system $(LDES_1^1)$ of first order in $(LDES_m^1)$ or any differential equation (LDE_1^n) of order n with complex coefficients in (LDE_m^n) are solvable with a solution basis $\mathscr{B} = \{\ \overline{\beta}_i(t) |\ 1 \leq i \leq n\}$ such that all general solutions are linear generated by elements in $\mathscr{B}$.

Denoted the solution basis of systems (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) of ordinary differential equations by $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_m$ and define a topological graph $G[DES_m^1]$ or $G[LDES_m^1]$ (or $G[LDE_m^n]$) in $\mathbb{R}^n$ by

$$\begin{aligned} V(G[DES_m^1]) &= V(G[LDES_m^1]) = V(G[LDE_m^n]) = \{\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_m\}; \\ E(G[DES_m^1]) &= E(G[LDES_m^1]) = E(G[LDE_m^n]) \\ &\qquad = \{(\mathscr{B}_i, \mathscr{B}_j) \ \text{ if } \ \mathscr{B}_i \bigcap \mathscr{B}_j \neq \emptyset,\ 1 \leq i, j \leq m\} \end{aligned}$$

with a labeling

$$L: \ \mathscr{B}_i \to \mathscr{B}_i, \qquad (\mathscr{B}_i, \mathscr{B}_j) \to \mathscr{B}_i \bigcap \mathscr{B}_j \text{ for } 1 \leq i \neq j \leq m.$$

Let T be a linear transformation on $\mathbb{R}^n$ determined by an invertible matrix $[a_{ij}]_{n\times n}$. Let

$$T: \ \{\mathscr{B}_i,\ 1 \leq i \leq m\} \to \{\mathscr{B}_i',\ 1 \leq i \leq m\}.$$

It is clear that $\mathscr{B}_i'$ is the solution basis of the ith transformed equation in (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)), and $\mathscr{B}_i' \bigcap \mathscr{B}_j' \neq \emptyset$ if and only if $\mathscr{B}_i \bigcap \mathscr{B}_j \neq \emptyset$. Thus T naturally induces an isomorphism T^* of graph with $T^* \circ L = L \circ T^*$ on labeling L.

Theorem 4.1 *A system (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) of ordinary differential equations inherits an invariant $G[DES_m^1]$ or $G[LDES_m^1]$ (or $G[LDE_m^n]$) under the action of invertible linear transformations on $\mathbb{R}^n$.*

Clearly, if the topological graph $G[DES_m^1]$ or $G[LDES_m^1]$ (or $G[LDE_m^n]$) are determined, the global behavior of solutions of systems (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)) in $\mathbb{R}^n$ are readily known. Such graphs are called respectively $G[DES_m^1]$*-solution* or $G[LDES_m^1]$*-solution* (or $G[LDE_m^n]$*-solution*) of systems of (DES_m^1) or $(LDES_m^1)$ (or (LDE_m^n)). Thus, for developing ordinary differential equation theory, an interesting problem should be:

Problem 4.2 *For a system of (DES_m^1) (or $(LDES_m^1)$, or (LDE_m^n)) of ordinary differential equations, determine its $G[DES_m^1]$-solution (or $G[LDES_m^1]$-solution, or $G[LDE_m^n]$-solution).*

For example, the topological graph $G[LDE_6^2]$ of system (LDE_6^2) of linear differential equation of order 2 in previous is shown in Fig.7.

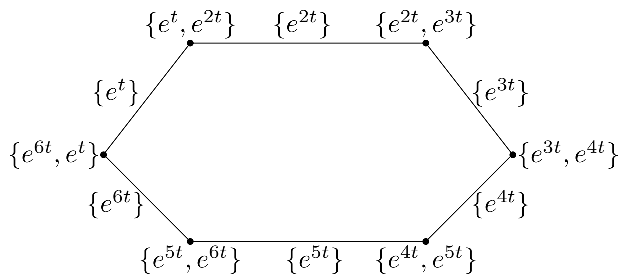


Fig.7

4.2 Non-Solvable Partial Differential Equations

Let $L_1, L_2, \cdots, L_m$ be m partial differential operators of first order (linear or non-linear) with

$$L_k = \sum_{i=1}^{n} a_{ki} \frac{\partial}{\partial x_i}, \ 1 \leq k \leq m.$$

Then the system of partial differential equations

$$L_i[u(x_1, x_2, \cdots, x_n)] = h_i, \qquad 1 \leq i \leq m, \qquad (PDES_m)$$

or the Cauchy problem

$$\begin{cases} L_i[u] = h_i \\ u(x_1, x_2, \cdots, x_{n-1}, x_n^0) = \varpi_i, \quad 1 \leq i \leq m \end{cases} \qquad (PDES_m^C)$$

is *non-solvable* if there are no function $u(x_1, \cdots, x_n)$ on a domain $D \subset \mathbb{R}^n$ with $(PDES_m)$ or $(PDES_m^C)$ holds, where h_i, $1 \leq i \leq m$ and ϖ_i $1 \leq i \leq m$ are all continuous functions on $D \subset \mathbb{R}^n$.

Clearly, the ith partial differential equation is solvable [3]. Denoted by S_i^0 the solution of ith equation in $(PDES_m)$ or $(DEPS_m^C)$. Then the system $(PDES_m)$ or $(DEPS_m^C)$ of partial differential equations is solvable only if $\bigcap\limits_{i=1}^{m} S_i^0 \neq \emptyset$. Because $u : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, so the $(PDES_m)$ or $(DEPS_m^C)$ is solvable only if $\bigcap\limits_{i=1}^{m} S_i^0$ is a non-empty functional set on a domain $D \subset \mathbb{R}^n$. Otherwise, non-solvable, i.e., $\bigcap\limits_{i=1}^{m} S_i^0 = \emptyset$ for any domain $D \subset \mathbb{R}^n$.

Define a topological graph $G[PDES_m]$ or $G[DEPS_m^C]$ in $\mathbb{R}^n$ by

$$\begin{aligned}
V(G[PDES_m]) &= V(G[DEPS_m^C]) = \{S_i^0,\ 1 \leq i \leq m\};\\
E(G[PDES_m]) &= E(G[DEPS_m^C])\\
&= \{(S_i^0, S_j^0)\ \text{ if }\ S_i^0 \bigcap S_j^0 \neq \emptyset,\ 1 \leq i, j \leq m\}
\end{aligned}$$

with a labeling

$$L:\ S_i^0 \to S_i^0, \qquad (S_i^0, S_j^0) \in E(G[PDES_m]) = E(G[DEPS_m^C]) \to S_i^0 \bigcap S_j^0$$

for $1 \leq i \neq j \leq m$. Similarly, if T is an invertible linear transformation on $\mathbb{R}^n$, then $T(S_i^0)$ is the solution of ith transformed equation in $(PDES_m)$ or $(DEPS_m^C)$, and $T(S_i^0) \bigcap T(S_j^0) \neq \emptyset$ if and only if $S_i^0 \bigcap S_j^0 \neq \emptyset$. Accordingly, T induces an isomorphism T^* of graph with $T^* \circ L = L \circ T^*$ holds on labeling L. We get the following result.

Theorem 4.3 *A system $(PDES_m)$ or $(DEPS_m^C)$ of partial differential equations of first order inherits an invariant $G[PDES_m]$ or $G[DEPS_m^C]$ under the action of invertible linear transformations on $\mathbb{R}^n$.*

Such a topological graph $G[PDES_m]$ or $G[DEPS_m^C]$ are said to be the $G[PDES_m]$-*solution* or $G[DEPS_m^C]$-*solution* of systems $(PDES_m)$ and $(DEPS_m^C)$, respectively. For example, the $G[DEPS_3^C]$-solution of Cauchy problem

$$\begin{cases} u_t + au_x = 0 \\ u_t + xu_x = 0 \\ u_t + au_x + e^t = 0 \\ u|_{t=0} = \phi(x) \end{cases} \qquad (DEPS_3^C)$$

is shown in Fig.8

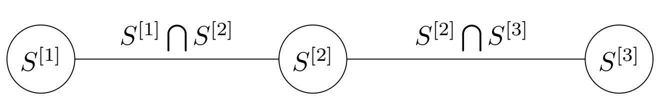


Fig.8

Clearly, system $(DEPS_3^C)$ is contradictory because $e^t \neq 0$ for t. However,

$$\begin{cases} u_t + au_x = 0 \\ u|_{t=0} = \phi(x) \end{cases} \quad \begin{cases} u_t + xu_x = 0 \\ u|_{t=0} = \phi(x) \end{cases} \quad \text{and} \quad \begin{cases} u_t + au_x + e^t = 0 \\ u|_{t=0} = \phi(x) \end{cases}$$

are solvable with respective solutions $S^{[1]} = \{\phi(x - at)\}$, $S^{[2]} = \{\phi(\frac{x}{e^t})\}$ and $S^{[3]} = \{\phi(x - at) - e^t + 1\}$, and $S^{[1]} \bigcap S^{[2]} = \{\phi(x - at) = \phi(\frac{x}{e^t})\}$, $S^{[2]} \bigcap S^{[3]} = \{\phi(\frac{x}{e^t}) = \phi(x - at) - e^t + 1\}$, but $S^{[1]} \bigcap S^{[3]} = \emptyset$.

Similar to ordinary case, an interesting problem on partial differential equations is the following:

Problem 4.4 *For a system of $(PDES_m)$ or $(DEPS_m^C)$ of partial differential equations, determine its $G[PDES_m]$-solution or $G[DEPS_m^C]$-solution.*

It should be noted that for an algebraically contradictory linear system

$$\begin{cases} F_i(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0 \\ F_j(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0, \end{cases}$$

if

$$F_k(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0$$

is contradictory to one of there two partial differential equations, then it must be contradictory to another. This fact enables one to classify equations in $(LPDES_m)$ by the contradictory property and determine $G[LPDES_m^C]$. Thus if $\mathscr{C}_1, \cdots, \mathscr{C}_l$ are maximal contradictory classes for equations in $(LPDES)$, then $G[LPDES_m^C] \simeq K(\mathscr{C}_1, \cdots, \mathscr{C}_l)$, i.e., an l-partite complete graph. Accordingly, all $G[LPDES_m^C]$-solutions of linear systems $(LPDES_m)$ are nothing else but $K(\mathscr{C}_1, \cdots, \mathscr{C}_s)$-solutions. More behaviors on non-solvable ordinary or partial differential equations of first order, for instance the global stability can be found in references [25]-[27].

4.3 Equation's Combinatorics

All these discussions in Sections 3 and $4.2 - 4.3$ lead to a conclusion that *a non-solvable system (ES) of equations in n variables inherits an invariant $G[ES]$ of topological graph labeled with those of individually solutions, if it is individually solvable*, i.e., equation's combinatorics by view it with the topological graph $G[ES]$ in $\mathbb{R}^n$. Thus, for holding the global behavior of a system (ES) of equations, the right way is not just to determine it is solvable or not, but its $G[ES]$-solution. Such a $G[ES]$-solution is existent by philosophy and enables one to include non-solvable equations, no matter what they are algebraic, differential, integral or operator equations to mathematics by G-system following:

Definition 4.5 *A G-system (ES_m) of equations $O_i(\overline{X}) = \overline{0}$, $1 \leq i \leq m$ with constraints $\mathcal{C}$ is a topological graph G with labeling $L: \ v \in V(G) \to L(v) \in \{S_{O_i};\ 1 \leq i \leq m\}$ and $L: \ (u, v) \in E(G) \to L(u) \bigcap L(v)$ with $L(u) \bigcap L(v) \neq \emptyset$, denoted by $G[ES_m]$, where, S_{O_i} is the solution space of equation $O_i(\overline{X}) = \overline{0}$ with constraints $\mathcal{C}$ for integers $1 \leq i \leq m$.*

Thus, holding the true face of a thing T characterized by a system (ES_m) of equations needs one to determine its G-system, i.e., $G[ES_m]$-solution, not only solvable or not for its objective reality.

Problem 4.6 *Determine $G[ES_m]$ for equation systems (ES_m), such as those of algebraic,* *differential, integral, operator equations, or their combination, or conversely, characterize G-systems of equations for given graphs G, foe example, these G-systems of equations for complete graphs $G = K_m$, complete bipartite graph $K(n_1, n_2)$ with $n_1 + n_2 = m$, path P_{m-1} or circuit C_m.*

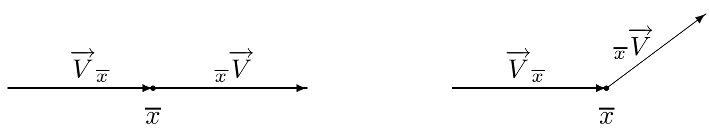
By this view, a solvable system (ES_m) of equations in classical mathematics is nothing else but such a K_m-system with $\bigcap\limits_{e \in E(K_m)} L(e) \neq \emptyset$. However, as we known, more systems of equations established on characters $\mu_i,\ 1 \leq i \leq n$ for a thing T are non-solvable with contradictions if $n \geq 2$. It is nearly impossible to solve all those systems in classical mathematics. Even so, its G-systems reveals behaviors of thing T to human beings.

§5. Geometry

As what one sees with an immediately form on things, the geometry proves to be one of applicable means for portraying things by its homogeneity with distinction. Nevertheless, the non-geometry can also contributes describing things complying with the Erlangen Programme that of Klein.

5.1 Non-Spaces

Let $\mathscr{K}^n = \{(x_1, x_2, \cdots, x_n)\}$ be an n-dimensional Euclidean (affine or projective) space with a normal basis $\overline{\epsilon}_i,\ 1 \leq i \leq n$, $\overline{x} \in \mathscr{K}^n$ and let $\overrightarrow{V}_{\overline{x}}$, ${}_{\overline{x}}\overrightarrow{V}$ be two orientation vectors with end or initial point at $\overline{x}$. Such as those shown in Fig.9.



(a) (b)

Fig.9

For point $\forall \overline{x} \in \mathscr{K}^n$, we associate it with an invertible linear mapping

$$\mu : \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\} \rightarrow \{\overline{\epsilon}'_1, \overline{\epsilon}'_2, \cdots, \overline{\epsilon}'_n\}$$

such that $\mu(\overline{\epsilon}_i) = \overline{\epsilon}'_i,\ 1 \leq i \leq n$, called its *weight*, i.e.,

$$(\overline{\epsilon}'_1, \overline{\epsilon}'_2, \cdots, \overline{\epsilon}'_n) = [a_{ij}]_{n \times n} (\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n)^t$$

where, $[a_{ij}]_{n \times n}$ is an invertible matrix. Such a space is a weighted space on points in $\mathscr{K}^n$, denoted by $(\mathscr{K}^n, \mu)$ with $\mu :\ \overline{x} \rightarrow \mu(\overline{x}) = [a_{ij}]_{n \times n}$. Clearly, if $\mu(\overline{x}_1) = \left[a'_{ij}\right]$, $\mu(\overline{x}_2) = \left[a''_{ij}\right]$, then $\mu(\overline{x}_1) = \mu(\overline{x}_2)$ if and only if there exists a constant λ such that $\left[a'_{ij}\right]_{n \times n} = \left[\lambda a''_{ij}\right]_{n \times n}$, and $(\mathscr{K}^n, \mu) = \mathbb{R}^n$ ($\mathbb{A}^n$ or $\mathbb{P}^n$), i.e., n-dimensional *Euclidean* (*affine* or *projective space*) if and only if $[a_{ij}]_{n \times n} = I_{n \times n}$ for $\forall \overline{x} \in \mathscr{K}^n$. Otherwise, *non-Euclidean, non-affine or non-projective space*, abbreviated to *non-space*.

Notice that $\left[a'_{ij}\right]_{n\times n} = \left[\lambda a''_{ij}\right]_{n\times n}$ is an equivalent relation on invertible $n\times n$ matrixes. Thus, for $\forall \overline{x}_0 \in \mathscr{K}^n$, define

$$\mathscr{C}(\overline{x}_0) = \{\overline{x} \in \mathscr{K}^n | \mu(\overline{x}) = \lambda\mu(\overline{x}_0),\ \lambda \in \mathbb{R}\},$$

an *equivalent set* of points to $\overline{x}_0$. Then there exist representatives $\mathscr{C}_\kappa, \kappa \in \Lambda$ constituting a partition of $\mathscr{K}^n$ in all equivalent sets $\mathscr{C}(\overline{x}), \overline{x} \in \mathscr{K}^n$ of points, i.e.,

$$\mathscr{K}^n = \bigcup_{\kappa\in\Lambda} \mathscr{C}_\kappa \quad \text{with} \quad \mathscr{C}_{\kappa_1} \bigcap \mathscr{C}_{\kappa_2} = \emptyset \ \text{ for } \ \kappa_1, \kappa_2 \in \Lambda \ \text{ if } \ \kappa_1 \neq \kappa_2,$$

where Λ maybe countable or uncountable.

Let $\mu(\overline{x}) = [a_{ij}]_{n\times n} = A_\kappa$ for $\overline{x} \in \mathscr{C}_\kappa$. For viewing behaviors of orientation vectors in an equivalent set $\mathscr{C}_\kappa$ of points, define $\mu_{A_\kappa}:\ \mathscr{K}^n \to \mu_{A_\kappa}(\mathscr{K}^n)$ by $\mu_{A_\kappa}(\overline{x}) = A_\kappa$. Then $(\mathscr{K}^n, \mu_{A_\kappa})$ is also a non-space if $A_\kappa \neq I_{n\times n}$. However, $(\mathscr{K}^n, \mu_{A_\kappa})$ approximates to $\mathscr{K}^n$ with homogeneity because each orientation vector only turns a same direction passing through a point. Thus, $(\mathscr{K}^n, \mu_{A_\kappa})$ can be viewed as space $\mathscr{K}^n$, denoted by $\mathscr{K}^n_{\mu_A}$. Define a topological graph $G[\mathscr{K}^n, \mu]$ by

$$V(G[\mathscr{K}^n, \mu]) = \{\mathscr{K}^n_{\mu_\kappa},\ \ \kappa \in \Lambda\};$$
$$E(G[\mathscr{K}^n, \mu]) = \{(\mathscr{K}^n_{\mu_{\kappa_1}}, \mathscr{K}^n_{\mu_{\kappa_2}}) \text{ if } \mathscr{K}^n_{\mu_{\kappa_1}} \bigcap \mathscr{K}^n_{\mu_{\kappa_2}} \neq \emptyset,\ \kappa_1, \kappa_2 \in \Lambda, \kappa_1 \neq \kappa_2\}$$

with labels

$$L:\ \mathscr{K}^n_{\mu_\kappa} \in V(G[\mathscr{K}^n, \mu]) \to \mathscr{K}^n_{\mu_\kappa},$$
$$L:\ (\mathscr{K}^n_{\mu_{\kappa_1}}, \mathscr{K}^n_{\mu_{\kappa_2}}) \in E(G[\mathscr{K}^n, \mu]) \to \mathscr{K}^n_{\mu_{\kappa_1}} \bigcap \mathscr{K}^n_{\mu_{\kappa_2}}, \ \ \kappa_1 \neq \kappa_2 \in \Lambda.$$

Then, we get an overview on $(\mathscr{K}^n, \mu)$ with Euclidean spaces $\mathscr{K}^n_{\mu_\kappa}, \kappa \in \Lambda$ by combinatorics. Clearly, $\mathscr{K}^n \bigcap \mathscr{K}^n_{\mu_\kappa} = \mathscr{C}_\kappa$ and $\mathscr{K}^n_{\mu_{\kappa_1}} \bigcap \mathscr{K}^n_{\mu_{\kappa_2}} = \emptyset$ if none of $\mathscr{K}^n_{\mu_{\kappa_1}}, \mathscr{K}^n_{\mu_{\kappa_2}}$ being $\mathscr{K}^n$. Thus, $G[\mathscr{K}^n, \mu] \simeq K_{1,|\Lambda|-1}$, a star with center $\mathscr{K}^n$, such as those shown in Fig.10. Otherwise, $G[\mathscr{K}^n, \mu] \simeq \overline{K}_{|\Lambda|}$, i.e., $|\Lambda|$ isolated vertices, which can be turned into $K_{1,|\Lambda|}$ by adding an imaginary center vertex $\mathscr{K}^n$.

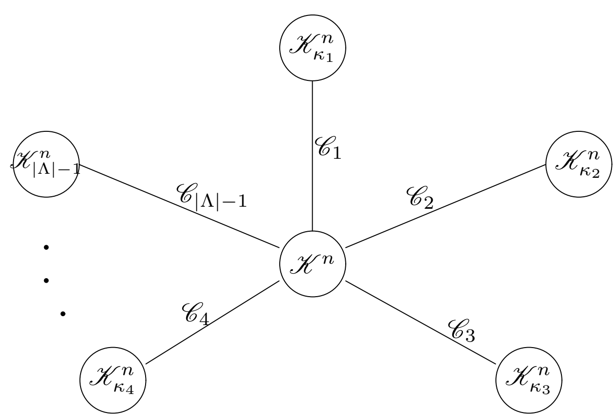


Fig.10

Let T be an invertible linear transformation on $\mathscr{K}^n$ determined by $(\overline{x}') = [\alpha_{ij}]_{n\times n}(\overline{x})^t$. Clearly, $T:\ \mathscr{K}^n \to \mathscr{K}^n,\ \ \mathscr{K}^n_{\mu_\kappa} \to T(\mathscr{K}^n_\kappa)$ and $T(\mathscr{K}^n_{\kappa_1})\bigcap T(\mathscr{K}^n_{\kappa_2}) \neq \emptyset$ if and only if $\mathscr{K}^n_{\kappa_1}\bigcap\mathscr{K}^n_{\kappa_2} \neq \emptyset$. Furthermore, one of $T(\mathscr{K}^n_{\kappa_1})$, $T(\mathscr{K}^n_{\kappa_2})$ should be $\mathscr{K}^n$. Thus T induces an isomorphism T^* from $G[\mathscr{K}^n,\mu]$ to $G[T(\mathscr{K}^n),\mu]$ of graph. Accordingly, we know the result following.

Theorem 5.1 *An n-dimensional non-space $(\mathscr{K}^n,\mu)$ inherits an invariant $G[\mathscr{K}^n,\mu]$, i.e., a star $K_{1,|\Lambda|-1}$ or $K_{1,|\Lambda|}$ under the action of invertible linear transformations on $\mathbb{K}^n$, where Λ is an index set such that all equivalent sets $\mathscr{C}_\kappa, \kappa\in\Lambda$ constitute a partition of space $\mathscr{K}^n$.*

5.2 Non-Manifolds

Let M be an n-dimensional manifold with an alta $\mathscr{A} = \{\ (U_\lambda;\varphi_\lambda) \mid \lambda\in\Lambda\}$, where $\varphi_\lambda:\ U_\lambda \to \mathbb{R}^n$ is a homeomorphism with countable Λ. A *non-manifold* $\neg M$ on M is such a topological space with $\varphi:\ U_\lambda \to \mathbb{R}^{n_\lambda}$ for integers $n_\lambda \geq 1,\ \lambda\in\Lambda$, which is a special but more applicable case of non-space $(\mathbb{R}^n,\mu)$. Clearly, if $n_\lambda = n$ for $\lambda\in\Lambda$, $\neg M$ is nothing else but an n-manifold.

For an n-manifold M, each U_λ is itself an n-manifold for $\lambda\in\Lambda$ by definition. Generally, let M_λ be an n_λ-manifold with an alta $\mathscr{A}_\lambda = \{\ (U_{\lambda\kappa};\varphi_{\lambda\kappa}) \mid \kappa\in\Lambda_\lambda\}$, where $\varphi_{\lambda\kappa}:\ U_{\lambda\kappa}\to\mathbb{R}^{n_\lambda}$. A *combinatorial manifold* $\widetilde{M}$ on M is such a topological space constituted by $M_\lambda,\ \lambda\in\Lambda$. Clearly, $\bigcup\limits_{\lambda\in\Lambda}\Lambda_\lambda$ is countable. If $n_\lambda = n$, i.e., all M_λ is an n-manifold for $\lambda\in\Lambda$, then the union $\mathscr{M}$ of $M_\lambda,\ \lambda\in\Lambda$ is also an n-manifold with alta

$$\widetilde{\mathscr{A}} = \bigcup_{\lambda\in\Lambda}\mathscr{A}_\lambda = \{(U_{\lambda\kappa};\varphi_{\lambda\kappa}) \mid \kappa\in\Lambda_\lambda,\ \lambda\in\Lambda\}.$$

Theorem 5.2 *A combinatorial manifold $\widetilde{M}$ is a non-manifold on $\mathscr{M}$, i.e.,*

$$\widetilde{M} = \neg\mathscr{M}.$$

Accordingly, we only discuss non-manifolds $\neg M$. Define a topological graph $G[\neg M]$ by

$$V(G[\neg M]) = \{U_\lambda,\ \ \lambda\in\Lambda\};$$
$$E(G[\neg M]) = \{(U_{\lambda_1},U_{\lambda_2}) \text{ if } U_{\lambda_1}\bigcap U_{\lambda_2}\neq\emptyset,\ \lambda_1,\lambda_2\in\Lambda, \lambda_1\neq\lambda_2\}$$

with labels

$$L:\ U_\lambda\in V(G[\neg M]) \to U_\lambda,$$
$$L:\ (U_{\lambda_1},U_{\lambda_2})\in E(\neg M]) \to U_{\lambda_1}\bigcap U_{\lambda_2},\quad \lambda_1\neq\lambda_2\in\Lambda,$$

which is an invariant dependent only on alta $\mathscr{A}$ of M.

Particularly, if each U_λ is a Euclidean spaces $\mathbb{R}^\lambda,\ \lambda\in\Lambda$, we get another topological graph $G[\mathbb{R}^\lambda,\lambda\in\Lambda]$ on Euclidean spaces $\mathbb{R}^\lambda,\ \lambda\in\Lambda$, a special non-manifold called *combinatorial Euclidean space*. The following result on $\neg M$ is easily obtained likewise the proof of Theorem 2.1 in [23].

Theorem 5.3 *A non-manifold $\neg M$ on manifold M with alta $\mathscr{A} = \{\ (U_\lambda;\varphi_\lambda) \mid \lambda\in\Lambda\}$ inherits an topological invariant $G[\neg M]$. Furthermore, if M is locally compact, $G[\neg M]$ is topological homeomorphic to $G[\mathbb{R}^\lambda, \lambda\in\Lambda]$ if $\varphi:\ U_\lambda\to\mathbb{R}^{n_\lambda},\ \lambda\in\Lambda$.*

It should be noted that Whitney proved that an n-manifold can be topological embedded as a closed submanifold of $\mathbb{R}^{2n+1}$ with a sharply minimum dimension $2n+1$ in 1936. Applying this result, one can easily show that a non-manifold $\neg M$ can be embedded into $\mathbb{R}^{2n_{\max}+1}$ if $n_{\max} = \max\{n_\lambda\in\Lambda\}<\infty$. Furthermore, let U_λ itself be a subset of Euclidean space $\mathbb{R}^{n_{\max}+1}$ for $\lambda\in\Lambda$, then $x_{n_{\max}+1} = \varphi_\lambda(x_1,x_2,\cdots,x_{n_\lambda})$ in $\mathbb{R}^{n_{\max}+1}$. Thus, one gets an equation

$$x_{n_{\max}+1} - \varphi_\lambda(x_1,x_2,\cdots,x_{n_\lambda}) = 0$$

in $\mathbb{R}^{n_{\max}+1}$. Particularly, if $\Lambda = \{1,2,\cdots,m\}$ is finite, one gets a system (ES_m) of equations

$$\begin{cases} x_{n_{\max}+1} - \varphi_\lambda(x_1,x_2,\cdots,x_{n_1}) = 0 \\ x_{n_{\max}+1} - \varphi_\lambda(x_1,x_2,\cdots,x_{n_2}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n_{\max}+1} - \varphi_\lambda(x_1,x_2,\cdots,x_{n_m}) = 0 \end{cases} \qquad (ES_m)$$

in $\mathbb{R}^{n_{\max}+1}$. Generally, this system (ES_m) is non-solvable, which enables one getting Theorem 3.1 once again.

5.3 Differentiable Non-Manifolds

For $\forall M_\lambda\in\neg M$, if M_λ is differentiable determined by a system of differential equations

$$(DES_{m_\lambda})\begin{cases} F_{\lambda 1}(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots) = 0 \\ F_{\lambda 2}(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_{\lambda m_\lambda}(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots) = 0 \end{cases}$$

Then the system (DES_m) consisting of systems $(DES_{m_\lambda}),\ 1\leq\lambda\leq m$ of differential equations with prescribed initial values x_{i_0},u_0,p_{i_0} for integers $i=1,2,\cdots,n$ is generally non-solvable with a geometrical figure of differentiable non-manifold $\neg M$.

Notice that a main characters for points p in non-manifold $\neg M$ is that the number of variables for determining its position in space is not a constant. However, it can also introduces differentials on non-manifolds constrained with $\varphi_\kappa|_{U_\kappa\bigcap U_\lambda} = \varphi_\lambda|_{U_\kappa\bigcap U_\lambda}$ for $\forall(U_\kappa,\varphi_\kappa),\ (U_\lambda,\varphi_\lambda)\in\mathscr{A}$, and smooth functions $f:\ \neg M\to\mathbb{R}$ at a point $p\in\neg M$. Denoted respectively by $\mathscr{X}_p$, $T_p\neg M$ all smooth functions and all tangent vectors $\overline{v}:\ \mathscr{X}_p\to\mathbb{R}$ at a point $p\in\neg M$. If $\varphi(p)\in\bigcap\limits_{i=1}^{s}\mathbb{R}^{n_i(p)}$ and $\widehat{s}(p) = \dim(\bigcap\limits_{i=1}^{s}\mathbb{R}^{n_i(p)})$, a simple calculation shows the dimension of tangent vector space

$$\dim T_p\neg M = \widehat{s}(p) + \sum_{i=1}^{s(p)}(n_i - \widehat{s}(p))$$

with a basis

$$\left\{ \left.\frac{\partial}{\partial x_{ij}}\right|_p,\ 1 \le i \le s(p), 1 \le j \le n_i \text{ with } x_{il} = x_{jl} \text{ if } 1 \le l \le \widehat{s}(p) \right\}$$

and similarly, for cotangent vector space $\dim T_p^*\neg M = \dim T_p\neg M$ with a basis

$$\left\{ \left.dx_{ij}\right|_p,\ 1 \le i \le s(p), 1 \le j \le n_i \text{ with } x_{il} = x_{jl} \text{ if } 1 \le l \le \widehat{s}(p) \right\},$$

which enables one to introduce vector field $\mathscr{X}(\neg M) = \bigcup\limits_{p\in\neg M} \mathscr{X}_p$, tensor field $T_s^r(\neg M) = \bigcup\limits_{p\in\neg M} T_s^r(p,\neg M)$, where,

$$T_s^r(p,\neg M) = \underbrace{T_p\neg M \otimes \cdots \otimes T_p\neg M}_{r} \otimes \underbrace{T_p^*\neg M \otimes \cdots \otimes T_p^*\neg M}_{s}$$

and connection $D: \ \mathscr{X}(\neg M) \times T_s^r(\neg M) \to T_s^r(\neg M)$ with $D_X\tau = D(X,\tau)$ such that for $\forall X, Y \in \mathscr{X}(\neg M)$, $\tau, \pi \in T_s^r(\neg M)$, $\lambda \in \mathbb{R}$, $f \in C^\infty(\neg M)$,

(1) $D_{X+fY}\tau = D_X\tau + fD_Y\tau$ and $D_X(\tau + \lambda\pi) = D_X\tau + \lambda D_X\pi$;

(2) $D_X(\tau \otimes \pi) = D_X\tau \otimes \pi + \sigma \otimes D_X\pi$;

(3) For any contraction C on $T_s^r(\neg M)$, $D_X(C(\tau)) = C(D_X\tau)$.

Particularly, let $g \in T_2^0(\neg M)$. If g is symmetrical and positive, then $\neg M$ is called a *Riemannian non-manifold*, denoted by $(\neg M, g)$. It can be readily shown that there is a unique connection D on Riemannian non-manifold $(\neg M, g)$ with equality

$$Z(g(X,Y)) = g(D_Z, Y) + g(X, D_Z Y)$$

holds. Such a D with $(\neg M, g)$, denoted by $(\neg M, g, D)$ is called a *Riemannian non-geometry*. Now let $D_{\frac{\partial}{\partial x_{kl}}}\frac{\partial}{\partial x_{ij}} = \Gamma_{(st)}^{(ij)(kl)}\frac{\partial}{\partial x_{ij}}$ on $(U_p;\varphi)$ for point $p \in (\neg M, g, D)$. Then $\Gamma_{(st)}^{(ij)(kl)} = \Gamma_{(st)}^{(kl)(ij)}$ and

$$\Gamma_{st}^{(kl)(ij)} = \frac{1}{2} g_{(st)(uv)}\Big(\frac{\partial g^{(kl)(uv)}}{\partial x_{ij}} + \frac{\partial g^{(uv)(ij)}}{\partial x_{kl}} - \frac{\partial g^{(kl)(ij)}}{\partial x_{uv}}\Big),$$

where $g = g^{(kl)(ij)} dx_{kl} dx_{ij}$ and $g_{(st)(uv)}$ is an element in matrix $[g^{(kl)(ij)}]^{-1}$.

Similarly, a *Riemannian curvature tensor*

$$R: \ \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \to C^\infty(\neg M)$$

of type $(0,4)$ is defined by $R(X,Y,Z,W) = g(R(Z,W)X,Y)$ for $\forall X,Y,Z,W \in \mathscr{X}(\neg M)$ and with a local form

$$R = R^{(ij)(kl)(st)(uv)} dx_{ij} \otimes dx_{kl} \otimes dx_{st} \otimes dx_{uv},$$

where

$$\begin{aligned} R^{(ij)(kl)(st)(uv)} \quad = \quad & \frac{1}{2}\left(\frac{\partial^2 g^{(st)(ij)}}{\partial x_{uv}\partial x_{kl}} + \frac{\partial^2 g^{(uv)(kl)}}{\partial x_{st}\partial x_{ij}} - \frac{\partial^2 g^{(st)(kl)}}{\partial x_{uv}\partial x_{ij}} - \frac{\partial^2 g^{(uv)(ij)}}{\partial x_{st}\partial x_{kl}}\right) \\ & + \Gamma_{ab}^{(st)(ij)}\Gamma_{cd}^{(uv)(kl)} g^{(cd)(ab)} - \Gamma_{ab}^{(st)(kl)}\Gamma^{(uv)(ij)_{cd}} g^{(cd)(ab)}, \end{aligned}$$

for $\forall p \in \neg M$ and $g^{(ij)(kl)} = g(\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}})$, which can be also used for measuring the curved degree of $(\neg M, g, D)$ at point $p \in \neg M$ (see [16] or [21] for details).

Theorem 5.4 *A Riemannian non-geometry $(\neg M, g, D)$ inherits an invariant, i.e., the curvature tensor $R: \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \times \mathscr{X}(\neg M) \to C^\infty(\neg M)$.*

5.4 Smarandache Geometry

A fundamental image of geometry $\mathscr{G}$ is that of *space* consisting of point p, line L, plane P, etc. elements with inclusions $P, L \ni p$ and $P \supset L$ and a geometrical axiom is a premise logic function T on geometrical elements $p, L, P, \cdots \in \mathscr{G}$ with $T(p, L, P, \cdots) = 1$ in classical geometry. Contrast to the classic, a *Smarandache geometry $S\mathscr{G}$* is such a geometry with at least one axiom behaves in two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways. Thus, $T(p, L, P, \cdots) = 1$, $\neg T(p, L, P, \cdots) = 1$ hold simultaneously, or $0 < \neg T(p, L, P, \cdots) = I_1, I_2, \cdots, I_k < 1$ for an integer $k \geq 2$ in $S\mathscr{G}$, which enables one to discuss Smarandache geometriy in two cases following:

Case 1. $T(p, L, P, \cdots) = 1 \wedge \neg T(p, L, P, \cdots) = 1$ in $S\mathscr{G}$.

Denoted by $U = T^{-1}(1) \subset S\mathscr{G}$, $V = \neg T^{-1}(1) \subset S\mathscr{G}$. Clearly, if $U \bigcap V \neq \emptyset$ and there are $p, L, P, \cdots \in U \bigcap V$. Then there must be $T(p, L, P, \cdots) = 1$ and $\neg T(p, L, P, \cdots) = 1$ in $U \bigcap V$, a contradiction. Thus, $U \bigcap V = \emptyset$ or $U \bigcap V \neq \emptyset$ but some of elements $p, L, P, \cdots \in S\mathscr{G}$ for T are missed in $U \bigcap V$.

Not loss of generality, let

$$U = \bigoplus_{k=1}^{m} U_C^k \quad \text{and} \quad V = \bigoplus_{i=1}^{n} V_C^i,$$

where U_C^k, V_C^i are respectively connected components in U and V. Define a topological graph $G[U, V]$ following:

$$V(G[U, V]) = \{U_C^k; 1 \leq k \leq m\} \bigcup \{V_C^i; 1 \leq i \leq n\};$$
$$E(G[U, V]) = \{(U_C^k, V_C^i) \text{ if } U_C^k \bigcap V_C^i \neq \emptyset,\ 1 \leq k \leq m, 1 \leq i \leq n\}$$

with labels

$$L: \ U_C^k \in V(G[U, V]) \to U_C^k, \quad V_C^i \in V(G[U, V]) \to V_C^i$$
$$L: \ (U_C^k, V_C^i) \in E(G[U, V]) \to U_C^k \bigcap V_C^i, \quad 1 \leq k \leq m, 1 \leq i \leq n.$$

Clearly, such a graph $G[U, V]$ is bipartite, i.e., $G[U, V] \leq K_{m,n}$ with labels.

Case 2. $0 < \neg T(p, L, P, \cdots) = I_1, I_2, \cdots, I_k < 1$, $k \geq 2$ in $S\mathscr{G}$.

Denoted by $A_1 = \neg T^{-1}(I_1) \subset S\mathscr{G}$, $A_2 = \neg T^{-1}(I_2) \subset S\mathscr{G}$, $\cdots$, $A_k = \neg T^{-1}(I_k) \subset S\mathscr{G}$. Similarly, if $A_i \bigcap A_j \neq \emptyset$ and there are $p, L, P, \cdots \in A_i \bigcap A_j$. Then there must be $A_i \bigcap A_j = \emptyset$ or $A_i \bigcap A_j \neq \emptyset$ but some of elements $p, L, P, \cdots \in S\mathscr{G}$ for T are missed in $A_1 \bigcap A_j$ for integers $1 \leq i \neq j \leq k$.

Let $A_i = \bigoplus\limits_{l=1}^{m_i} A_C^{i_l}$ with $A_C^{i_l},\ 1 \leq l \leq m_i$ connected components in A_i. Define a topological graph $G[A_i,[1,k]]$ following:

$$\begin{aligned} V(G[A_i,[1,k]]) &= \bigcup_{i=1}^{k} \{A_C^{i_l}; 1 \leq l \leq m_i\}; \\ E(G[A_i,[1,k]]) &= \bigcup_{\substack{i,j=1 \\ i \neq j}}^{k} \{(A_C^{i_l}, A_C^{j_s}) \text{ if } A_C^{i_l} \bigcap A_C^{j_s} \neq \emptyset,\ 1 \leq l \leq m_i, 1 \leq s \leq m_j\} \end{aligned}$$

with labels

$$\begin{aligned} &L:\ A_C^{i_l} \in V(G[A_i,[1,k]]) \to A_C^{i_l},\quad A_C^{j_s} \in V(G[A_i,[1,k]]) \to A_C^{j_s} \\ &L:\ (A_C^{i_l}, A_C^{j_s}) \in E(G[A_i,[1,k]]) \to A_C^{i_l} \bigcap A_C^{j_s},\quad 1 \leq l \leq m_i, 1 \leq s \leq m_j \end{aligned}$$

for integers $1 \leq i \neq j \leq k$. Clearly, such a graph $G[A_i,[1,k]]$ is k-partite, i.e., $G[A_i,[1,k]] \leq K_{m_1,m_2,\cdots,m_k}$ with labels.

For an invertible transformation T on geometry $S\mathscr{G}$, it is clear that $T(p)$, $T(L)$, $T(P), \cdots$ also constitute the elements of $S\mathscr{G}$ with graphs $G[U,V]$ and $G[A_i,[1,k]]$ invariant. Thus, we know

Theorem 5.5 *A Smarandache geometry $S\mathscr{G}$ inherits a bipartite invariant $G[U,V]$ or k-partite $G[A_i,[1,k]]$ under the action of its linear invertible transformations.*

5.5 Geometrical Combinatorics

All previous discussions on non-space $(\mathscr{K}^n,\mu)$, non-manifold $\neg M$ or differentiable non-manifold $\neg M$ and Smarandache geometry $S\mathscr{G}$ allude a philosophical notion that *any non-geometry can be decomposed into geometries inheriting an invariant $G[\mathscr{K}^n,\mu]$, $G[\neg M]$, $G[U,V]$ or $G[A_i,[1,k]]$ of topological graph labeled with those of geometries*, i.e., geometrical combinatorics accordant with that notion of Klein's. Accordingly, for extending field of geometry, one needs to determine the inherited invariants $G[\mathscr{K}^n,\mu]$, $G[\neg M]$, $G[U,V]$ or $G[A_i,[1,k]]$ and then know geometrical behaviors on non-geometries. But this approach is passive for including non-geometry to geometry. A more initiative way with realization is geometrical G-systems following:

Definition 5.6 *Let $(\mathscr{G}_1;\mathcal{A}_1),(\mathscr{G}_2;\mathcal{A}_2,\cdots,(\mathscr{G}_m;\mathcal{A}_m)$ be m geometrical systems, where $\mathscr{G}_i$, $\mathcal{A}_i$ be respectively the geometrical space and the system of axioms for an integer $1 \leq i \leq m$. A geometrical G-system is a topological graph G with labeling $L:\ v \in V(G) \to L(v) \in \{\mathscr{G}_1,\mathscr{G}_2,\cdots,\mathscr{G}_m\}$ and $L:\ (u,v) \in E(G) \to L(u)\bigcap L(v)$ with $L(u)\bigcap L(v) \neq \emptyset$, denoted by $G[\mathscr{G},\mathcal{A}]$, where $\mathscr{G} = \bigcup\limits_{i=1}^{m} \mathscr{G}_i$ and $\mathcal{A} = \bigcup\limits_{i=1}^{m} \mathcal{A}_i$.*

Clearly, a geometrical G-system can be applied for holding on the global behavior of systems $\mathscr{G}_1,\mathscr{G}_2,\cdots,\mathscr{G}_m$. For example, a geometrical $K_4-\{e\}$-system is shown in Fig.11, where, $\mathbb{R}_i^3$, $1 \leq i \leq 4$ are Euclidean spaces with dimensional 3 and $\mathbb{R}_i^3 \bigcap \mathbb{R}_j^3$ maybe homeomorphic to $\mathbb{R},\mathbb{R}^2$ or $\mathbb{R}^3$ for $1 \leq i,j \leq 4$.

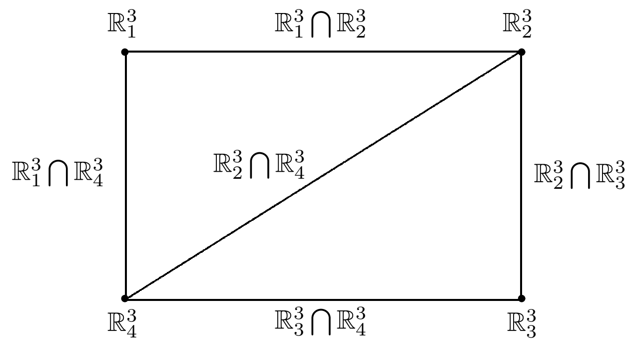


Fig.11

Problem 5.7 *Characterize geometrical G-systems $G[\mathscr{G},\mathcal{A}]$. Particularly, characterize these geometrical G-systems, such as those of Euclidean geometry, Riemannian geometry, Lobachevshy-Bolyai-Gauss geometry for complete graphs $G = K_m$, complete k-partite graph $K_{m_1,m_2,\cdots,m_k}$, path P_m or circuit C_m.*

Problem 5.8 *Characterize geometrical G-systems $G[\mathscr{G},\mathcal{A}]$ for topological or differentiable manifold, particularly, Euclidean space, projective space for complete graphs $G = K_m$, complete k-partite graph $K_{m_1,m_2,\cdots,m_k}$, path P_m or circuit C_m.*

It should be noted that classic geometrical system are mostly K_1-systems, such as those of Euclidean geometry, projective geometry,$\cdots$, etc., also a few K_2-systems. For example, the topological group and Lie group are in fact geometrical K_2-systems, but neither K_m-system with $m \geq 3$, nor $G \not\simeq K_m$-system.

§6. Applications

As we known, mathematical non-systems are generally faced up human beings in scientific fields. Even through, the mathematical combinatorics contributes an approach for holding on their global behaviors.

6.1 Economics

A *circulating economic system* is such a overall balance input-output $M(t) = \bigcup\limits_{i=1}^{m} M_i(t)$ underlying a topological graph $G[M(t)]$ that there are no rubbish in each producing department. Whence, there is a circuit-decomposition $G\left[M(t)\right] = \bigcup\limits_{i=1}^{l} \overrightarrow{C}_s$ such that each output of a producing department $M_i(t)$, $1 \leq i \leq m$ is on a directed circuit $\overrightarrow{C}_s$ for an integer $1 \leq s \leq l$, such as those shown in Fig.12.

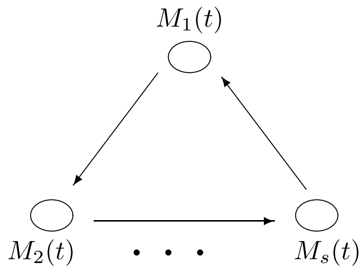


Fig.12

Assume that there are m producing departments $M_1(t)$, $M_2(t), \cdots, M_m(t)$, x_{ij} output values of $M_i(t)$ for the department $M_j(t)$ and d_i for the social demand. Let $F_i(x_{1i}, x_{2i}, \cdots, x_{ni})$ be the producing function in $M_i(t)$. Then the input-output model of a circulating economic system can be characterized by a system of equations

$$\begin{cases} F_1(\overline{x}) = \sum\limits_{j=1}^{m} x_{1j} + d_1 \\ F_2(\overline{x}) = \sum\limits_{j=1}^{m} x_{2j} + d_2 \\ \dots\dots\dots\dots\dots\dots \\ F_m(\overline{x}) = \sum\limits_{j=1}^{m} x_{mj} + d_m \end{cases}$$

Generally, this system is non-solvable even if it is a linear system. Nevertheless, it is a G-system of equations. The main task is not finding its solutions, but determining whether it runs smoothly, i.e., a macro-economic behavior of system.

6.2 Epidemiology

Assume that there are three kind groups in persons at time t, i.e., infected $I(t)$, susceptible $S(t)$ and recovered $R(t)$ with $S(t) + I(t) + R(t) = 1$. Then one established the *SIR model* of infectious disease as follows:

$$\begin{cases} \frac{dS}{dt} = -kIS, \\ \frac{dI}{dt} = kIS - hI, \\ S(0) = S_0, I(0) = I_0, R(0) = 0, \end{cases},$$

which are non-linear equations of first order.

If the number of persons in an area is not constant, let $C_1, C_2, \cdots, C_m$ be m segregation areas with respective $N_1, N_2, \cdots, N_m$ persons. Assume at time t, there are $U_i(t)$, $V_i(t)$ persons moving in or away C_i. Thus $S_i(t) + I_i(t) - U_i(t) + V_i(t) = N_i$. Denoted by $c_{ij}(t)$ the persons moving from C_i to C_j for integers $1 \le i, j \le m$. Then

$$\sum_{s=1}^{m} c_{si}(t) = U_i(t) \quad \text{and} \quad \sum_{s=1}^{m} c_{is}(t) = V_i(t).$$

A combinatorial model of infectious disease is defined by a topological graph G following:

$$V(G) = \{C_1, C_2, \cdots, C_m\},$$
$$E(G) = \{(C_i, C_j) |\text{ there are traffic means from } C_i \textit{ to } C_j,\ 1 \leq i, j \leq m\},$$
$$L(C_i) = N_i, \quad L^+(C_i, C_j) = c_{ij} \text{ for } \forall (C_i, C_j) \in E\left(G^l\right),\ 1 \leq i, j \leq m,$$

such as those shown in Fig.13.

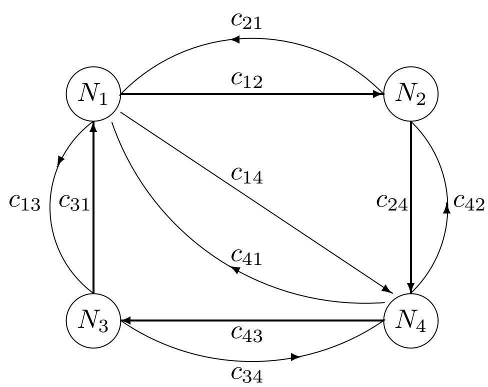


Fig.13

In this case, the SIR model for areas $C_i,\ 1 \leq i \leq m$ turns to

$$\left.\begin{aligned} &\frac{dS_i}{dt} = -kI_iS_i, \\ &\frac{dI_i}{dt} = kI_iS_i - hI_i, \\ &S_i(0) = S_{i0}, I_i(0) = I_{i0}, R(0) = 0, \end{aligned}\right\} 1 \leq i \leq m,$$

which is a non-solvable system of differential equations.

Even if the number of an area is constant, the SIR model works only with the assumption that a healed person acquired immunity and will never be infected again. If it does not hold, the SIR model will not immediately work, such as those of cases following:

Case 1. there are m known virus $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$ and an person infected a virus $\mathscr{V}_i$ will never infects other viruses $\mathscr{V}_j$ for $j \neq i$.

Case 2. there are m varying $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ from a virus $\mathscr{V}$ with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$ such as those shown in Fig.14.

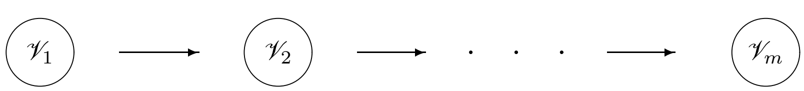


Fig.14

However, it is easily to establish a non-solvable differential model for the spread of viruses following by combining SIR model:

$$\begin{cases} \dot{S} = -k_1 SI \\ \dot{I} = k_1 SI - h_1 I \\ \dot{R} = h_1 I \end{cases} \quad \begin{cases} \dot{S} = -k_2 SI \\ \dot{I} = k_2 SI - h_2 I \\ \dot{R} = h_2 I \end{cases} \quad \cdots \quad \begin{cases} \dot{S} = -k_m SI \\ \dot{I} = k_m SI - h_m I \\ \dot{R} = h_m I \end{cases}$$

Consider the equilibrium points of this system enables one to get a conclusion ([27]) for globally control of infectious diseases, i.e., they decline to 0 finally if

$$0 < S < \sum_{i=1}^{m} h_i \Big/ \sum_{i=1}^{m} k_i \ ,$$

particularly, these infectious viruses are globally controlled if each of them is controlled in that area.

6.3 Gravitational Field

What is the true face of gravitation? Einstein's equivalence principle says that *there are no difference for physical effects of the inertial force and the gravitation in a field small enough*, i.e., considering the curvature at each point in a spacetime to be all effect of gravitation, called *geometrization of gravitation*, which finally resulted in Einstein's gravitational equations ([2])

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

in $\mathbb{R}^4$, where $R^{\mu\nu} = R^{\mu\alpha\nu}_{\alpha} = g_{\alpha\beta} R^{\alpha\mu\beta\nu}$, $R = g_{\mu\nu} R^{\mu\nu}$ are the respective Ricci tensor, Ricci scalar curvature, $G = 6.673 \times 10^{-8} cm^3/gs^2$, $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$ and Schwarzschild spacetime with a spherically symmetric Riemannian metric

$$ds^2 = f(t)\left(1 - \frac{r_g}{r}\right) dt^2 - \frac{1}{1 - \frac{r_g}{r}} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

for $\lambda = 0$. However, a most puzzled question faced up human beings is *whether the dimension of the universe is really* 3? if not, *what is the meaning of one's observations*? Certainly, if the dimension≥ 4, all these observations are nothing else but a projection of the true faces on our six organs, a pseudo-truth.

For a gravitational field $\mathbb{R}^n$ with $n \geq 4$, decompose it into dimensional 3 Euclidean spaces $\mathbb{R}^3_u$, $\mathbb{R}^3_v$, $\cdots$, $\mathbb{R}^3_w$. Then there are Einstein's gravitational equations:

$$R^{\mu_u\nu_u} - \frac{1}{2} g^{\mu_u\nu_u} R = -8\pi G T^{\mu_u\nu_u},$$

$$R^{\mu_v\nu_v} - \frac{1}{2} g^{\mu_v\nu_v} R = -8\pi G T^{\mu_v\nu_v},$$

$$\cdots\cdots\cdots\cdots\cdots ,$$

$$R^{\mu_w\nu_w} - \frac{1}{2} g^{\mu_w\nu_w} R = -8\pi G T^{\mu_w\nu_w}$$

for each $\mathbb{R}^3_u$, $\mathbb{R}^3_v, \cdots$, $\mathbb{R}^3_w$, such as a K_4-system shown in Fig.15,

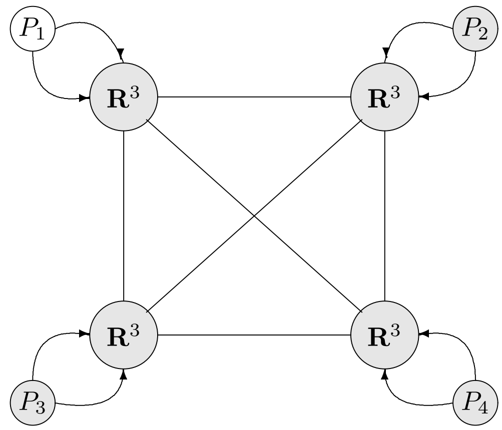


Fig.15

where P_1, P_2, P_3, P_4 are the observations. In this case, these gravitational equations can be represented by

$$R^{(\mu\nu)(\sigma\tau)} - \frac{1}{2}g^{(\mu\nu)(\sigma\tau)}R = -8\pi G T^{(\mu\nu)(\sigma\tau)}$$

with a coordinate matrix

$$[\overline{x}_p] = \begin{bmatrix} x_{11} & \cdots & x_{1\widehat{m}} & \cdots & x_{13} \\ x_{21} & \cdots & x_{2\widehat{m}} & \cdots & x_{23} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m1} & \cdots & x_{m\widehat{m}} & \cdots & x_{m3} \end{bmatrix}$$

for $\forall p \in \mathbb{R}^n$, where $\widehat{m} = \dim\left(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}\right)$ a constant for $\forall p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ and $x^{il} = \dfrac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$. Then, by the *Projective Principle*, i.e., *a physics law in a Euclidean space* $\mathbb{R}^n \simeq \widetilde{\mathbb{R}} = \bigcup\limits_{i=1}^{n} \mathbb{R}^3$ *with* $n \geq 4$ *is invariant under a projection on* $\mathbb{R}^3$ *from* $\mathbb{R}^n$, one can determines its combinatorial Schwarzschild metric. For example, if $\widehat{m} = 4$, i.e., $t_\mu = t, r_\mu = r, \theta_\mu = \theta$ and $\phi_\mu = \phi$ for $1 \leq \mu \leq m$, then ([18])

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r}\right)dt^2 - \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r}\right)^{-1}dr^2 - mr^2(d\theta^2 + \sin^2\theta d\phi^2)$$

and furthermore, if $m_\mu = M$ for $1 \leq \mu \leq m$, then

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)mdt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}mdr^2 - mr^2(d\theta^2 + \sin^2\theta d\phi^2),$$

which is the most enjoyed case by human beings. If so, all the behavior of universe can be realized finally by human beings. But if $\widehat{m} \leq 3$, there are infinite underlying connected graphs, one can only find an approximating theory for the universe, i.e., “Name named is not the eternal Name”, claimed by Lao Zi.

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On Cosets and Normal Subgroups

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Abstract: The paper [5] has worked on fuzzy cosets and fuzzy normal subgroups of a group, [8] has extended the idea to fuzzy middle coset. In addition to what has been done, we make a link between fuzzy coset and fuzzy middle coset and investigate some more properties of the fuzzy middle coset. [7] made attempt with some results needing adjustment. [2], [8] and [9] have shown that if $f \in F(S_n)$, the set of all fuzzy subgroups of S_n, is such that Imf has the highest order and f is constant on the conjugacy classes of S_n, then it is co-fuzzy symmetric subgroup of S_n. Then, using some results of [5], we get another result.

Key Words: Middle cosets, fuzzy normal, normal subgroups, fuzzy μ-commutativity.

AMS(2010): 20N25

§1. Introduction

This paper seeks to contribute to the body of knowledge existing in the area of fuzzy normal subgroup without any damage to the existing one.

§2. Preliminaries

Definition 2.1 *Let X be a non-empty set. A fuzzy subset μ of the set G is a function $\mu : G \to [0, 1]$.*

Definition 2.2 *Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if*

(*i*) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$;
(*ii*) $\mu(x^{-1}) = \mu(x)$;
(*iii*) *μ is called a fuzzy normal subgroup if $\mu(xy) = \mu(yx)$ for all x and y in G.*

Definition 2.3 *Let μ be a fuzzy subset (subgroup) of X. Then, for some t in $[0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset (subgroup) of the fuzzy subset (subgroup) μ.*

Definition 2.4 *Let μ be a fuzzy subgroup of a group G. For a in G, the fuzzy left (or right) coset*

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$a\mu$ (or μa) of G determined by a and μ is defined by $(a\mu)(x) = \mu(a^{-1}x)$ (or $(\mu a)(x) = \mu(xa^{-1})$) for all x in G.

Definition 2.5 *Let μ be a fuzzy subgroup of a group G. For a and b in G, the fuzzy middle coset $a\mu b$ of G is defined by $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$ for all x in G.*

Proposition 2.6 *Let G be a group and μ a fuzzy subset of G. Then μ is a fuzzy subgroup of G if and only if G_μ^t is a level subgroup of G for every t in $[0, \mu(e)]$, where e is the identity of G.*

Theorem 2.7 *Let μ be a fuzzy normal subgroup of a group G. Let $t \in [0,1]$ such that $t \leq \mu(e)$, where e is the identity of G. Then G_μ^t is a normal subgroup of G.*

Remark 2.8 The paper [5] have also shown that the collection $\{G_\mu^t\}$ form a chain of normal subgroups of G.

Theorem 2.9 *Let μ and λ be fuzzy subgroups of G. Then they are conjugate if for some $a \in G$ we have $\mu(a^{-1}xa) = \lambda(x)\ \ \forall x \in G$.*

Theorem 2.10 *Let μ and λ be any two fuzzy subgroup of any group G. Then, μ and λ are conjugate fuzzy subgroup of G if and only if $\mu = \lambda$.*

Theorem 2.11([8]) *Let μ be a fuzzy normal subgroup of G, Then for any $g \in G$, $\mu(gxg^{-1}) = \mu(g^{-1}xg)$ for every $x \in G$.*

§3. Some Results on Fuzzy Normal and Cosets

Theorem 3.1 *Let $a^{-1}\mu a$ be a fuzzy middle coset of G for some $a \in G$. Then all such a form the normalizer $N(\mu)$ of fuzzy subgroup μ of G if and only if μ is fuzzy normal.*

Proof The paper [5] defined the normalizer of μ by $N(\mu) = \{a \in G : \mu(axa^{-1}) = \mu(x)\}$. Then, $\mu(axa^{-1}) = \mu(x) \Leftrightarrow \mu$ is fuzzy normal so that $\mu(axa^{-1}a) = \mu(xa) \Leftrightarrow \mu(ax) = \mu(xa)$.

Conversely, let μ be fuzzy normal and $a^{-1}\mu a$ a middle coset in G. Then, for all $x \in G$ and some $a \in G$,

$$(a^{-1}\mu a)(x) = \mu(axa^{-1}) = \mu(aa^{-1}x) = \mu(x).$$

This implies that

$$\mu(axa^{-1}) = \mu(x).$$

Hence,

$$\{a\} = N(\mu).$$

□

Proposition 3.2 *Let μ be a fuzzy normal subgroup of G by a and b. Then every fuzzy middle coset $a\mu b$ coincides with some left and right cosets $c\mu$ and μc respectively, where c^{-1} is the product $b^{-1}a^{-1}$.*

Proof By associativity in G and 2.2(iii), we have that

$$(a\mu b)(x) = \mu((a^{-1}x)b^{-1}) = \mu(b^{-1}(a^{-1}x))$$

$$\mu(b^{-1}a^{-1}x) = \mu(c^{-1}x) = \mu(xc^{-1}) \text{ still by 2.2(iii).}$$

Thus,

$$(a\mu b) = c\mu = \mu c. \qquad \square$$

Theorem 3.3 *Let G be a group of order 2 and μ a fuzzy normal subgroup of G. Then, for some $a \in G$ and $\forall x \in G$, the middle coset $a\mu a$ coincides with fuzzy subgroup μ.*

Proof In the middle coset $a\mu b$, take $a = b$. By associativity in G, we have

$$(a\mu a)(x) = \mu((a^{-1}x)a^{-1})$$

. By 3.2,

$$\mu((a^{-1}x)a^{-1}) = \mu(a^{-2}x)$$

. Since $a^{-1} \in G$ and G is of order 2,

$$\mu(a^{-2}x) = \mu((a^{-1})^2x) = \mu(ex) = \mu(x).$$

Therefore,

$$a\mu a = \mu. \qquad \square$$

Now we introduce the notion of fuzzy μ-commutativity.

Definition 3.4 *Let μ be a fuzzy subgroup of G. Two elements a and b in G are said to be fuzzy μ-commutative if $a\mu b = b\mu a$.*

Theorem 3.5 *Let μ be a fuzzy normal subgroup of G. Then any two elements a and b in G are fuzzy μ-commutative.*

Proof Notice that

$$(a\mu b)(x) = \mu(a^{-1}xb^{-1}).$$

Then, by 2.11,

$$\mu(a^{-1}xb^{-1}) = \mu(b^{-1}xa^{-1}) = (b\mu a)(x).$$

Thus,

$$a\mu b = b\mu a. \qquad \square$$

In [7], it is claimed that every middle coset of a group G is a fuzzy subgroup. But here is a counter example.

Example 3.6 Let $G = (\mathbb{Z}_4, +)$ and choose $a = b = 1$ then $a^{-1} = b^{-1} = 3$.

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0 = e \\ 0.6, & \text{otherwise.} \end{cases}$$

It can be seen that μ is a fuzzy subgroup of G. But the middle coset $a\mu b$ defined by

$$(a\mu b)(x) = \begin{cases} 1, & \text{if } x = 2 \\ 0.6, & \text{otherwise.} \end{cases}$$

is such that $(a\mu b)(2) > (a\mu b)(e)$. But this is a contradiction, since, usually, if μ is a fuzzy subgroup of a group G, $\mu(e) \geq \mu(x)\ \forall x \in G$.

In the following theorem, a necessary condition for middle coset of a group to be fuzzy subgroup is given.

Theorem 3.7 *Every middle coset $a\mu b$ of a group G is a fuzzy subgroup if μ is fuzzy conjugate to some fuzzy subgroup λ of G.*

Proof Let $b = a^{-1}$ for some $a, b \in G$ and μ and λ be fuzzy conjugate subgroups of G.

$$(a\mu b)(xy^{-1}) = (a\mu a^{-1})(xy^{-1}) = \mu(a^{-1}xy^{-1}a) = \lambda(xy^{-1}) \geq min\{\lambda(x), \lambda(y)\}$$

This implies that

$$min\{\lambda(x), \lambda(y)\} = min\{\mu(a^{-1}xa), \mu(a^{-1}ya)\} = min\{(a\mu a^{-1})(x), (a\mu a^{-1})(y)\}.$$

Hence,

$$(a\mu b)(xy^{-1}) \geq min\{(a\mu b)(x), (a\mu b)(y)\}.$$ □

Remark 3.8 If $b = a^{-1}$, the middle coset $a\mu a^{-1}$ is a fuzzy subgroup since μ is self conjugate. Hence, the result of 3.7 generalizes the theorem 1.2.10 of [8].

Proposition 3.8 of [7] says that fuzzy middle cosets form normal subgroup of G. But here is a counter example.

Example 3.9 Let $G = S_3$ and $a = (123),\ b = (12),\ x = (12),\ y^{-1} = (123)$. Also, define the fuzzy group μ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = e \\ 0.5, & \text{if } x = (123), (132) \\ 0.3, & \text{otherwise.} \end{cases}$$

Then, $(a\mu b)(xy^{-1}) = 0.3$ and $(a\mu b)(y^{-1}x) = 1$. Thus,

$$(a\mu b)(xy^{-1}) \neq (a\mu b)(y^{-1}x),$$

which implies that $a\mu b$ is not fuzzy normal.

We now give a characterization for $a\mu b$ to be fuzzy normal. It is noteworthy that [8] has shown that $a\mu a^{-1}$ and μ are conjugates.

Theorem 3.10 *A fuzzy middle coset $a\mu b$ is fuzzy normal if and only if $b = a^{-1}$ and μ is fuzzy normal.*

Proof Let μ be fuzzy normal. By definition,

$$(a\mu b)(xy) = \mu(a^{-1}xyb^{-1}).$$

By 2.9 and 2.10, if we take $b = a^{-1}$, $a\mu b$ and μ are conjugate so that

$$\mu(a^{-1}xyb^{-1}) = (a\mu b)(xy) = \mu(xy).$$

Since, μ is fuzzy normal,

$$\mu(xy) = \mu(yx) = \mu(a^{-1}yxb^{-1}) = (a\mu b)(yx).$$

Thus,

$$(a\mu b)(xy) = (a\mu b)(yx).$$

Conversely, assume that $a\mu b$ is fuzzy normal. Then,

$$(a\mu b)(xy) = (a\mu b)(yx).$$

It follows from 2.10 that

$$(a\mu b)(xy) = \mu(a^{-1}xyb^{-1}) = \mu(xy) \Leftrightarrow b = a^{-1}$$

and

$$(a\mu b)(yx) = \mu(a^{-1}yxb^{-1}) = \mu(yx) \Leftrightarrow b = a^{-1}.$$

This implies that

$$\mu(xy) = (a\mu b)(xy) = (a\mu b)(yx) = \mu(yx) \Leftrightarrow b = a^{-1}.$$

Hence, μ is fuzzy normal and $b = a^{-1}$. □

Proposition 3.11 *Let $\mu \in F(S_n)$ be co-fuzzy symmetric subgroup of S_n. Then μ is fuzzy normal.*

Proof Since μ is co-fuzzy, it is constant on $x^{-1}\Pi x$, the conjugate class of S_n containing Π. Hence $\mu(x^{-1}\Pi x) = \mu(\Pi)$ for $\forall \Pi \in C(\Pi)$ and some $x \in S_n$. By 2.2(iii), μ is fuzzy normal. □

Theorem 3.12 *Every symmetric group with co-fuzzy symmetric subgroup μ is such that the level subgroups μ_t are normal subgroups of the symmetric group so that for $t \in [0, 1]$ and $t < \mu(e)$, the collection $\{\mu_t\}$ is a chain of normal subgroups of S_n.*

Proof Let $\mu \in F(S_n)$ be such that μ is co-fuzzy. Then it is fuzzy normal by 3.11. Then every level subgroup μ_t (which is a subgroup of S_n) of μ is a normal subgroup by 2.7. Then, by 2.8, the collection $\{\mu_t\}$ is a chain of normal subgroups of S_n. □

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On Radio Mean Number of Some Graphs

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Abstract: A radio mean labeling of a connected graph G is a one to one map f from the vertex set $V(G)$ to the set of natural numbers N such that for each distinct vertices u and v of G, $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam }(G)$. The radio mean number of f, $rmn(f)$, is the maximum number assigned to any vertex of G.The radio mean number of G, $rmn(G)$ is the minimum value of $rmn(f)$ taken over all radio mean labeling f of G. In this paper we find the radio mean number of some graphs which are related to complete bipartite graph and cycles.

Key Words: Carona, path, complete bipartite graph, cycle, Smarandache radio mean number, radio mean number.

AMS(2010): 05C78

§1. Introduction

We considered finite, simple undirected and connected graphs only. Let $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of G. Chatrand et al.[1] defined the concept of radio labeling of G in 2001. Radio labeling of graphs is applied in channel assignment problem [1]. Radio number of several graphs determined [2,7,5,9]. In this sequal Ponraj et al. [8] introduced the radio mean labeling in G. A radio mean labeling is a one to one mapping f from $V(G)$ to N satisfying the condition

$$d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam }(G) \tag{1.1}$$

for every $u, v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of Graph G. For any subgraph $H \leq G$, a *Smarandache radio mean number* of G on H is the lowest span taken over al such labelings of the graph G that its constraint on H is a radio mean labeling. Particularly, if $H = G$, such a Smarandache radio mean number is called the *radio mean number* of G, denoted by $rmn(G)$. The condition (1.1) is called radio mean condition. In [8] we determined the radio mean number of some graphs like graphs with diameter three,

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lotus inside a circle, gear graph, Helms and Sunflower graphs. In this paper we determine radio mean number of subdivision of complete bipartite, corona complete graph with path and one point union of cycle C_6. The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uwv. The corona of G with H, $G \odot H$ is the graph obtained by taking one copy of G and p copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H. Let x be any real number. Then $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms and definitions not defined here are follow from Harary [6].

§2. Main Results

Theorem 2.1 $rmn(S(K_{m,n})) = (m+1)(n+1) - 1$, $m > 1, n > 1$.

Proof Let $V(S(K_{m,n})) = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{w_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(S(K_{m,n})) = \{u_i w_{i,j}, w_{i,j} v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Note that $diam(S(K_{m,n})) = 4$. Here we display $S(K_{2,2})$ with a vertex labeling in Figure 1.

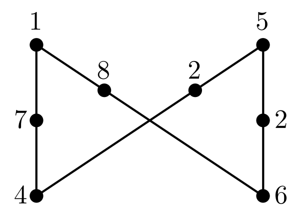

Figure 1

One can easily verify that the above vertex labeling satisfies the radio mean condition. We now explain a method for labeling the vertices of $S(K_{m,n})$ where $n \geq 3$. Consider the vertex $w_{i,j}$. Assign the label 2 to the vertex $w_{m,n}$. Put the label 3 to $w_{m,(n-1)}$. Similarly for $w_{m,(n-2)}$ we can label it by by 4. Proceeding like this $w_{m,1}$ is labeled by $n+1$. Next we label the neighbours of u_{m-1}. Allocate the labels $2n+3-j$ to the vertices $w_{(m-1),j}$ $(1 \leq j \leq n)$. Then we move to the vertices which are adjacent to w_{m-2}. Put the labels $3n+4-j$ to the vertices $w_{(m-2),j}$ $(1 \leq j \leq n)$. Proceeding like this the labels of the neighbours of u_1 are $mn+m+1-j$, $1 \leq j \leq n$. Now consider the vertices u_i $(1 \leq i \leq m)$. Put the label 1 to u_1. Then the vertices u_i $(2 \leq i \leq m)$ are labeled by $n+2+(n+1)(m-i)$. Then the integers from $\{mn+m+1, mn+m+2, \ldots, mn+m+n\}$ are assigned to the remaining vertices in any order.

Claim 1 The labeling f is a valid radio mean labeling. We must show that the condition

$$d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + diam(S(K_{m,n})) = 5,$$

holds for all pairs of vertices (u,v) where $u \neq v$.

Case 1. Check the pair (u_i, v_j).

$$d(u_i, v_j) + \left\lceil \frac{f(u_i)+f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+mn+m+1}{2} \right\rceil \geq 6.$$

Case 2. Verify the pair (u_i, u_j).

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{1+n+2}{2} \right\rceil \geq 7.$$

Case 3. Consider the pair $(u_1, w_{1,j})$, $n \geq 3$.

$$d(u_1, w_{1,j}) + \left\lceil \frac{f(u_1) + f(w_{1,j})}{2} \right\rceil \geq 1 + \left\lceil \frac{1+mn+m-n+1}{2} \right\rceil \geq 5.$$

Case 4. Examine the pair $(u_1, w_{i,j})$, $i \neq 1$.

$$d(u_1, w_{i,j}) + \left\lceil \frac{f(u_1) + f(w_{i,j})}{2} \right\rceil \geq 3 + \left\lceil \frac{1+2}{2} \right\rceil \geq 5.$$

Case 5. Examine the pair $(u_i, w_{i,j})$, $u_i \neq u_1$, $n \geq 3$.

$$d(u_i, w_{i,j}) + \left\lceil \frac{f(u_i) + f(w_{i,j})}{2} \right\rceil \geq 1 + \left\lceil \frac{n+2+2}{2} \right\rceil \geq 5.$$

Case 6. Check the pair $(v_i, w_{i,j})$.

$$d(v_i, w_{i,j}) + \left\lceil \frac{f(v_i) + f(w_{i,j})}{2} \right\rceil \geq 1 + \left\lceil \frac{mn+m+1+2}{2} \right\rceil \geq 6.$$

Case 7. Consider the pair $(w_{i,j}, w_{r,t})$.

$$d(w_{i,j}, w_{r,t}) + \left\lceil \frac{f(w_{i,j}) + f(w_{r,t})}{2} \right\rceil \geq 2 + \left\lceil \frac{2+3}{2} \right\rceil \geq 5.$$

Case 8. Verify the pair (v_i, v_j).

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{mn+m+1+mn+m+2}{2} \right\rceil \geq 12.$$

Hence $rmn(S(K_{m,n})) = (m+1)(n+1) - 1$. □

Theorem 2.2 *$rmn(K_{m,n} \odot P_t) = (m+n)(t+1)$, $m \geq 2$, $n \geq 2$, $t \geq 2$.*

Proof Let $V(K_{m,n}) = \{x_i, y_i : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{x_i y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Let $u_1^i u_2^i \cdots u_t^i$ be the path P_t^i and $v_1^j v_2^j \cdots v_t^j$ be the path P_t^{*j}, where $1 \leq i \leq m$, $1 \leq j \leq n$. The vertex set and edge set of the corona graph $K_{m,n} \odot P_t$ is given below. Let $V(K_{m,n} \odot P_t) = V(K_{m,n}) \cup (\bigcup\limits_{i=1}^{m} V(P_t^i)) \cup (\bigcup\limits_{j=1}^{n} V(P_t^{*j}))$ and $E(K_{m,n} \odot P_t) = E(K_{m,n}) \cup (\bigcup\limits_{i=1}^{m} E(P_t^i)) \cup (\bigcup\limits_{j=1}^{n} E(P_t^{*j})) \cup \{x_i u_j^i : 1 \leq i \leq m, 1 \leq j \leq t\} \cup \{y_i v_j^i : 1 \leq i \leq n, 1 \leq j \leq t\}$. Assign the label $1, 2, \cdots, m$ to the vertices $u_1^1, u_1^2, \cdots, u_1^m$ respectively. Then we move to the path vertices of P_t^{*j}. Assign the label $m+1, m+2, \cdots, m+t$ to the vertices $v_1^1, v_2^1, \cdots, v_t^1$ respectively. Then assign $m+t+1, m+t+2, \cdots, m+2t$ to the vertices $v_1^2, v_2^2, \cdots, v_t^2$ respectively. Proceeding like this until we reach the vertices of P_t^{*n}. Note that $v_1^n, v_2^n, \cdots, v_t^n$ received the labels $m+(n-1)t+1, m+(n-1)t+2, \cdots, m+nt$. Again we move to the vertices of the path P_t^i. Assign the label $m+nt+1, m+nt+2, \cdots, m+nt+t-1$ to the vertices $u_2^1, u_3^1, \cdots, u_t^1$ respectively.

Then assign the label $m+nt+t, m+nt+t+1, \cdots, m+nt+2t-2$ to the vertices $u_2^2, u_3^2, \cdots, u_t^2$ respectively. Proceed in the same way, assign the labels to the remaining vertices. Clearly the vertices $u_2^m, u_3^m, \cdots, u_t^m$ respectively received the labels $nt+mt-t+1, nt+mt-t+2, \cdots, nt+mt$. Finally assign the labels $nt+mt+1, nt+mt+2, \ldots, nt+mt+m$ to the vertices $x_1, x_2, \cdots, x_m$ and $nt+mt+m+1, nt+mt+2, \cdots, nt+mt+m+n$ to the vertices $y_1, y_2, \ldots, y_n$ respectively. We now check the radio mean condition for every pair of vertices.

Case 1. Consider the pair (u_i^j, u_s^r).

Subcase 1.1 $j \neq r$.

$$d(u_i^j, u_s^r) + \left\lceil \frac{f(u_i^j) + f(u_s^r)}{2} \right\rceil \geq 4 + \left\lceil \frac{1+2}{2} \right\rceil \geq 6$$

Subcase 1.2 $j = r$.

$$d(u_i^j, u_s^j) + \left\lceil \frac{f(u_i^j) + f(u_s^j)}{2} \right\rceil \geq 1 + \left\lceil \frac{1+m+nt+1}{2} \right\rceil \geq 5$$

Case 2 Check the pair (u_i^j, x_r).

$$d(u_i^j, x_r) + \left\lceil \frac{f(u_i^j) + f(x_r)}{2} \right\rceil \geq 1 + \left\lceil \frac{1+nt+mt+1}{2} \right\rceil \geq 6$$

Case 3 Verify the pair (u_i^j, y_r).

$$d(u_i^j, y_r) + \left\lceil \frac{f(u_i^j) + f(y_r)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+nt+m}{2} \right\rceil \geq 6$$

Case 4 Examine the pair (u_i^j, v_r^s).

$$d(u_i^j, v_r^s) + \left\lceil \frac{f(u_i^j) + f(v_r^s)}{2} \right\rceil \geq 3 + \left\lceil \frac{1+m+t+1}{2} \right\rceil \geq 6$$

Case 5 Consider the pair (v_i^j, v_r^s).

$$d(v_i^j, v_r^s) + \left\lceil \frac{f(v_i^j) + f(v_r^s)}{2} \right\rceil \geq 1 + \left\lceil \frac{m+t+1+m+t+2}{2} \right\rceil \geq 7$$

Case 6 Verify the pair (v_i^j, x_r).

$$d(v_i^j, x_r) + \left\lceil \frac{f(v_i^j) + f(x_r)}{2} \right\rceil \geq 2 + \left\lceil \frac{m+t+1+nt+mt+1}{2} \right\rceil \geq 9$$

Case 7 Verify the pair (v_i^j, y_r).

$$d(v_i^j, y_r) + \left\lceil \frac{f(v_i^j) + f(y_r)}{2} \right\rceil \geq 1 + \left\lceil \frac{m+t+1+nt+mt+m+1}{2} \right\rceil \geq 9$$

Case 8 Consider the pair (x_i, x_j).

$$d(x_i, x_j) + \left\lceil \frac{f(x_i) + f(x_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{nt + mt + 1 + nt + mt + 2}{2} \right\rceil \geq 12$$

Case 9 Examine the pair (y_i, y_j).

$$d(y_i, y_j) + \left\lceil \frac{f(y_i) + f(y_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{nt + mt + m + 1 + nt + mt + m + 2}{2} \right\rceil \geq 14$$

Case 10 Check the pair (x_i, y_j).

$$d(x_i, y_j) + \left\lceil \frac{f(x_i) + f(y_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{nt + mt + 1 + nt + mt + m + 1}{2} \right\rceil \geq 11$$

Hence $rmn\,(K_{m,n} \odot P_t) = (m+n)\,(t+1)$. □

The one point union of t cycles of length n is called the friendship graph and it is denoted by $C_n^{(t)}$.

Theorem 2.3 *For any integer $t \geq 2$,*

$$rmn\left(C_6^{(t)}\right) = \begin{cases} 5t+3 & \text{if} \quad t = 2 \\ 5t+2 & \text{if} \quad t = 3 \\ 5t+1 & \text{otherwise} \end{cases}$$

Proof Let $u_1^i u_2^i u_3^i u_4^i u_4^i u_5^i u_1^i$ be the i^{th} copy of the cycle $C_6^{(i)}$. Identify the vertex u_1^i $(1 \leq i \leq t)$. It is easy to verify that

$$diam\left(C_6^{(t)}\right) = \begin{cases} 3 & \text{if} \quad t = 1 \\ 6 & \text{otherwise} \end{cases}$$

Case 1 $t = 2$.

Claim 1 $rmn(C_6^{(2)}) \neq 5t + 1$.

Suppose $rmn(C_6^2) = 5t + 1$. Let f be the radio mean labeling of $C_6^{(2)}$ for which $rmn(f) = 5t + 1$. Then the vertices are labeled from the set $\{1, 2, \cdots, 5t + 1\}$. Clearly 1 and 2 should be labeled to the vertices with a distance at least 5. The possible vertices with label 1 and 2 are indicated in Figures 2 and 3.

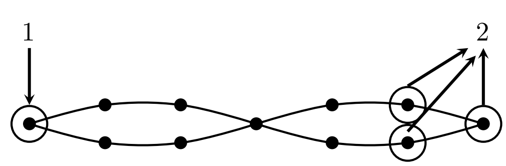

Figure 2

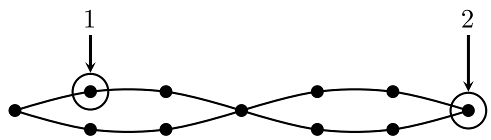

Figure 3

Clearly 2 and 3 are labeled at a distance at least 4 and 3 and 1 are labeled at a distance at least 5. There is no such vertex. Hence $rmn(C_6^{(2)}) \neq 5t+1$.

Claim 2 $rmn(C_6^{(2)}) \neq 5t+2$.

Suppose $rmn(C_6^{(2)}) = 5t+2$ then the vertices are labeled from the set $\{1, 2, \cdots, 5t+2\}$. If 1 is a label of a vertex then 3 and 4 are not labels of any vertices. Therefore the vertices are labeled from the set $\{2, 3, \cdots, 5t+2\}$. Note that 2 and 3 should be labeled to the vertices which are at a distance at least 4. Therefore 2 can not be a label of the identified vertex u_1^i. Suppose 2 is a label of the vertex u_2^i. This implies 3 should be a label of the vertex u_4^2. Then 4 can not be a label of any of the remaining vertices. If we put the label 2 to the vertex u_3^i, then 3 should be a label of either of the vertices u_3^2, u_4^2, u_5^2. In this case also 4 can not be a label of the remaining vertices. The same fact arises when 2 is a label of the vertex u_4^1. By symmetry, this is true for the other cases also. Hence we can not label the vertices of $C_6^{(2)}$ with the labels from the set $\{2, 3, \cdots, 5t+2\}$. Therefore $rmn(C_6^{(2)}) \neq 5t+2$.

Claim 3 $rmn(C_6^{(2)}) = 5t+3$.

The Figure 4 given below shows that the vertex labels are satisfies the radio mean condition.

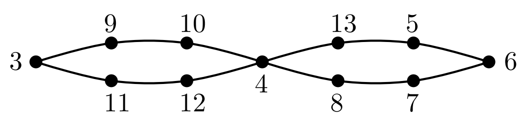

Figure 4

This implies $rmn(C_6^{(2)}) = 5t+3$.

Case 2. $t = 3$.

Claim 4 $rmn(C_6^{(3)}) \geq 5t+1$.

We observe that, for satisfying the radio mean condition, the labels 1, 2 and 3 are labels of the vertices of different cycles. Without loss of generality assume that 1 is a vertex label of the first copy of C_6, 2 is a vertex label of the second copy of C_6 and 3 is a vertex label of the third copy of C_6. Note that if 1 is a label of u_1^1 or u_2^1 then 24 can not be a label. If $f(u_3^1) = 1$ then 2 should be a label of u_4^3 . This implies 4 can not be a label of any of the remaining vertices. Suppose u_4^1 is labeled by 1. Then 2 is labeled by either one of the vertices u_3^2, u_5^2 or u_4^2. It follows that 3 should be a label of either u_3^3, u_5^3 or u_5^2 according as 2 is labeled. In either case 4 can not be a label of any of the vertices. Thus $rmn(C_6^{(3)}) \geq 5t+1$.

Claim 5 $rmn(C_6^{(3)}) \leq 5t+2$.

The vertex labeling given in figure 5 establish that it satisfies the radio mean condition and hence $rmn(C_6^{(3)}) \leq 5t + 2$.

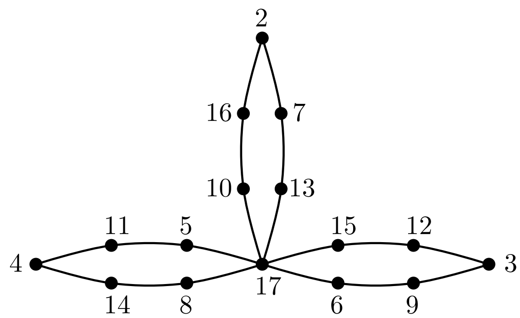

Figure 5

Therefore $rmn(C_6^{(3)}) = 5t + 2$.

Case 3. $t \neq 2, 3$.

When $t = 1$, the vertex labels given in Figure 6 satisfies the requirements.

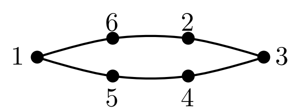

Figure 6

Hence $rmn(C_6^1) = 8$. Assume $t \geq 4$. Here we describe a labeling f as follows.

$$\begin{array}{lcll} f\left(u_4^i\right) & = & i & 1 \leq i \leq t \\ f\left(u_2^{t-i+1}\right) & = & t+i & 1 \leq i \leq t \\ f\left(u_6^{t-i+1}\right) & = & 2t+i & 1 \leq i \leq t \\ f\left(u_3^{t-i+1}\right) & = & 3t+i & 1 \leq i \leq t \\ f\left(u_5^{t-i+1}\right) & = & 4t+i & 1 \leq i \leq t \\ f\left(u_1^i\right) & = & 5t+1. & \end{array}$$

We now check whether the vertex labeling f is a valid labeling.

Case 3.1 Consider the pair (u_1^i, u_j^r).

$$d(u_1^i, u_j^r) + \left\lceil \frac{f(u_1^i) + f(u_j^r)}{2} \right\rceil \geq 2 + \left\lceil \frac{5t+1+1}{2} \right\rceil \geq 12$$

Case 3.2 Consider the pair (u_4^i, u_4^j).

$$d(u_4^i, u_4^j) + \left\lceil \frac{f(u_4^i) + f(u_4^j)}{2} \right\rceil \geq 6 + \left\lceil \frac{1+2}{2} \right\rceil \geq 8$$

Case 3.3 Consider the pair (u_4^i, u_2^i).

$$d(u_4^i, u_2^i) + \left\lceil \frac{f(u_4^i) + f(u_2^i)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+2t}{2} \right\rceil \geq 7$$

It is easy to verify that all the other pair of distinct vertices are also satisfies the radio mean condition. Hence $rmn(C_6^{(t)}) = 5t + 1$ where $t \neq 2, 3$. □

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Semientire Equitable Dominating Graphs

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Abstract: The semientire equitable dominating graph $SE_qD(G)$ of a graph $G = (V, E)$ is the graph with vertex set $V \cup S$, where S is the collection of all minimal equitable dominating sets of G and with two vertices $u, v \in V \cup S$ adjacent if $u, v \in D$, where D is the minimal equitable dominating set or $u \in V(G)$ and $v = D$ is a minimal equitable dominating set of G containing u. In this paper, some necessary and sufficient conditions are given for $SE_qD(G)$ to be connected and Eulerian. Finally, some bounds on domination number of $SE_qD(G)$ are obtained in terms of vertices and edges of G.

Key Words: Dominating set, equitable dominating set, semientire equitable dominating graph.

AMS(2010): 05C69

§1. Introduction

All graphs considered here are finite, undirected with no loops and multiple edges. As usual $p = |V(G)|$ and $q = |E(G)|$ denote the number of vertices and edges of a graph $G = (V, E)$ respectively. For any graph theoretic terminology and notations we refer to Harary [3] and for more details about parameters of domination number, we refer [4] and [6].

A set D of vertices in a graph G is called a *dominating* set of G if every vertex in $V - D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all minimal dominating sets of G. (See Ore [7]).

A subset D of V is called an *equitable dominating set* if for every $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such dominating sets is denoted by $\gamma^e(G)$ and called the *equitable domination number* of G [8].

In this paper, we use this idea to introduce a new graph valued function in the field of domination theory in graphs.

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§2. Semientire Equitable Dominating Graph

Definition 1 *Let $G = (V, E)$ be a graph. Let S be the collection of all minimal equitable dominating sets of G. The semientire equitable dominating graph $SE_qD(G)$ of a graph G is the graph with vertex set $V \cup S$ and two vertices $u, v \in V \cup S$ adjacent if $u, v \in D$, where D is a minimal equitable dominating set or $u \in V(G)$ and $v = D$ is a minimal equitable dominating set containing u.*

In Fig.1, a graph G and its semientire equitable dominating graph $SE_qD(G)$ are shown. Here $D_1 = \{1, 4, 5\}$, $D_2 = \{2, 4, 5\}$ and $D_3 = \{3, 4, 5\}$ are minimal equitable dominating sets of G.

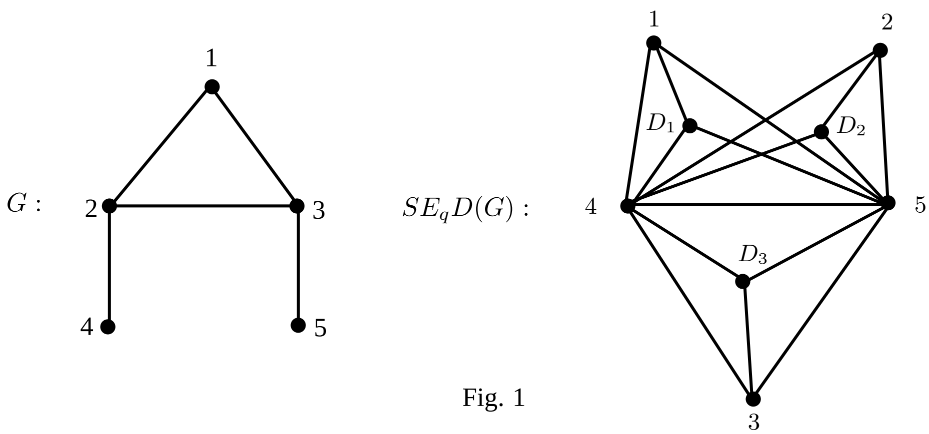

Fig. 1

§3. Results

Observation 1 In any graph G, the degree of a vertex D in $SE_qD(G)$ is the cardinality of minimal equitable dominating set D of G.

The following will be useful in the proof of our results.

Theorem A([2]) *Let G be a graph. If D is a maximal equitable independent set of G, then D is also a minimal equitable dominating set of G.*

Theorem 3.1 *For any nontrivial connected graph G, $\overline{G} \subseteq SE_qD(G)$.*

Proof Let u and v be any two adjacent vertices in $\overline{G}$ but which are not adjacent in G, then we can extend the set {u, v} into maximal equitable independent set D in G which is also a minimal equitable dominating set that is u and v are adjacent vertices in $SE_qD(G)$. Hence $\overline{G} \subseteq SE_qD(G)$. □

A subset D of V is called an *equitable independent set*, if for any $u \in D$, $v \notin N(u)$, for all $v \in D - \{u\}$. If a vertex $u \in V(G)$ be such that $|deg(u) - deg(v)| \geq 2$ for all $v \in N(u)$ then u is in each equitable dominating set. Such vertices are called *equitable isolates*.

First we obtain a necessary and sufficient condition on a graph G such that the semientire equitable dominating graph $SE_qD(G)$ is connected.

Theorem 3.2 *For any nontrivial connected graph G, the semientire equitable dominating graph $SE_qD(G)$ is connected if and only if $\Delta(G) \leq p-1$ and $\gamma^e(G) \geq 2$.*

Proof Let $\Delta(G) \leq p-1$ and u, v be any two vertices in G. Then we have the following cases.

Case 1 If u and v are not adjacent in G, then by Theorem 3.1, u is adjacent to v in $SE_qD(G)$.

Case 2 If u and v are adjacent in G and there is a vertex w in G which is not adjacent to both u and v, then u and v are joined by a path uwv in $SE_qD(G)$.

Case 3 Let u and v are adjacent in G and w is another vertex in G which is adjacent to both u and v, then there exist two maximal equitable independent sets D_1 and D_2 are minimal equitable dominating sets in G. Hence u and v connected through w in $SE_qD(G)$. From the above cases, we get $SE_qD(G)$ is connected.

Suppose $\gamma^e(G) = 1$. Then every vertex of G has $\Delta(G) = p-1$ and forms a minimal equitable dominating set except one vertex which is adjacent to all the other vertices in G. Therefore by definition, the semientire equitable dominating graph is disconnected, a contradiction. Hence $\gamma^e(G) \geq 2$.

Conversely, suppose $SE_qD(G)$ is connected. On the contrary $\gamma^e(G) = 1$. If G is a graph having $\Delta(G) \leq p-1$ with no equitable isolated vertices, then every vertex of G forms a minimal equitable dominating set D of G. This implies $SE_qD(G)$ is disconnected, a contradiction. Hence $\gamma^e(G) \geq 2$. □

Let k and $k+1$ be any two positive integers, $1 \leq k \leq k+1$. A graph G is said to be $(k, k+1)$ *bi-regular graph*, if its vertices have degree either k or $k+1$.

Theorem 3.3 *For any unicyclic graph G without isolated vertices, then $SE_qD(G)$ is a $(p+2, p-2)$ bi-regular graph.*

Proof Let G be a unicyclic graph of order p and contain no isolated vertices. Then from the definition of semientire equitable dominating graph, every vertex of $SE_qD(G)$ has the degree either $p+2$ or $p-2$. Hence $SE_qD(G)$ is a $(p+2, p-2)$ bi-regular graph. □

Remark 1 If T is a tree of order p, then $SE_qD(T)$ is a p-regular graph.

Proposition 3.1 *The semientire equitable dominating graph $SE_qD(G)$ is pK_2 if and only if $G = K_p$; $p \geq 2$.*

Proof Suppose $G = K_p; p \geq 2$. Then clearly each vertex of G will form a minimal equitable dominating set. Hence $SE_qD(G) = pK_2$.

Conversely, suppose $SE_qD(G) = pK_2$ and $G \neq K_p$. Then there exists at least one minimal equitable dominating set D containing two vertices of G. Then by the definition of semientire equitable dominating graph, D will form C_3 in $SE_qD(G)$. Hence $G = K_p; p \geq 2$. □

Theorem 3.4 *Let the semientire equitable dominating graph $SE_qD(G)$ is a graph with $2p$ vertices and p edges if and only if $G = K_p; p \geq 2$.*

Proof Suppose $G = K_p; p \geq 2$. Then by definition of $SE_qD(G)$, it is clear that $SE_qD(G)$ is a graph with $2p$ vertices and q edges.

Conversely, suppose $SE_qD(G)$ is a $(2p, p)$ graph. Then the graph pK_2 is the only graph with $2p$ vertices and p edges. Then by Proposition 3.1, $G = K_p; p \geq 2$. □

Corollary 1 *If $G = K_{1,n}$; $n \geq 3$, then $SE_qD(G) = K_{n+2}$.*

Theorem 3.5 *If G is a connected graph with $\Delta(G) < p - 1$, then $diam(SE_qD(G)) \leq 2$, where $diam(G)$ is the diameter of a graph G.*

Proof Let G be a nontrivial connected graph and by Theorem 3.2, $SE_qD(G)$ is connected. Let $u, v \in V(SE_qD(G))$ be any two arbitrary vertices. We consider the following cases.

Case 1 Suppose $u, v \in V(G)$, u and v are nonadjacent vertices in G.
Then $d_{SE_qD(G)}(u, v) = 1$. If u and v are adjacent in G and there is no minimal equitable dominating set containing both u and v. Then there exists another vertex w in $V(G)$, which is not adjacent to both u and v. Let D_1 and D_2 be any two equitable dominating sets containing u, w and v, w respectively. Hence u and v are connected in $SE_qD(G)$ by a path uwv. Thus $d_{SE_qD(G)}(u, v) \leq 2$.

Case 2 Suppose $u \in V(G)$ and $v \notin V(G)$. Then $v = D$ is a minimal equitable dominating set of G. If $u \in D$ then $d_{SE_qD(G)}(u, v) = 1$. If $u \notin D$, then there exist a vertex $w \in D$ which is adjacent to both u and v. Hence $d_{SE_qD(G)}(u, v) = d(u, w) + d(w, v) = 2$.

Case 3 Suppose $u, v \in V(G)$. Then $u = D$ and $v = D'$ are two minimal equitable dominating sets of G. If D and D' are disjoint, then every vertex in $w \in D$ is adjacent to some vertex $z \in D'$ and vice versa. This implies that

$$d_{SE_qD(G)}(u, v) = d(u, w) + d(w, z) + d(z, v) = 3.$$

If D and D' have a vertex in common, then $d_{SE_qD(G)}(u, v) = d(u, w) + d(w, v) = 2$. Thus from all these cases the result follows. □

The *equitable dominating graph* $E_qD(G)$ of a graph $G = (V, E)$ is the graph with vertex set $V \cup D$, where D is the set of all minimal equitable dominating sets of G and with two adjacent vertices $u, v \in V \cup D$ if $u \in V$ and v is a minimal equitable dominating set of G containing u.

Proposition 3.2([1]) *The equitable dominating graph $E_qD(G)$ is pK_2 if and only if $G = K_p; p \geq 2$.*

Theorem 3.6 *The equitable dominating graph is isomorphic to the semientire equitable dominating graph if and only if G is a nontrivial complete graph.*

Proof Let G be a nontrivial complete graph K_p. Then from Proposition 3.2, $E_qD(G) = pK_2$, and we have Proposition 3.1, Hence $E_qD(G) = SE_qD(G) = pK_2$.

Conversely, suppose $E_qD(G) = SE_qD(G)$, Propositions 3.1 and 3.2, G must be complete graph. Hence $G = K_p; p \geq 2$. $\square$

We need the following theorem for the proof of our next results.

Theorem B([3]) *A connected graph G is eulerian if and only if every vertex of G has even degree.*

Next, we prove the necessary and sufficient condition for $SE_qD(G)$ to be Eulerian.

Theorem 3.7 *For any graph G with no isolated vertices, $SE_qD(G)$ is Eulerian if and only if the cardinality of each minimal equitable dominating set is even.*

Proof Let $\Delta(G) \leq p-1$ and $\gamma^e(G) \geq 2$, by Theorem 3.2, $SE_qD(G)$ is connected. Suppose $SE_qD(G)$ is Eulerian. On the contrary, every minimal equitable dominating set contains odd number of vertices and by observation 1, hence $SE_qD(G)$ has a vertex of odd degree, therefore by Theorem B, $SE_qD(G)$ is not Eulerian. Hence the cardinality of minimal equitable dominating set is even.

Conversely, suppose the cardinality of minimal equitable dominating set is even. Then degree of each vertex in $SE_qD(G)$ is even. Therefore by Theorem B, $SE_qD(G)$ is Eulerian. $\square$

\$4. **Domination in** $SE_qD(G)$

We first calculate the domination number of $SE_qD(G)$ of some standard class of graphs.

Theorem 4.1 *Let G be a graph without isolated vertices. Then,*

1. if $G = K_p; p \geq 2$, then $\gamma(SE_qD(K_p) = p$.
2. if $G = K_{1,p}; p \geq 1$, then $\gamma(SE_qD(K_{1,p}) = 1$.
3. if $G = P_p$, $p \geq 2$, then $\gamma(SE_qD(P_p) = 2$.
4. if $G = C_p; p \geq 4$, then $\gamma(SE_qD(C_p) = 3$.
5. if $G = W_p; p \geq 5$, then $\gamma(SE_qD(W_p) = 1$.

Theorem 4.2 *Let G be any graph of order p and $S = \{S_1, S_2, S_3, \cdots S_n\}$ be the minimal equitable dominating set of G, then $\gamma(SE_qD(G)) \leq \gamma(\overline{G}) + |S|$.*

Proof Let G be a connected graph. Let $D = \{v_1, v_2, v_3, \cdots v_i\}$; $1 \leq i \leq p$ be the set of all minimal equitable dominating sets of $\overline{G}$. By the definition of $SE_qD(G)$, each S_i; $1 \leq i \leq p$ is independent in $SE_qD(G)$. Hence $D' = D \cup S$ will form a dominating set in $SE_qD(G)$. Therefore $\gamma(SE_qD(G)) \leq |D'| = |D \cup S| = \gamma(\overline{G}) + |S|$. $\square$

Further, we get the Nordhaus-Gaddum type result for semientire equitable dominating graph.

Theorem 4.3 *Let G be a graph such that both G and $\overline{G}$ are connected of order $p \geq 2$. Then*

1. $\gamma(SE_qD(G)) + \gamma(SE_qD(\overline{G})) \leq p$.

2. $\gamma(SE_qD(G)).\gamma(SE_qD(\overline{G})) \leq 2p$.

Further, the equality holds good if and only if $G = P_4$.

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Friendly Index Sets and Friendly Index Numbers of Some Graphs

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Consider the set $A = \{0, 1\}$. A labeling $f : V(G) \rightarrow A$, induces a partial edge labeling $f^* : E(G) \rightarrow A$, defined by $f^*(xy) = f(x)$ if and only if $f(x) = f(y)$ for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and we denote $e_{Bf^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. In this paper we define friendly index number(FIN) and full friendly index number(FFIN) of graph G as the cardinality of the distinct elements of friendly index set and full friendly index set respectively and obtaining these numbers along with their sets of some families graphs.

Key Words: Friendly index set, full friendly index set, friendly index number and full friendly index number, Smarandache friendly index number.

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§1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V, E)$. Here elements of set V and E are known as vertices and edges respectively. For all other terminologies and notations we follow Harary [2].

In 1986 Cahit [1] introduced cordial graph labeling. A function f from $V(G)$ to $\{0, 1\}$, where for each edge xy, $f^*(xy) = |f(x) - f(y)|$, $v_f(i)$ is the number of vertices v with $f(v) = i$ and $e_{f^*}(i)$ is the number of edges e with $f^*(e) = i$, is called friendly if $|v_f(1) - v_f(0)| \leq 1$. A friendly labeling f is called cordial if $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$.

In [6] Lee and Ng defined the friendly index set of a graph G as $FI(G) = \{|e_{f^*}(1) - e_{f^*}(0)| :$ f^* runs over all friendly labeling f of $G\}$. The concept was extended by Harris and Kwong [7] to full friendly index set for the graph G, denoted $FFI(G)$, defined as $FFI(G) = \{e_{f^*}(1) - e_{f^*}(0)$ $: f^*$ runs over all friendly labeling f of $G\}$.

Lee, Liu and Tan [5] considered a new labeling problem of graph theory. A vertex labeling of G is a mapping f from $V(G)$ into the set $\{0, 1\}$. For each vertex labeling f of G, a partial edge labeling f^* of G is defined in the following way.

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For each edge uv in G,

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0 \\ 1, & \text{if } f(u) = f(v) = 1 \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge uv is not labeled by f^*. Thus f^* is a partial function from $E(G)$ into the set $\{0, 1\}$. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_{f^*}(0)$ and $e_{f^*}(1)$ denote the number of edges of G that are labeled by 0 and 1 under the induced partial function f^* respectively.

In [4] Kim, Lee, and Ng defined the balance index set of a graph G as $\mathrm{BI}(G) = \{|e_{f^*}(1) - e_{f^*}(0)| : f^*$ runs over all friendly labelings f of G $\}$.

Definition 1.1 *The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as a graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 1.2 *The crown $C_n \odot K_1$ is obtained by joining a pendant edge to each vertex of C_n.*

Definition 1.3 *A chord of cycle C_n is an edge joining two non-adjacent vertices of cycle C_n.*

Definition 1.4 *The shell S_n is the graph obtained by taking $n-3$ concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan f_{n-1}. Thus $S_n = f_{n-1} = P_{n-1} + K_1$.*

Definition 1.5 *The wheel W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as apex vertex, the vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges and edges joining apex and vertices of cycle are spoke edges.*

Definition 1.6 *The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 1.7 *The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.*

More details of known results of graph labelings given in Gallian [3].

In number theory and combinatorics, a partition of a positive integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered to be the same partition; if order matters then the sum becomes a composition. For example, 4 can be partitioned in five distinct ways

$$4 + 0, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.$$

In this paper we are using the idea of integer partition of numbers. Let G be any graph with p vertices. Partition of p in to (p_0, p_1), where p_0 and p_1 are the number of vertices labeled by 0 and 1 respectively.

§2. Main Results

Here we are introducing two new parameters $e_{Bf^*}(i)$ and $e_{Ff^*}(i)$, which are the number of edges labeled i under balanced labeling and cordial labeling respectively. While proving our results, FI(G) and FFI(G) are used as below:

$FI(G) = \{|e_{Ff^*}(1) - e_{Ff^*}(0)| : Ff^*$ runs over all friendly labeling f of $G\}$;
$FFI(G) = \{e_{Ff^*}(1) - e_{Ff^*}(0) : Ff^*$ runs over all friendly labeling f of $G\}$.

Theorem 2.1 *Let $G(V,E)$ be a graph with $|E(G)| = q$ and $e_{Bf^*}(i)$ is the number of edges labeled i under the balanced labeling, where $i = 0, 1$. Then*

(1) $FI(G) = \{|q - 2(e_{Bf^*}(0) + e_{Bf^*}(1))|$: *the partial edge labeling Bf^* runs over all friendly labeling f of $G\}$;*

(2) $FFI(G) = \{q - 2(e_{Bf^*}(0) + e_{Bf^*}(1))$: *the partial edge labeling Bf^* runs over all friendly labeling f of $G\}$.*

Definition 2.2 *For a graph G with a subgraph $H \leq G$, the Smarandache friendly index number $SFIN$ is the number of distinct elements runs over all labeling $f : V(G) \to A$ with friendly index set $FIN(H)$, particularly, if $H = G$, such number is called friendly index number on G and denoted by FIN.*

Definition 2.2 *The full friendly index number is the number of distinct elements in the full friendly index set and it is denoted as FFIN.*

We are using Theorem 2.1 to prove the following results.

Theorem 2.4 *In a shell graph S_n with $n \geq 4$ vertices,*

$$FI(S_n) = \begin{cases} \{1, 3, 5, \cdots, n-2\}, & \text{if } n \text{ is odd} \\ \{1, 3, 5, \cdots, n-1\}, & \text{if } n \text{ is even} \end{cases}$$

Proof In a shell graph S_n, $|V(S_n)| = n$ and $|E(S_n)| = 2n - 3$.

Case 1 n is odd.

To satisfy friendly labeling, the possible compositions of n are

$$\left(\frac{n-1}{2}, \frac{n+1}{2}\right) \text{ and } \left(\frac{n+1}{2}, \frac{n-1}{2}\right).$$

Consider the composition $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$ of n. If the apex vertex labeled 0, then $e_{Bf^*}(0) = \frac{n-3}{2} + i$, where $i = 0, 1, 2, \cdots, \frac{n-5}{2}$; $e_{Bf^*}(1) = j$, where $j = i+1, i+2, i+3, \cdots, \frac{n-1}{2}$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = \left|(2n-3) - 2\left(\frac{n-3}{2} + i + j\right)\right| = |n - 2(i+j)|,$$

where $i=0,1,2,\cdots,\dfrac{n-5}{2}$ and $j=i+1,i+2,i+3,\cdots,\dfrac{n-1}{2}$. If we consider the composition $\left(\dfrac{n+1}{2},\dfrac{n-1}{2}\right)$ of n and the apex vertex labeled 0, then $e_{Bf^*}(0)=\dfrac{n-1}{2}+i$, where $i=0,1,2,\cdots,\dfrac{n-3}{2}$; $e_{Bf^*}(1)=j$, where if $i=0,1,2,\cdots,\dfrac{n-5}{2}$, then $j=0,1,2,\cdots,\dfrac{n-3}{2}$; if $i=\dfrac{n-3}{2}$, then $j=0,1,2,\cdots,\dfrac{n-5}{2}$. Therefore,

$$|e_{Ff^*}(1)-e_{Ff^*}(0)|=\left|(2n-3)-2\left(\frac{n-1}{2}+i+j\right)\right|=|n-2(i+j+1)|,$$

where if $i=0,1,2,\cdots,\dfrac{n-5}{2}$, then $j=0,1,2,\cdots,\dfrac{n-3}{2}$; if $i=\dfrac{n-3}{2}$, then $j=0,1,2,\cdots,\dfrac{n-5}{2}$.

Considering all possible values of i and j, we get $\mathrm{BI}(S_n)=\{1,3,5,\cdots,n-2\}$. Also if the apex vertex labeled 1, then $\mathrm{FI}(S_n)$ will be same.

Case 2 n is even.

To satisfy friendly labeling, the possible partition of n is $\left(\dfrac{n}{2},\dfrac{n}{2}\right)$. If the apex vertex labeled 0, then, $e_{Bf^*}(0)=\dfrac{n}{2}-1+i$, where $i=0,1,2,\cdots,\dfrac{n}{2}-2$; $e_{f^*}(1)=j$, where if $i=0$, then $j=i,i+1,i+2,\cdots,\dfrac{n}{2}-1$; if $i=1,2,\cdots,\dfrac{n}{2}-2$, then $j=i+1,i+2,i+3,\cdots,\dfrac{n}{2}-1$. Therefore,

$$|e_{Ff^*}(1)-e_{Ff^*}(0)|=\left|(2n-3)-2\left(\frac{n}{2}-1+i+j\right)\right|=|n-(2i+2j+1)|,$$

where if $i=0$, then $j=i,i+1,i+2,\cdots,\dfrac{n}{2}-1$; if $i=1,2,\cdots,\dfrac{n}{2}-2$, then $j=i+1,i+2,i+3,\cdots,\dfrac{n}{2}-1$.

Considering all possible values of i and j, we get $\mathrm{FI}(S_n)=\{1,3,5,\cdots,n-1\}$. Also if the apex vertex labeled 1, then $\mathrm{FI}(S_n)$ will be same. □

Corollary 2.5 *The graph S_n is cordial.*

Corollary 2.6 *The friendly index set of the graph S_n forms an arithmetic progression with common difference* 2.

Corollary 2.7

$$FIN(S_n)=\begin{cases}\dfrac{n-1}{2}, & \text{if } n \text{ is odd}\\[2ex] \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases}$$

Corollary 2.8 *In a shell graph S_n with $n\geq 4$ vertices,*

$$FFI(S_n)=\begin{cases}\{-n+6,-n+8,-n+10,...,n-2\}, & \text{if } n \text{ is odd}\\ \{-n+5,-n+7,-n+9,...,n-1\}, & \text{if } n \text{ is even}\end{cases}$$

Corollary 2.9

$$FFIN(S_n) = \begin{cases} n-3, & \textit{if } n \textit{ is odd} \\ n-2, & \textit{if } n \textit{ is even} \end{cases}$$

Corollary 2.10 *The full friendly index set of the graph S_n forms an arithmetic progression with common difference* 2.

Example 2.11 Friendly index set of shell graph S_5 is $\{1,3\}$.

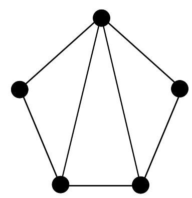

Figure 1: The shell graph S_5

Table 1: Compositions of integer 5 for friendly labeling with elements of friendly index set.

Compositions of integer 5	Corresponding elements friendly index set
(2, 3)	1, 3
(3, 2)	1, 3

Theorem 2.12 *In a crown graph $C_n \odot K_1$ with $n \geq 3$,*

$$FI(C_n \odot K_1) = \begin{cases} \{0,4,8,\cdots,2n\}, & \textit{if } n \textit{ is even} \\ \{0,4,8,\cdots,2n-2\}, & \textit{if } n \textit{ is odd} \end{cases}$$

Proof Consider the crown graph $C_n \odot K_1$, $|V(C_n \odot K_1)| = 2n$ and $|E(C_n \odot K_1)| = 2n$.

Case 1 n is even.

To satisfy friendly labeling, the possible partitions of number of vertices of cycle and pendent vertices of $C_n \odot K_1$ are $(n-i,i)$ and $(i,n-i)$, where $i = 0,1,2,\cdots,\frac{n}{2}$.

If i=0, then $e_{Ff^*}(0) = n$ and $e_{Ff^*}(1) = n$. Therefore friendly index is '0'. If $i = 1,2,3,\cdots,\frac{n}{2}$, then $e_{Bf^*}(0) = n-i-1-j+k$, where $j = 0,1,2,\cdots,i-1$ and $k = 0,1,2,\cdots,i$; $e_{Bf^*}(1) = l+k$, where $l = 0,1,2,\cdots,i-1$ and $k = 0,1,2,\cdots,i$ such that $j+l = i-1$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |2n - 2[(n-i-1-j+k)+(l+k)]| = |4(i-l-k)|,$$

where if $i = 1, 2, \cdots, \frac{n}{2}$, then $l = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$.

Considering all possible values of i, l and k, we get FI$=\{0, 4, 8, \cdots, 2n\}$.

Case 2 n is odd.

To satisfy friendly labeling, the possible partitions of number of vertices of cycle and pendent vertices of $C_n \odot K_1$ are $(n-i, i)$ and $(i, n-i)$, where $i = 0, 1, 2, \cdots, \frac{n-1}{2}$.

If i=0, then $e_{Ff^*}(0) = n$ and $e_{Ff^*}(1) = n$. Therefore friendly index is $'0'$. If $i = 1, 2, 3, \cdots, \frac{n-1}{2}$, then $e_{Bf^*}(0) = n-i-1-j+k$, where $j = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$, $e_{Bf^*}(1) = l+k$, where $l = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$ such that $j+l = i-1$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |2n - 2[(n-i-1-j+k) + (l+k)]| = |4(i-l-k)|,$$

where $i = 1, 2, \cdots, \frac{n-1}{2}$, $l = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$.

Considering all possible values of i, l and k, we get FI$(C_n \odot K_1) = \{0, 4, 8, ..., 2n-2\}$. $\square$

Corollary 2.13 *The graph $C_n \odot K_1$ is cordial.*

Corollary 2.14 *The friendly index set of the graph $C_n \odot K_1$ forms an arithmetic progression with common difference* 4.

Corollary 2.15

$$FIN(C_n \odot K_1) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}+1, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.16 *In a crown graph $C_n \odot K_1$ with $n \geq 3$,*

$$FFI(C_n \odot K_1) = \begin{cases} \{-2n+4, -2n+8, -2n+12, ..., 2n\}, & \text{if } n \text{ is even} \\ \{-2n+6, -2n+10, -2n+14, ..., 2n-2\}, & \text{if } n \text{ is odd} \end{cases}$$

Corollary 2.17 *The full friendly index set of the graph $C_n \odot K_1$ forms an arithmetic progression with common difference* 4.

Corollary 2.18

$$FFIN(C_n \odot K_1) = \begin{cases} n, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$$

Example 2.19 Friendly index set of crown graph $C_5 \odot K_1$ is $\{0, 4, 8\}$.

Theorem 2.20 *In a helm graph H_n,*

$$FI(H_n) = \begin{cases} \{1, 3, 5, \cdots, 2n-1\}, & \text{if } n \text{ is odd} \\ \{0, 2, 4, \cdots, 2n\}, & \text{if } n \text{ is even} \end{cases}$$

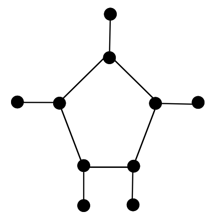

Figure 2: The crown graph $C_5 \cdot K_1$

Table 2: Compositions of integer 5 for friendly labeling with elements of friendly index set.

Partition of integers 5 and 5	Corresponding elements of friendly index set
(5, 0) and (0, 5)	0
(4, 1) and (1, 4)	0, 4
(3, 2) and (2, 3)	0, 4, 8

Proof Consider the helm graph H_n. $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$.

Case 1 n is odd.

First we label the apex vertex as 0.

Subcase 1.1 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n, 0)$ and $(0, n)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = n$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = n.$$

Subcase 1.2 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-i, i)$ and $(i, n-i)$, where $i = 1, 2, 3, \cdots, n-1$, respectively. Then $e_{Bf^*}(0) = (n-j)+(n-i)+l$, where if $i = 1, 2, 3, \cdots, \frac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-i$; $e_{Bf^*}(1) = k + l$, where if $i = 1, 2, 3, \cdots, \frac{n-1}{2}$, then $k = 0, 1, 2, \cdots, i-1$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n-1$, then $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0, 1, 2, \cdots, n-i$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |2(i+j-k-2l) - n|,$$

where if $i = 1, 2, 3, \cdots, \frac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$,$k = 0, 1, 2, \cdots, i-1$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$, $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0, 1, 2, \cdots, n-i$ such that $j + k = 2i$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |n + 2i - 4j + 4l|,$$

where if $i = 1, 2, 3, \cdots, \dfrac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n+1}{2}, \dfrac{n+3}{2}, \dfrac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-i$.

Subcase 1.3 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n, 0)$ respectively, then $e_{Ff^*}(0) = n$ and $e_{Ff^*}(1) = 2n$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = n.$$

Subcase 1.4 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-(i+1), i+1)$ and $(i, n-i)$, where $i = 0, 1, 2, \cdots, n-1$ respectively. Then $e_{Bf^*}(0) = (n-j)+(n-(i+1))+l$, where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $j = i+2, i+3, i+4, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$; $e_{Bf^*}(1) = k+l+1$, where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |3n - 2[(n-j)+(n-(i+1))+k+2l+1]|,$$

where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$, $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $j = i+2, i+3, i+4, \cdots, n$, $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$ such that $j+k = 2(i+1)$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |n + 2i - 4j + 4l + 4|,$$

where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-2$, then $j = i+2, i+3, i+4, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$.

Subcase 1.5 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n-1, 1)$ respectively, then $e_{Ff^*}(0) = n+1$ and $e_{Ff^*}(1) = 2n-1$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = n-2.$$

Considering all the above sub cases and all possible values of i, j and l, we get $BI(H_n) = \{1, 3, 5, \cdots, 2n-1\}$. If we label the apex vertex as 1 and considering all possible compositions of number of vertices for friendly labeling, then also the friendly index set will be same.

Case 2 n is even.

First we label the apex vertex as 0.

Subcase 2.1 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n, 0)$ and $(0, n)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = n$.

Subcase 2.2 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-i,i)$ and $(i,n-i)$, where $i=1,2,3,\cdots,n-1$, respectively. Then $e_{Bf^*}(0)=(n-j)+(n-i)+l$, where if $i=1,2,3,\cdots,\frac{n}{2}$, then $j=i+1,i+2,i+3,\cdots,2i$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2}+1,\frac{n}{2}+2,\frac{n}{2}+3,\cdots,n-1$, then $j=i+1,i+2,i+3,\cdots,n$ and $l=0,1,2,\cdots,n-i$; $e_{Bf^*}(1)=k+l$, where if $i=1,2,3,\cdots,\frac{n}{2}$, then $k=0,1,2,\cdots,i-1$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2}+1,\frac{n}{2}+2,\frac{n}{2}+3,\cdots,n-1$, then $k=2i-n,2i-(n-1),2i-(n-2),\cdots,i-1$ and $l=0,1,2,\cdots,n-i$ such that $j+k=2i$. Therefore, $|e_{Ff^*}(1)-e_{Ff^*}(0)|$ $=|2(i+j-k-2l)-n|$, where if $i=1,2,3,\cdots,\frac{n}{2}$, then $j=i+1,i+2,i+3,\cdots,2i$, $k=0,1,2,\cdots,i-1$ and $l=0,1,2,3,\cdots,i$; if $i=\frac{n}{2}+1,\frac{n}{2}+2,\frac{n}{2}+3,\cdots,n-1$, then $j=i+1,i+2,i+3,\cdots,n$, $k=2i-n,2i-(n-1),2i-(n-2),\cdots,i-1$ and $l=0,1,2,\cdots,n-i$ such that $j+k=2i$. Therefore, $|e_{Ff^*}(1)-e_{Ff^*}(0)|=|n+2i-4j+4l|$, where if $i=1,2,3,\cdots,\frac{n}{2}$, then $j=i+1,i+2,i+3,\cdots,2i$ and $l=0,1,2,3,\cdots,i$; if $i=\frac{n}{2}+1,\frac{n}{2}+2,\frac{n}{2}+3,\cdots,n-1$, then $j=i+1,i+2,i+3,\cdots,n$ and $l=0,1,2,\cdots,n-i$.

Subcase 2.3 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0,n)$ and $(n,0)$, respectively, then $e_{Ff^*}(0)=n$ and $e_{Ff^*}(1)=2n$. Therefore, $|e_{Ff^*}(0)-e_{Ff^*}(1)|$ $=n$.

Subcase 2.4 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-(i+1),i+1)$ and $(i,n-i)$, where $i=0,1,2,\cdots,n-1$, respectively. Then $e_{Bf^*}(0)=(n-j)+(n-(i+1))+l$, where if $i=0,1,2,\cdots,\frac{n}{2}-1$, then $j=i+2,i+3,i+4,\cdots,2(i+1)$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-1$, then $j=i+2,i+3,\cdots,n$ and $l=0,1,2,\cdots,n-(i+1)$; $e_{Bf^*}(1)=k+l+1$, where if $i=0,1,2,\cdots,\frac{n}{2}-1$, then $k=0,1,2,\cdots,i$; if $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-1$, then $k=2(i+1)-n,2i-(n-1),2i-(n-2),\cdots,i$ and $l=0,1,2,\cdots,n-(i+1)$. Therefore, $|e_{Ff^*}(1)-e_{Ff^*}(0)|=|3n-2[(n-j)+(n-(i+1))+k+2l+1]|$, where if $i=0,1,2,\cdots,\frac{n}{2}-1$, then $j=i+2,i+3,i+4,\cdots,2(i+1)$, $k=0,1,2,\cdots,i$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-1$, then $j=i+2,i+3,i+4,\cdots,n$, $k=2(i+1)-n,2(i+1)-(n-1),2(i+1)-(n-2),\cdots,i$ and $l=0,1,2,\cdots,n-(i+1)$ such that $j+k=2(i+1)$. Therefore, $|e_{Ff^*}(1)-e_{Ff^*}(0)|=|n+2i-4j+4l+4|$, where if $i=0,1,2,\cdots,\frac{n}{2}-1$, then $j=i+2,i+3,i+4,\cdots,2(i+1)$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-2$, then $j=i+2,i+3,i+4,\cdots,n$ and $l=0,1,2,\cdots,n-(i+1)$.

Subcase 2.5 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0,n)$ and $(n-1,1)$, respectively, then $e_{Ff^*}(0)=n+1$ and $e_{f^*}(1)=2n-1$. Therefore $|e_{Ff^*}(1)-e_{Ff^*}(0)|=n-2$. Considering all the above sub cases and all possible values of i, j and l, we get $BI(H_n)=\{0,2,4,\cdots,2n\}$.

If we label the apex vertex as 1 and considering all possible compositions of number of vertices for friendly labeling , then also the friendly index set will be same. □

Corollary 2.21 *The graph H_n is cordial.*

Corollary 2.22 *The friendly index set of helm graph H_n forms an arithmetic progression with* *common difference* 2.

Corollary 2.23 *In a helm graph H_n,*

$$FIN(H_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.24 *In a helm graph H_n,*

$$FFI(H_n) = \begin{cases} \{-2n+5, -2n+7, -2n+9, \cdots, 2n-1\}, & \text{if } n \text{ is odd} \\ \{-2n+6, -2n+8, -2n+10, \cdots, 2n\}, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.25 *The full friendly index set of helm graph H_n forms an arithmetic progression with common difference* 2.

Corollary 2.26 *In a Helm graph H_n, $FFIN(H_n) = 2n-2$.*

Table 3: Compositions of number of rim vertices of wheel and pendent vertices of H_5 for friendly labeling with elements of friendly index set.

Compositions of integers 5 and 5	Corresponding elements of friendly index set
(5, 0) and (0, 5)	5
(4, 1) and (1, 4)	1, 3
(3, 2) and (2, 3)	1, 3, 5, 7
(2, 3) and (3, 2)	1, 3, 5, 9
(1, 4) and (4, 1)	3, 7
(0, 5) and (5, 0)	5
(4, 1) and (0, 5)	1
(3, 2) and (1, 4)	1, 3, 5
(2, 3) and (2, 3)	1, 3, 5, 7
(1, 4) and (3, 2)	1, 5
(0, 5) and (4, 1)	3

Example 2.27 Friendly index set of helm graph H_5 is $\{1, 3, 5, 7, 9\}$.

Theorem 2.28 *In a flower graph Fl_n,*

$$FI(Fl_n) = \begin{cases} \{0, 4, 8, \cdots, 2n-2\}, & \text{if } n \text{ is odd} \\ \{0, 4, 8, \cdots, 2n\}, & \text{if } n \text{ is even} \end{cases}$$

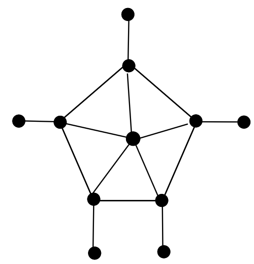

Figure 3: The helm graph H_5

Proof Consider the flower graph Fl_n. $|V(Fl_n)| = 2n+1$ and $|E(Fl_n)| = 4n$.

Case 1 n is odd.

First we label the apex vertex as 0.

Subcase 1.1 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n,0)$ and $(0,n)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Subcase 1.2 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-i,i)$ and $(i,n-i)$, where $i = 1,2,3,\cdots,n-1$, respectively. Then $e_{Bf^*}(0) = (n-j)+(n-i)+i+l$, where if $i = 1,2,3,\cdots,\dfrac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0,1,2,\cdots,i$; if $i = \dfrac{n+1}{2}, \dfrac{n+3}{2}, \dfrac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0,1,2,\cdots,n-i$; $e_{Bf^*}(1) = k+l$, where if $i = 1,2,3,\cdots,\dfrac{n-1}{2}$, then $k = 0,1,2,\cdots,i-1$ and $l = 0,1,2,\cdots,i$; if $i = \dfrac{n+1}{2}, \dfrac{n+3}{2}, \dfrac{n+5}{2}, \cdots, n-1$, then $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0,1,2,\cdots,n-i$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4n - 2((n-j)+(n-i)+i+k+2l)|$, where if $i = 1,2,3,\cdots,\dfrac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$, $k = 0,1,2,\cdots,i-1$ and $l = 0,1,2,\cdots,i$; if $i = \dfrac{n+1}{2}, \dfrac{n+3}{2}, \dfrac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$, $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0,1,2,\cdots,n-i$ such that $j+k = 2i$. Therefore $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 4\,|i-j+l|$, where if $i = 1,2,3,\cdots,\dfrac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0,1,2,\cdots,i$; if $i = \dfrac{n+1}{2}, \dfrac{n+3}{2}, \dfrac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0,1,2,\cdots,n-i$.

Subcase 1.3 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0,n)$ and $(n,0)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Subcase 1.4 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-(i+1), i+1)$ and $(i, n-i)$, where $i = 0,1,2,\cdots,n-1$, respectively. Then $e_{Bf^*}(0) = (n-j)+(n-(i+1))+i+l$, where if $i = 0,1,2,\cdots,\dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0,1,2,\cdots,i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $j = i+2, i+3, \cdots, n$ and $l = 0,1,2,\cdots,n-(i+1)$; $e_{Bf^*}(1) = k+l+1$, where if $i =$

$0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $k = 2(i+1) - n, 2(i+1) - (n-1), 2(i+1) - (n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$. Therefore $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4n - 2((n-j) + (n-(i+1)) + i + 2l + k + 1)|$, where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$, $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $j = i+2, i+3, i+4, \cdots, n$, $k = 2(i+1) - n, 2(i+1) - (n-1), 2(i+1) - (n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$ such that $j + k = 2(i+1)$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4(i - j + l + 1)|$, where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-2$, then $j = i+2, i+3, i+4, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$.

Subcase 1.5 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n-1, 1)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Considering all the above sub cases and all possible values of i, j and l, we get $FI(Fl_n) = \{0, 4, 8, \cdots, 2n-2\}$. If we label the apex vertex '$1'$' and consider all possible compositions of number of vertices for friendly labeling , then the balance index set will be same.

Case 2. n is even.

First we label the apex vertex as '$0'$'.

Subcase 2.1 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n, 0)$ and $(0, n)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Subcase 2.2 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-i, i)$ and $(i, n-i)$, where $i = 1, 2, 3, \cdots, n-1$, respectively. Then $e_{Bf^*}(0) = (n-j) + (n-i) + i + l$, where if $i = 1, 2, 3, \cdots, \dfrac{n}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n}{2}+1, \dfrac{n}{2}+2, \dfrac{n}{2}+3, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-i$; $e_{Bf^*}(1) = k + l$, where if $i = 1, 2, 3, \cdots, \dfrac{n}{2}$, then $k = 0, 1, 2, \cdots, i-1$ and $l = 0, 1, 2, ..., i$; if $i = \dfrac{n}{2}+1, \dfrac{n}{2}+2, \dfrac{n}{2}+3, \cdots, n-1$, then $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0, 1, 2, \cdots, n-i$. Therefore, $|e_{Ff^*}(0) - e_{Ff^*}(1)| = |4n - 2((n-j) + (n-i) + i + k + 2l)|$, where if $i = 1, 2, 3, \cdots, \dfrac{n}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$, $k = 0, 1, 2, \cdots, i-1$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n}{2}+1, \dfrac{n}{2}+2, \dfrac{n}{2}+3, \cdots, n-1$, then $j = i+1, i+2, i+3, ..., n$, $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0, 1, 2, \cdots, n-i$ such that $j + k = 2i$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 4\,|i - j + l|$, where if i=1, 2, 3, . . .,$\dfrac{n}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n}{2}+1, \dfrac{n}{2}+2, \dfrac{n}{2}+3, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-i$.

Subcase 2.3 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n, 0)$, respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Subcase 2.4 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-(i+1), i+1)$ and $(i, n-i)$, where $i = 0, 1, 2, \cdots, n-1$, respectively. Then $e_{Bf^*}(0) = (n-j)+(n-(i+1))+i+l$, where if $i = 0, 1, 2, \cdots, \frac{n}{2}-1$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$, then $j = i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$; $e_{Bf^*}(1) = k+l+1$, where if $i = 0, 1, 2, \cdots, \frac{n}{2}-1$, then $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$, then $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4n - 2((n-j) + (n-(i+1)) + i + 2l + k + 1)|$, where if $i = 0, 1, 2, \cdots, \frac{n}{2} - 1$, then $j = i + 2, i+3, i+4, \cdots, 2(i+1)$, $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, ..., i$; if $i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$, then $j = i+2, i+3, i+4, \cdots, n$, $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$ such that $j+k = 2(i+1)$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4(i-j+l+1)|$, where if $i = 0, 1, 2, \cdots, \frac{n}{2} - 1$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-2$, then $j = i+2, i+3, i+4, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$.

Subcase 2.5 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n-1, 1)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Considering all the above subcases and all the possible values of i, j and l, we get $FI(Fl_n) = \{0, 4, 8, \cdots, 2n\}$. If we label the apex vertex '1' and consider all possible compositions of number of vertices for friendly labeling, then the balance index set will be same. □

Corollary 2.29 *The flower graph Fl_n is cordial.*

Corollary 2.30 *The friendly index set of the graph Fl_n forms an arithmetic progression with common difference* 4.

Corollary 2.31 *In a flower graph Fl_n,*

$$FIN(Fl_n) = \begin{cases} \dfrac{n+3}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n+2}{2}, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.32 *In a flower graph Fl_n,*

$$FFI(Fl_n) = \begin{cases} \{-2n+6, -2n+10, -2n+14, \cdots, 2n-2\}, & \text{if } n \text{ is odd} \\ \{-2n+4, -2n+8, -2n+12, \cdots, 2n\}, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.33 *The full friendly index set of the flower graph Fl_n forms an arithmetic progression with common difference* 4.

Corollary 2.34 *In a flower graph Fl_n,*

$$FFIN(Fl_n) = \begin{cases} n-1, & \text{if } n \text{ is odd} \\ n, & \text{if } n \text{ is even} \end{cases}$$

Example 2.35 Friendly index set of flower graph Fl_5 is $\{0, 4, 8\}$.

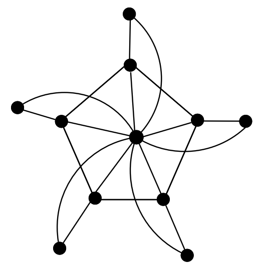

Figure 4: The flower graph Fl_5

Table 4: Compositions of number of rim vertices of wheel and number of vertices with degree two of Fl_5 for friendly labeling and corresponding elements of friendly index set.

Compositions of integers 5 and 5	Corresponding elements of friendly index set
(5, 0) and (0, 5)	0
(4, 1) and (1, 4)	0,4
(3, 2) and (2, 3)	0, 4, 8
(2, 3) and (3, 2)	0, 4, 8
(1, 4) and (4, 1)	0, 4
(0, 5) and (5, 0)	0
(4, 1) and (0, 5)	4
(3, 2) and (1, 4)	0, 4, 8
(2, 3) and (2, 3)	0, 4, 8
(1, 4) and (3, 2)	0, 4
(0, 5) and (4, 1)	0

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Necessary Condition for Cubic Planar 3-Connected Graph to be Non-Hamiltonian with Proof of Barnette's Conjecture

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Abstract: A conjecture of Barnette states that, every three connected cubic bipartite planar graph is Hamiltonian. This problem has remained open since its formulation. This paper has a threefold purpose. The first is to provide survey of literature surrounding the conjecture. The second is to give the necessary condition for cubic planar three connected graph to be non-Hamiltonian and finally, we shall prove near about 50 year Barnett's conjecture. For the proof of different results using to prove the results we illustrate most of the results by using counter examples.

Key Words: Cubic graph, hamiltonian cycle, planar graph, bipartite graph, faces, subgraphs, degree of graph.

AMS(2010): 05C25

§1. Introduction

It is not an easy task to prove the Barnette's conjecture by direct method because it is very difficult process to prove or disprove it by direct method. In this paper, we use alternative method to prove the conjecture. It must be noted that if any one property of the Barnette's graph is deleted graph is non Hamiltonian. A planar graph is an undirected graph that can be embedded into the Euclidean plane without any crossings. A planar graph is called polyhedral if and only if it is three vertex connected, that is, if there do not exists two vertices the removal of which would disconnect the rest of the graph. A graph is bipartite if its vertices can be colored with two different colors such that each edge has one end point of each color. A graph is cubic if each vertex is the end point of exactly three edges. And a graph is Hamiltonian if there exists a cycle that pass exactly once through each of its vertices. Self-loops and parallel edges are not allowed in these graphs. Barnett's conjecture states that every cubic polyhedral graph is Hamiltonian. P.G.Tait in (1884) conjectured that every cubic polyhedral graph is Hamiltonian; this came to be known as Tait's conjecture. It was disproved by W.T. Tutte (1946), who constructs a counter example with 46 vertices; other researchers later found even smaller counterexamples, however, none of these counterexamples is bipartite. Tutte himself

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conjectured that every cubic 3-connected bipartite graph is Hamiltonian but this was shown to be false by discovery of a counterexample, the Horton graph [16] .David W. Barnett (1969) proposed a weakened combination of Tait's and Tutte's conjecture, stating that every cubic bipartite polyhedral graph is Hamiltonian this conjecture first announced in [12] and later in [3]. In [10], Tutte proved that all planar 4-connected graphs are Hamiltonian, and in [9] Thomassen extended this result by showing that every planar 4-connected graph is Hamiltonian connected, that is for any pair of vertices, there is a Hamiltonian path with those vertices as endpoints.

§2. Supports for the Conjecture

In [5] Holton confirmed through a combination of clever analysis and computer search that all Barnett graphs with up to and including 64 vertices are Hamiltonian. In an announcement [14,11], McKay used computer search to extend this result to 84 vertices this implies that if Barnett conjecture is indeed false than a minimal counterexample must contain at least 86 vertices, and is therefore considerable larger than the minimal counterexample to Tait and Tutte conjecture. This is not all we know about a possible counterexample; another interesting result is that of Fowler, who in an unpublished manuscript [15] provided a list of subgraphs that cannot appear in any minimal counterexample to Barnett's conjecture.

Goody in [2] consider proper subsets of the Barnett graphs and proved the following.

Theorem 2.1 *Every Barnett graph which has faces consisting exclusively of quadrilaterals, and hexagons is Hamiltonian, and further more in all such graphs, any edge that is common to both a quadrilateral and a hexagon is a part of some Hamiltonian cycle.*

Theorem 2.2 *Every Barnett graph which has faces consisting of 7 quadrilaterals, 1 octagon and any number of hexagons is Hamiltonian, and any edge that is common to both a quadrilateral and an octagon is a part of some Hamiltonian cycle.*

In [6] Jensen and Toft reported that Barnett conjecture is equivalent to following.

Theorem 2.3 *Barnett conjecture is true if and only if for every Barnett graph G, it is possible to partition its vertices in to two subsets so that each induced an acyclic subgraph of G. (This theorem is not correct)*

Theorem 2.4([8]) *The edges of any bipartite graph G can be colored with $\delta(G)$ colors, where $\delta(G)$ is the minimum degree of vertices in G.*

Theorem 2.5([4]) *Barnett conjecture holds if and only if any arbitrary edge in a Barnett graph is a part of some Hamiltonian cycle.*

Theorem 2.6([13]) *Barnett conjecture holds if and only if for any arbitrary face in a Barnett graph there is a Hamiltonian cycle which passes through any two arbitrary edges on that face.*

Theorem 2.7([7]) *Barnett conjecture holds if and only if for any arbitrary face in a Barnett graph and for any arbitrary edges e_1 and e_2 on that face there is a Hamiltonian cycle which* *passes through e_1 and avoids e_2.*

It is difficult to say whether any of the technique described above will aid in settling Barnett conjecture. Certainly many of them seems to be useful and worth extending. One strategy is to keep chipping away at it; if Barnett conjecture is true then Godey's result can be extended to show that successively large and large subsets of Barnett graphs are Hamiltonian.

The Grinberg's Theorem [1] is not useful to find the counter example to Barnett's conjecture because all faces in Barnett graphs have even number of sides.

§3. New Results Supporting the Conjecture

Definition 3.1 *Any closed subgraph H of cubic planar three connected graph G is called complete cubic planar* 3*-connected subgraph H^C if all possible edges in that subgraph H are drawn then it also becomes cubic planar* 3*-connected graph. Thus we say H^C is cubic planar 3-connected graph.*

We illustrate by counter example following.

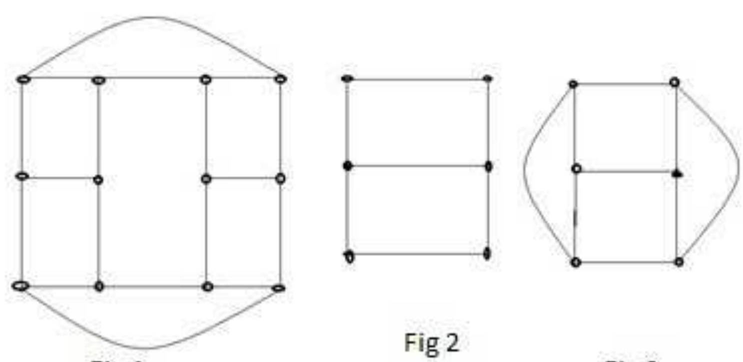


Fig 1 Fig 2 Fig 3

Let G be any cubic planar three connected graph as shown in Fig.1 we take its subgraph H shown in Fig.2 then we draw all possible edges in the subgraph as shown in Fig.3 the subgraph graph becomes complete cubic planar three connected H^C subgraph.

Definition 3.2 *Any closed subgraph H of cubic planar 3-connected graph G is called complete planar $n-1$ cubic 3-connected subgraph and is denoted by H^{C+} if all possible edges in that subgraph H are drawn then it becomes planar $n-1$ cubic 3-connected graph. i.e. Only one vertex has degree two and remaining graph is cubic planar 3-connected.*

Illustrate by counter example.

Let G be any cubic planar three connected graph as shown in Fig.4 H be its subgraph as shown in Fig.5 we draw all possible edges in the subgraph as shown in Fig.6 but still there exist a vertex having degree two only thus we say the subgraph H^{C+} be its complete planar $n-1$ cubic three connected graph.

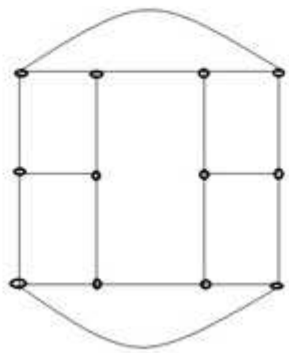


Fig. 4

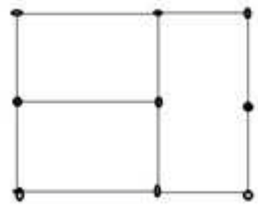


Fig. 5

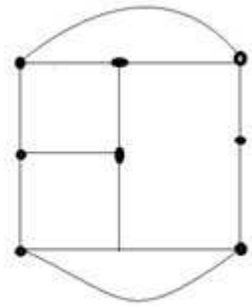


Fig. 6

Remark 3.1 A vertex can not have degree one in closed cubic planar three connected subgraphs, then it should be pendent vertex which is not possible in closed graphs so the degree of remaining vertices is two and degree cannot be more than three because it is the subgraph of cubic planar three connected graph so only possibility is that degree of remaining vertex is two.

Definition 3.3 *A closed subgraph H of cubic planar 3-connected graph is called complete planar $n-r$ cubic and 3-connected if all possible edges in that subgraph H are draw then it becomes cubic planar 3-connected, but it is still planar $n-r$ cubic and 3-connected, i,e its r vertices have degree two and remaining all vertices are cubic and three connected. It can be represented by H^{Cr+}.*

Lemma 3.1 *A planar bipartite 3-connected and $n-3$ cubic is non hamiltonian. In other words a planar graph which is bipartite 3-connected and $n-3$ cubic, i.e., only three of its vertices are of degree four and remaining graph is cubic then such a graph is non hamiltonian.* (Only encircle vertices is of degree four and rest of the graph is cubic and 3-connected)

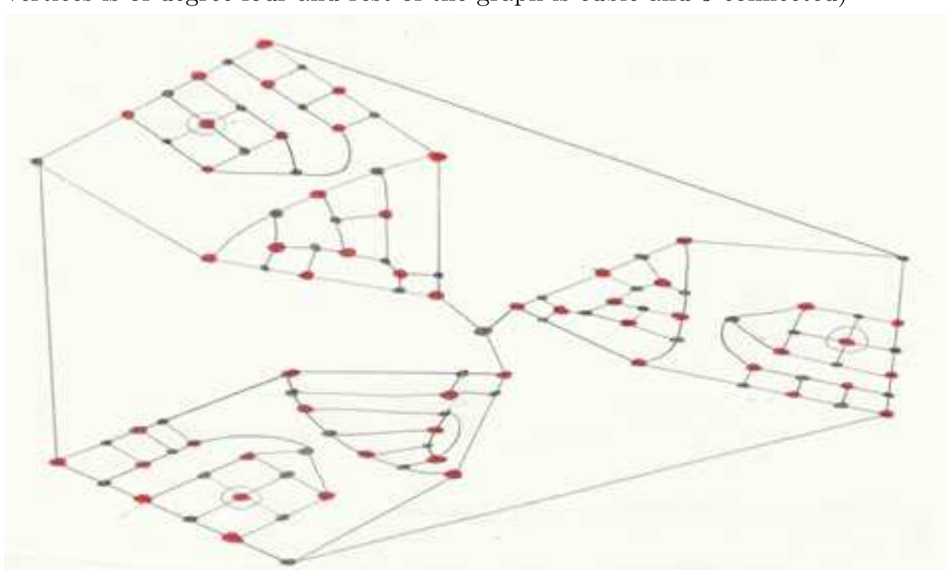


Fig. 7

We shall prove this result by counter example. The main aim behind the result is to prove that if a single property is deleted in cubic planar three connected bipartite graphs then it is non-Hamiltonian. This graph can be divided in to three closed subgraphs and an isolated vertex such that these closed subgraphs are H^{C+} sub graphs. Later we use this result in the main theorem.

Lemma 3.2 A cubic planar bipartite 2-connected graph is non-hamiltonian. It can be seen in this example. (It is not possible for me to give number of counter examples even though we can construct number of such examples) Fig.8 below is the cubic planar bipartite 2-connected graph but non-hamiltonian.

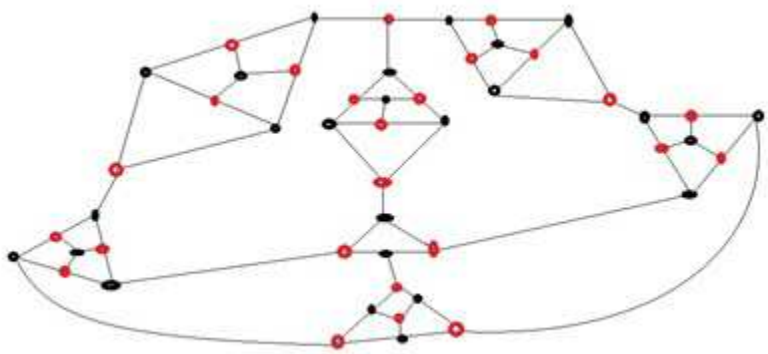


Fig.8

Remark 3.2 In every cubic planar three connected bipartite graph if any one of the property is deleted then the graph is non Hamiltonian.

Remark 3.3 Let $[\cdot]$ denotes the greatest integer function.

(1) If a and b are any two positive integers then $[a+b]=[a]+[b]$;

(2) If a is any positive integer and b is any positive real number then $[a+b]=[a]+[b]$.

Remark 3.4 Let G be any graph and if G is cubic planar three connected, we know that every cubic planar three connected graph, the Degree of each vertex is exactly equal to three. Thus the sum of all the degree of the Graph is $3n$ that is

$$\sum_{i=1}^{n} d_i = 3n.$$

Since each edge contributes two to the degrees thus the number of edges in the graph is

$$E = \frac{\sum\limits_{i=1}^{n} d_i}{2} = \frac{3n}{2},$$

where n is the number of vertices of the graph. Thus we conclude that if number of nodes is n number of edges is $\frac{3n}{2}$ and if number of edges is $\frac{3n}{2}$ the number of nodes is $\frac{2}{3}E = \frac{2}{3} \times \frac{3n}{2} = n$. Thus we conclude that in any cubic planar three connected graph edges and nodes are connected by certain relation.

The number of edges of any cubic planar three connected graph is always divisible by three if we take any planar cubic three connected graph and number of edges is not divisible by three then given graph is not H^C it is planar $n-1$ cubic and three connected, i.e., it contain a vertex of degree two only such a graph is denoted by H^{C+}. There does not exist any two vertices of degree two because we can draw an edge between them.

Lemma 3.3 *The number of regions in every cubic three connected planar graph and every cubic planar bipartite three connected graph of n vertices is* $\frac{n+4}{2}$*, where n is the number of vertices of the graph.*

Proof Since in every cubic planar three connected graph and every cubic planar three connected bipartite graph the degree of each vertices is exactly equal to three as graph is cubic. Thus the sum of all the degree of the Graph is $3n$ that is

$$\sum_{i=1}^{n} d_i = 3n.$$

Since each edge contributes two to the degrees thus the number of edges in these graph is

$$E = \frac{\sum\limits_{i=1}^{n} d_i}{2} = \frac{3n}{2},$$

where n is the number vertices of the graph. Thus we conclude that if number of vertices is n number of edges is $\frac{3n}{2}$ and if number of edges is $\frac{3n}{2}$ the number of vertices is $\frac{2}{3}e = \frac{2}{3} \times \frac{3n}{2} = n$, i.e., the number of vertices and edges are connected by certain relation. We know by Euler's theorem on planar graphs the number of regions is equal to

$$r = e - v + 2.$$

Since we have a graph of n vertices as we know it is cubic planar three connected or cubic planar three connected bipartite graph the number of edges in such graph's is $\frac{3n}{2}$ as shown above, now substitute these values in equation (i) we get

$$r = e - n + 2 = \frac{3n}{2} - n + 2 = \frac{n+4}{2}.$$

That proves the result. □

Thus from the above result we conclude that in every cubic planar three connected and every cubic planar bipartite three connected graph it is true that

$$e - v + 2 = \frac{n+4}{2}$$

The above result is not true for other planar graphs as we can take a counter example of three connected bipartite planar graph known as Herschel graph which contain 11 vertices and 18 edges. Contain 9 regions does not satisfy the above result.

Note 3.1 In every cubic planar three connected graph G and every cubic planar bipartite three connected graph G^+

1. The order of graphs G and G^+ is even;

2. The number of regions in both the graphs G and G^+ are equal to $\frac{n+4}{2}$, where n is the total number of vertices (See lemma 3);

3. The edges and vertices in both the graphs are connected by certain relation i.e

$$E = \frac{3n}{2} \text{ and } V = \frac{2E}{3}.$$

4. In G odd cycles are allowed but in G^+ it is bipartite thus odd cycles are not allowed.

§4. Necessary Condition for a Cubic Planar 3-Connected Graph to be Non-hamiltonian

Theorem A *A cubic planar 3-connected graph is non-hamiltonian if the graph is divided into three closed subgraphs of any order and an arbitrary isolated vertex such that these three closed subgraphs are planar $n-1$ cubic three connected I.e. they are H^{C+}subgraphs in other words a planar 3-connected graph is non-hamiltonian if these three subgraphs are such that*

$$\left[\frac{3n}{2}\right] \not\equiv 0(mod3),$$

where $[\cdot]$ *denotes the greatest integer function and* $\frac{3n}{2}$ *is the number of edges in these subgraphs (Remark* 3.4 *above).*

Proof Let G be any cubic planar 3-connected graph of order n number of edges is $\frac{3n}{2}$. Let us suppose that all the three closed subgraphs of G are complete closed planar cubic 3-connected, i.e. H^C subgraphs then

$$\left[\frac{3n}{2}\right] \equiv 0(mod3)$$

Since odd cycles are allowed so we can take any closed subgraph of any order in such a way that these closed subgraphs are necessarily H^{C+} first of all we shall take order of all closed subgraphs is odd if these closed subgraphs are H^{C+} then we have to stop the process of searching as such subgraphs exist but if such closed subgraphs are not H^{C+} then we try for different orders.

Let order of closed subgraph be odd, i.e., n is odd say $n = 2m+1$ or $n = 2m-1$ and

$$\left[\frac{3n}{2}\right] \equiv 0(mod3), \quad \left[\frac{3(2m+1)}{2}\right] \equiv 0(mod3), \quad \left[\frac{6m+3}{2}\right] \equiv 0(mod3).$$

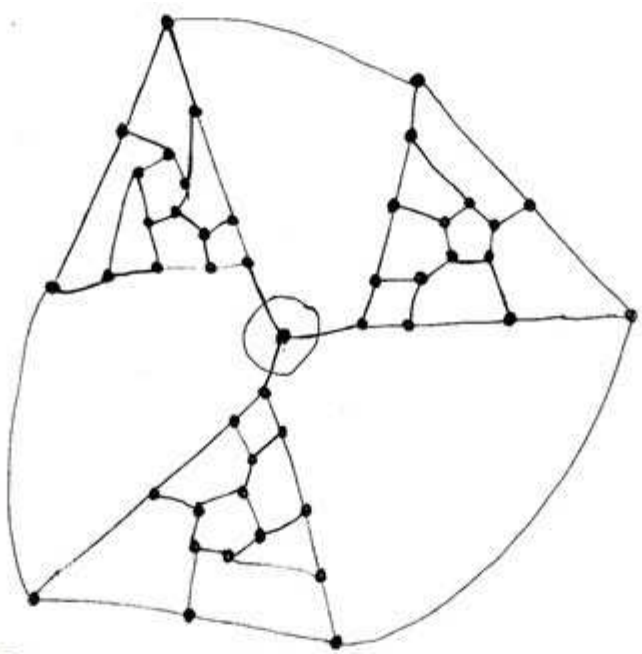
Since in graphs the number of vertices and edges represent positive integers so

$$\begin{aligned}\left[\frac{6m}{2}\right] + \left[\frac{3}{2}\right] \equiv 0(mod3) \quad &\Rightarrow \quad 3m+1 \equiv 0(mod3)\\ &\Rightarrow \quad 3/3m+1 \text{ and } 3/-3m\\ &\Rightarrow \quad 3/3m+1-3m \Rightarrow 3/1,\end{aligned}$$

which is contradiction similarly if $n = 2m-1$. We get $3/3m-1-3m$ which gives $3/-1$. This again gives contradiction.

Thus we conclude that
$$\left[\frac{3n}{2}\right] \ncong 0(mod3)$$
(Since such subgraphs exists we shell stop our search). Thus there exists one vertex in all the three closed subgraphs having degree 2 only (the degree cannot be one are more than three discussed above remark 4) that is these three subgraphs are H^{C+} subgraphs. If two vertices are of degree two we can draw an edge between them and subgraph becomes H^C only that is these subgraphs are cubic planar three connected which is not possible. When graph satisfy these conditions we first of all delete all those edge which we have added in the subgraphs then we join these sub graphs together with the arbitrary isolated vertex (it must be noted that such graphs does not contain only one arbitrary vertex it may contain more than one arbitrary vertex) with those vertices of the subgraphs having degree 2 in such a way that graph becomes cubic planar three connected. since odd cycles are allowed when we start from any arbitrary vertex it is not possible to travel all the vertices once and reaches back at the stating vertex because an arbitrary vertex can be traveled only at once so we can travel at most two of these H^{C+} subgraphs which we have joined to make the graph cubic planar three connected thus the graph so obtained is non-Hamiltonian.

Now we shall illustrate the result with following graphs and prove that this condition is satisfied by these graphs. All these graphs are cubic planar three connected and non Hamiltian satisfy the above conditions Fig.9 to 26 below.

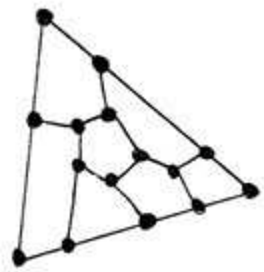


G

Fig.9

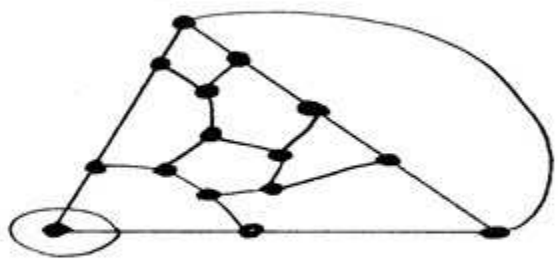
1) First of all let us take Tuttle graph, in which we take an encircle vertex as an isolated vertex and divide the remaining graph in three closed subgraphs not necessary of same order we shall show that these closed subgraphs are H^{C+} subgraphs.

Let H be its subgraph shown below



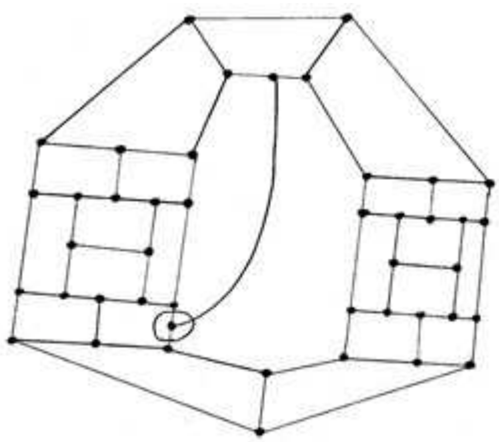
H Fig. 10

Again if we draw all possible edges in this closed subgraph the subgraph becomes planar $n-1$ cubic and three connected, i.e., H^{C+} subgraph as shown below and a vertex having degree two only has been shown by encircling the vertex.



H^{C+} Fig. 11

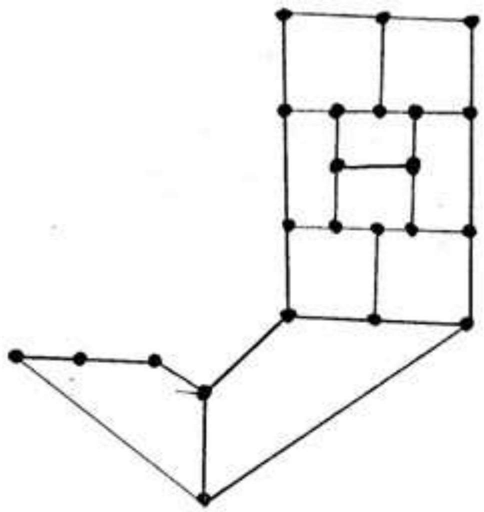
Since all the three closed subgraphs of this graph are of same order so other two subgraphs have same property as discussed above.



G Fig. 12

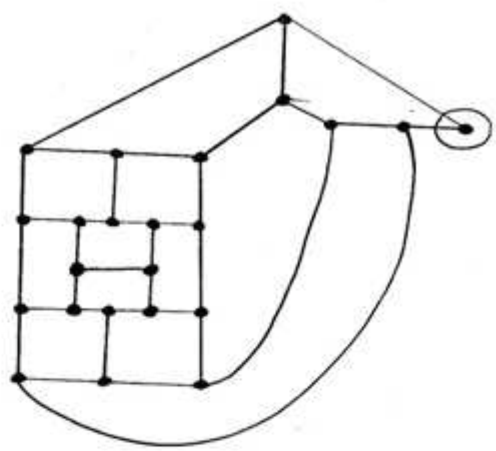
2) Now let us take younger graph of 44 vertices which is cubic planar three connected non-Hamiltonian. And isolated vertex is shown by encircle it, and remaining graph is divided into three closed subgraphs not necessarily of same order all these closed subgraphs are H^{C+}.

Let H_1 be its one closed subgraph as shown



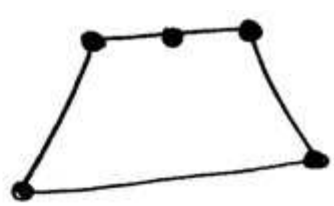
H_1 Fig.13

If we draw all possible edges in this subgraph it becomes planar $n-1$ cubic three connected as shown below H^{C+} only encircle vertex is of degree two.



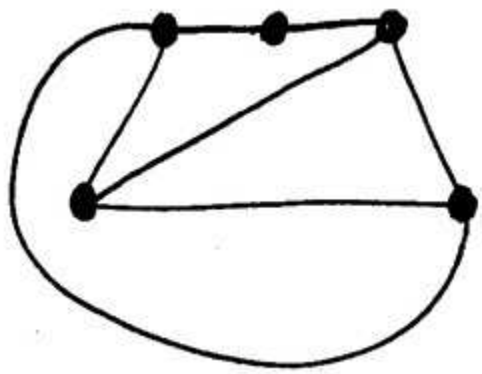
H^{C+} Fig. 14

Let another subgraph H_2 of the graph is given below.



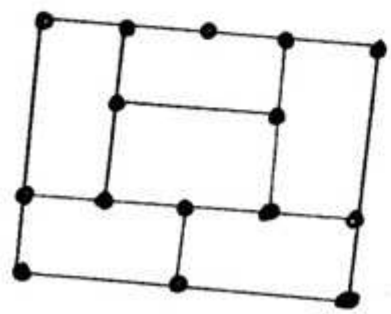
H_2 Fig. 15

If we draw all possible edges in the subgraph it also becomes H^{C+} subgraph as shown below.



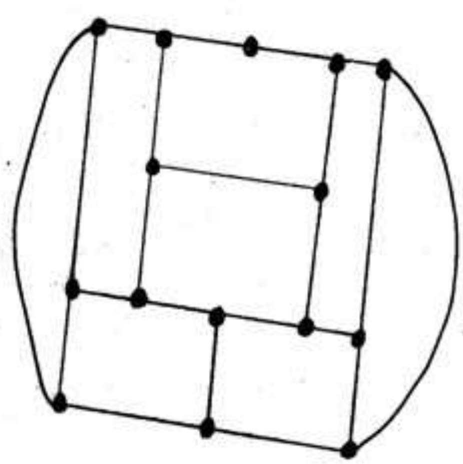
H^{C+} Fig. 16

Let another subgraph H_3 of a graph is given as



H_3 Fig. 17

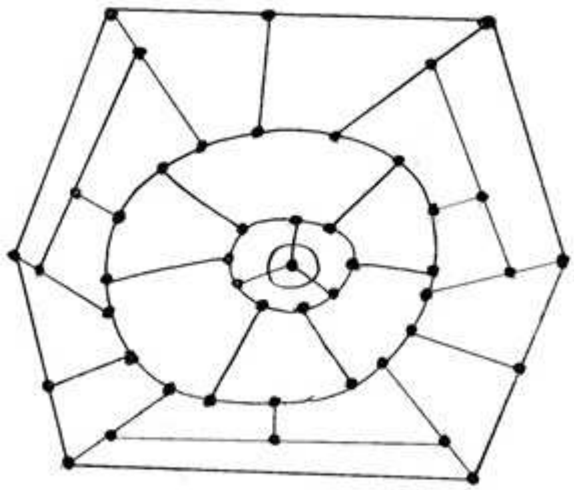
If we draw all possible edges in the subgraph it becomes H^{C+} subgraph as shown below



H^{C+} Fig. 18

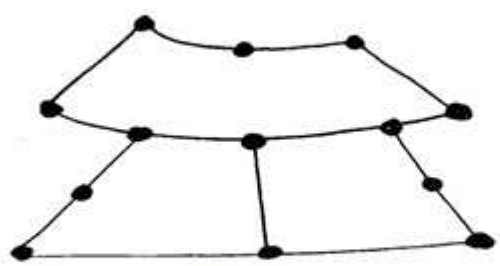
3) Now let us take another example of cubic planar three connected non Hamiltonian graph known as Grin berg graph 0f 46 vertices as shown below in which encircle vertex is an arbitrary

vertex.



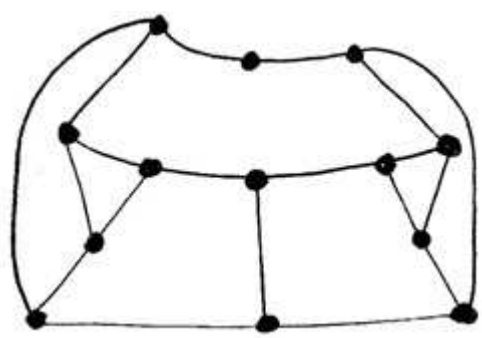
G Fig. 19

4) Let us take its closed subgraph H_1 as shown below



H_1 Fig. 20

Now if we draw all possible edges in the subgraph it becomes H^{C+} subgraph as shown below



H^{C+} Fig.21

Let us take another subgraph H_2 of the graph given below

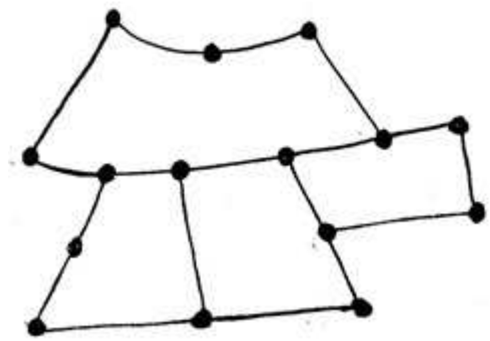


Fig. 22

If we draw all possible edges in the graph it becomes planar $n-1$ cubic and three connected as shown below

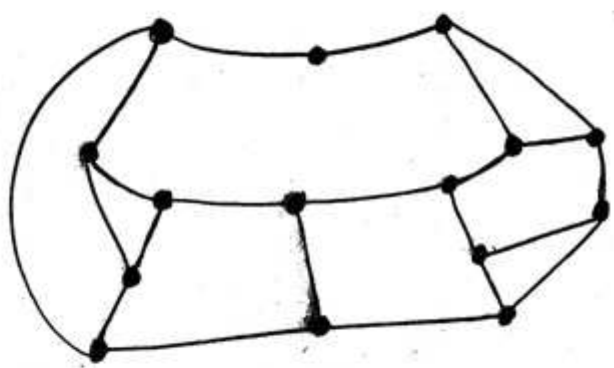


Fig. 23

Now again if we take another closed subgraph H_3 as below

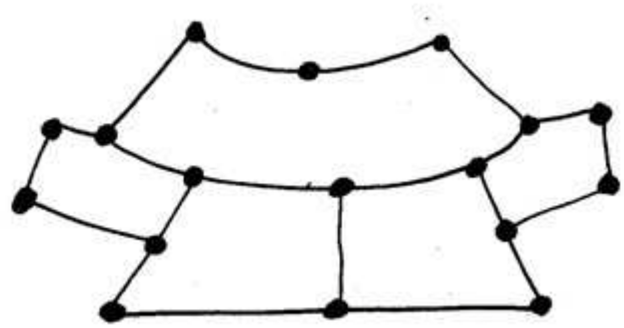
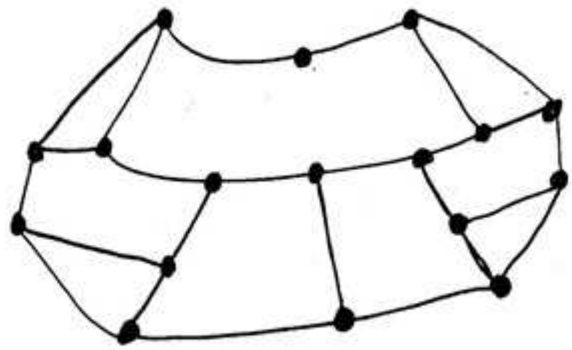


Fig. 24

If we again draw all possible edges in the closed subgraph it becomes H^{C+} subgraph as shown below



H^{C+} Fig. 25

Now all other planar cubic three connected non Hamiltonian graphs satisfy this condition these graphs are shown in Fig.26 − 1 to Fig.26 − 3.

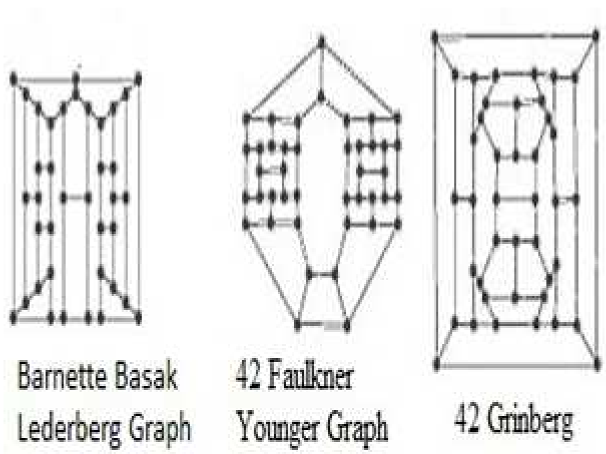


Fig.26-1

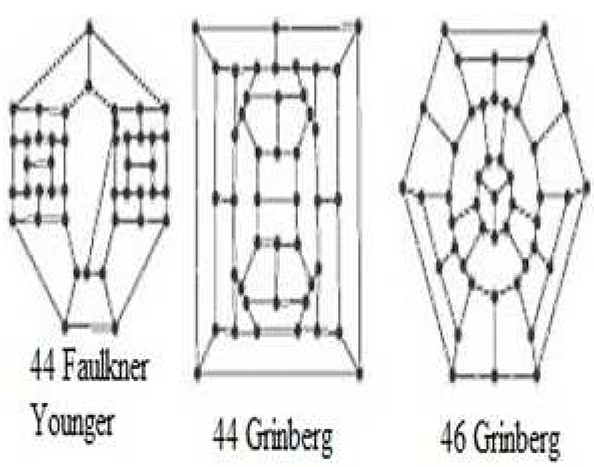


Fig.26-2

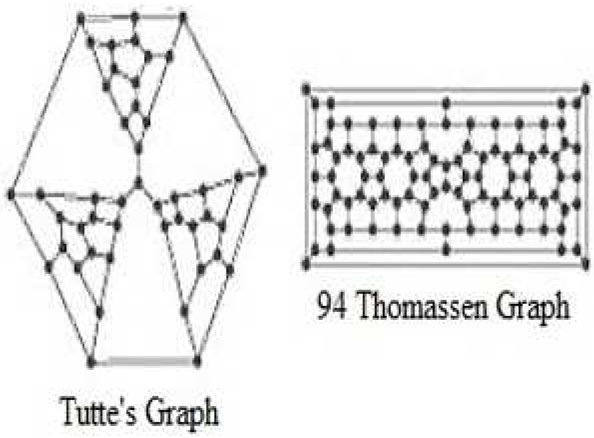
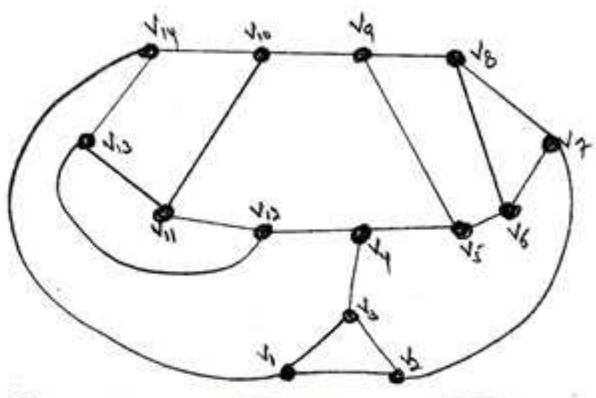


Fig.26-3

Note 4.1 One of the most important thing regarding the cubic planar three connected non-Hamiltonian graphs which was proved by professors Linfan Mao and Yanpei Liu in 2001 [17] there exists infinite three connected non-Hamiltonian cubic maps on every surface (orient able or non-orient able) not only the above graphs but also these infinite graphs satisfy the condition which we have proved above.

Now we shall show that above result is sharp. We use counter example to prove this sharpness. Below example Fig.27 is a graph which is cubic planar three connected contain Hamiltonian cycle start from $V_1, V_2, V_3, V_4, \cdots, V_{14}, V_1$. In this graph if we take V_4 as arbitrary vertex all the three closed subgraphs are not H^{C+} as shown below, it is not necessary we take V_4 as arbitrary vertex we can take any vertex as arbitrary vertex in such a way that remaining graph is divided into three closed subgraphs of any order but all such closed subgraphs are H^{C+} which is not possible in this graph.



G Fig. 27

Thus we conclude that all cubic planar three connected non Hamiltonian graphs can be divided in to three closed subgraphs of any order and an isolated vertex satisfying the property that all the three closed subgraphs are H^{C+}, But if graph is cubic planar three connected and Hamiltonian it is not necessary that all the three closed subgraphs satisfy H^{C+} property as shown below, thus the condition which we use to prove the theorem is sharp. In other words every cubic planar three connected graph which is Hamiltonian and can be divided into three closed subgraphs of any order and an isolated vertex all the three subgraphs may are may not be H^{C+} subgraphs, but if graph is non Hamiltonian all such closed subgraphs are H^{C+} (There are other examples as well but it is not possible to draw all in this paper).

Take a closed subgraph H and its H^{C+} subgraph

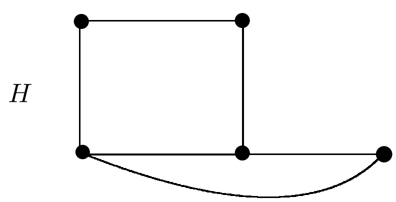


Fig.28

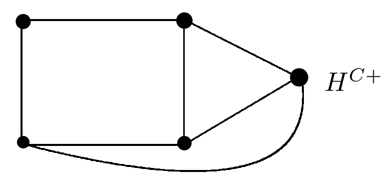


Fig.29

Take another closed subgraph H and its H^{C+} subgraph

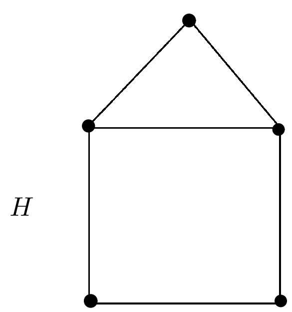


Fig.30

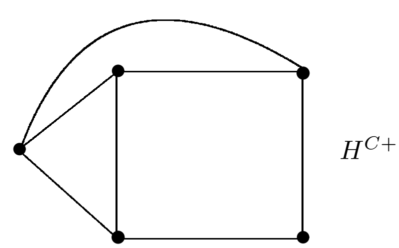


Fig.31

And finally take a closed subgraph H of order three so it is not H^{C+} because we cannot draw any more edge in this subgraph (these edges are parallel edges). □

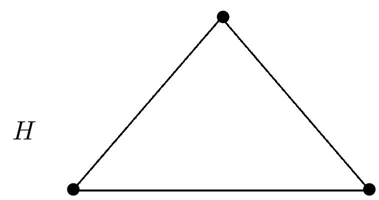


Fig.32

Remark 4.1 It has been discussed above that number of regions in cubic planar three connected graphs and cubic planar three connected bipartite graphs are $\frac{n+4}{2}$, thus it is necessary that every cubic planar three connected bipartite graph is non Hamiltonian if it has at least one closed subgraph which is H^{C+} also in lemma 1. we have given a counter example of $n-3$ cubic planar three connected bipartite non Hamiltonian graph satisfy H^{C+} property.

Theorem B *Every cubic planar bipartite three connected graph is Hamiltonian (Barnett's conjecture).*

Proof Since every bipartite graph is two colorable and thus without odd cycles so it contains only even cycles and number of vertices is also even, we cannot take any closed subgraph of odd order because it is not connected as odd cycles are not allowed, so every closed subgraph of cubic planar bipartite three connected graph is of even length. Thus in this type of graph n is always even, such graphs are non Hamiltonian only if there exist at least one subgraph H of any order which is planar $n-1$ cubic and three connected, i.e., H^{C+} subgraph, Then conjecture is not true because counter example can be constructed to disprove it if such a condition is satisfied. (See theorem A above) and Fig.7 of Lemma 1.

Let it is true that such graphs have at least one closed subgraph of H^{C+} then it must satisfy the following condition

$$\left[\frac{3n}{2}\right] \ncong 0(mod3)$$

Since n is necessarily even i.e. order of every subgraph is even because graph is bipartite and odd cycles are not allowed. Let $n = 2m$. Then

$$\begin{aligned}\frac{3n}{2} &= 3 \times \frac{(2m)}{2} = 3m \\ \frac{3n}{2} &\equiv 0(mod3) \\ 3m &\equiv 0(mod3) \\ &\Rightarrow 3/3m \text{ and} 3/-3m \\ &\Rightarrow 3/3m - 3m \Rightarrow 3/0,\end{aligned}$$

which contradicts the given statement that

$$\left[\frac{3n}{2}\right] \not\equiv 0(mod3).$$

Thus we conclude that there does not exist any subgraph H of cubic planar bipartite three connected graph G which is planar $n-1$ cubic and three connected, i.e., which is H^{C+} thus there does not exists any counter example which proves that Barnett's conjecture does not hold, thus every cubic planar bipartite three connected graph is Hamiltonian that proves the conjecture. □

The above theorem can be verified by Lemma 3.1 above the non Hamiltonian graph G of Lemma 3.1 can be divided into an arbitrary vertex and three closed subgraphs H^{C+} even though graph is bipartite(without odd cycles) three connected planar $n-3$ cubic, only three vertices are of degree four and contain only even cycles.

Below is the graph which is cubic planar three connected contains Hamiltonian cycle. This cannot be divided into an arbitrary vertex and three closed subgraphs H^{C+}.

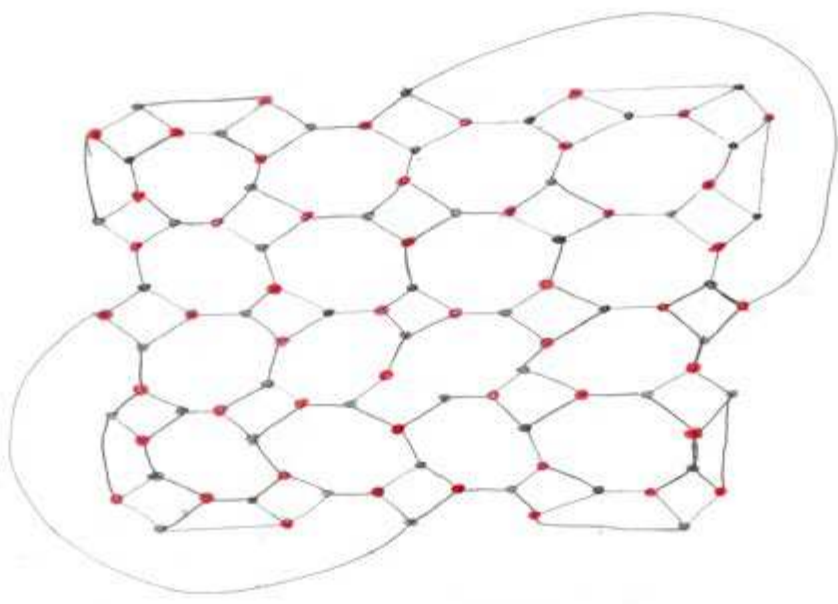


Fig. 33

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Odd Sequential Labeling of Some New Families of Graphs

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Abstract: A graph $G = (V(G), E(G))$ with p vertices and q edges is said to be an *odd sequential graph* if there is an injection $f : V(G) \to \{0, 1, \cdots, q\}$ or if G is a tree then f is an injection $f : V(G) \to \{0, 1, \cdots, 2q-1\}$ such that when each edge xy is assigned the label $f(x)+f(y)$, the resulting edge labels are $\{1, 3, \cdots, 2q-1\}$. In this paper we initiate a study on some new families of odd sequential graphs generated by some graph operations on some standard graphs.

Key Words: Odd sequential labeling, Smarandachely odd sequential labeling, super subdivisions of a graph, shadow graph.

AMS(2010): 05C78

§1. Introduction

By a graph $G = (V(G), E(G))$ with p vertices and q edges we mean a simple, connected and undirected graph in this paper. A brief summary of definitions and other information is given in order to maintain compactness. The terms not defined here are used in the sense of Harary [3].

Definition 1.1 *The super subdivisions of a graph G produces a new graph by replacing each edge of G by a complete bipartite graph $K_{2,m}$ (where m is any positive integer) in such a way that the ends of each e_i are merged with two vertices of 2-vertices part of $K_{2,m}$ after removing the edge e_i from the graph G. It is denoted by $SS(G)$.*

Definition 1.2 *A comb is a caterpillar in which each vertex in the path is joined to exactly one pendant vertex.*

Definition 1.3 *For a graph G, its split graph is obtained by adding to each vertex v, a new vertex v' so that v' is adjacent to every vertex that is adjacent to v in G.*

Definition 1.4 *The shadow graph $D_2(G)$ of a connected graph G is obtained by taking two copies of G say G' and G'', then join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.*

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Definition 1.5 *A bistar is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$ by an edge.*

Definition 1.6([6]) *Let $G = (V(G), E(G))$ be a graph and $G_1, G_2, \cdots, G_n$ be n copies of graph G. Then the graph obtained by adding an edge between G_i and G_{i+1}, for $i = 1, 2, \cdots, n-1$ is called a path union of G.*

Definition 1.7 *If the vertices are assigned values subject to certain conditions then it is known as graph labeling.*

Graph labeling introduced by Rosa in [5] is now one of the fascinating areas of research with applications ranging from social sciences to computer science and from neural network to bio-technology to mention a few. A systematic study on various applications of graph labeling is carried out by Bloom and Golomb [1]. The famous Ringel-Kotzig [4] graceful tree conjecture and many illustrious works on it brought a tide of different labeling techniques like harmonious labeling, odd graceful labeling, edge graceful labeling etc. For detailed survey on graph labeling and related results we refer to Gallian [2]. The concept of odd sequential labeling was introduced by Singh and Varkey [7] which is defined as follows.

Definition 1.8 *A graph $G = (V(G), E(G))$ with p vertices and q edges is said to be an odd sequential graph if there is an injection $f : V(G) \to \{0, 1, \cdots, q\}$ or if G is a tree then f is an injection $f : V(G) \to \{0, 1, \cdots, 2q-1\}$ such that when each edge xy is assigned the label $f(x) + f(y)$, the resulting edge labels are $\{1, 3, \cdots, 2q-1\}$.*

The graph which admits odd sequential labeling is known as an *odd sequential graph*. Generally, a graph G is called *Smarandachely odd sequential* if there is a subset $V' \subset V(G)$ such that the resulting edge labels of $G \setminus \langle V' \rangle$ are $\{1, 3, \cdots, 2q'-1\}$, where $q' \leq q$. Clearly, if $V' = \emptyset$, such a Smarandachely odd sequential graph is nothing else but an odd sequential graph. In[7] it has been also proved that the graphs such as combs, grids, stars and rooted trees of level 2 are odd sequential while odd cycles are not.

Here we investigate odd sequential labeling of some new families of graphs generated by some graph operations on some standard graphs.

§2. Results on Odd Sequential Labeling

Theorem 2.1 *The graph $C_n \odot nK_{1,m}$, where n is even admits odd sequential labeling.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of C_n, where n is even. Let u_{ij} be the newly added vertices in C_n to form $C_n \odot nK_{1,m}$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. To define $f : V(C_n \odot nK_{1,m}) \to \{0, 1, \cdots, q\}$ two cases are to be considered.

Case 1. $n \equiv 0(mod 4)$

Consider the following 4 subcases:

Subcase 1.1 $1 \leq i \leq \frac{n}{2}$

In this caes, let $f(v_i) = (m+1)(i-1)$ if i is odd and $i(m+1)-1$ if i is even.

Subcase 1.2 $\dfrac{n}{2}+1 \le i \le n$

In this case, let $f(v_i) = (m+1)(i-1)+2$ if i is odd and $i(m+1)-1$ if i is even.

Subcase 1.3 $1 \le i \le \dfrac{n}{2}$ and $1 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd and $(m+1)(i-2)+2j$ if i is even.

Subcase 1.4 $\dfrac{n}{2}+1 \le i \le n$ and $1 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd and $(m+1)(i-2)+2(j+1)$ if i is even.

Case 2. $n \equiv 2(mod 4)$

Consider the following 5 subcases.

Subcase 2.1 $1 \le i \le \dfrac{n}{2}$

In this case, let $f(v_i) = (m+1)(i-1)$ if i is odd, $i(m+1)-1$ if i is even and $f(v_i) = i(m+1)+1$ if $i = \dfrac{n}{2}+1$.

Subcase 2.2 $\dfrac{n}{2}+2 \le i \le n$

In this case, let $f(v_i) = (m+1)(i-1)+2$ if i is odd and $i(m+1)-1$ if i is even.

Subcase 2.3 $1 \le i \le \dfrac{n}{2}+1$ and $1 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd, $(m+1)(i-2)+2j$ if i is even and $f(u_{ij}) = (m+1)(i-1)-1$ if $i = \dfrac{n}{2}+2$ and $j = 1$.

Subcase 2.4 $i = \dfrac{n}{2}+2$ and $2 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd and $(m+1)(i-2)+2(j+1)$ if i is even.

Subcase 2.5 $\dfrac{n}{2}+3 \le i \le n$ and $1 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd and $(m+1)(i-2)+2(j+1)$ if i is even.

In view of the above defined labeling pattern f satisfies the conditions for odd sequential labeling. That is $C_n \odot nK_{1,m}$ is an odd sequential graph. $\square$

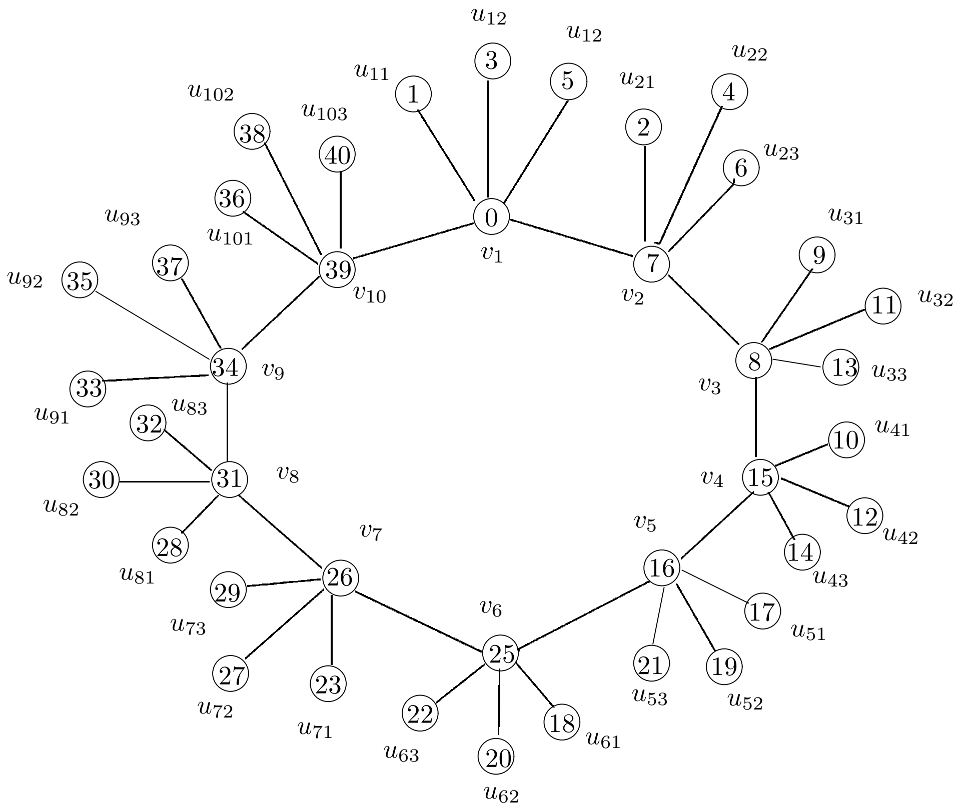

Figure 1

Illustration 2.2 The Figure 1 shows an odd sequential labeling of $C_{10} \odot 10K_{1,3}$.

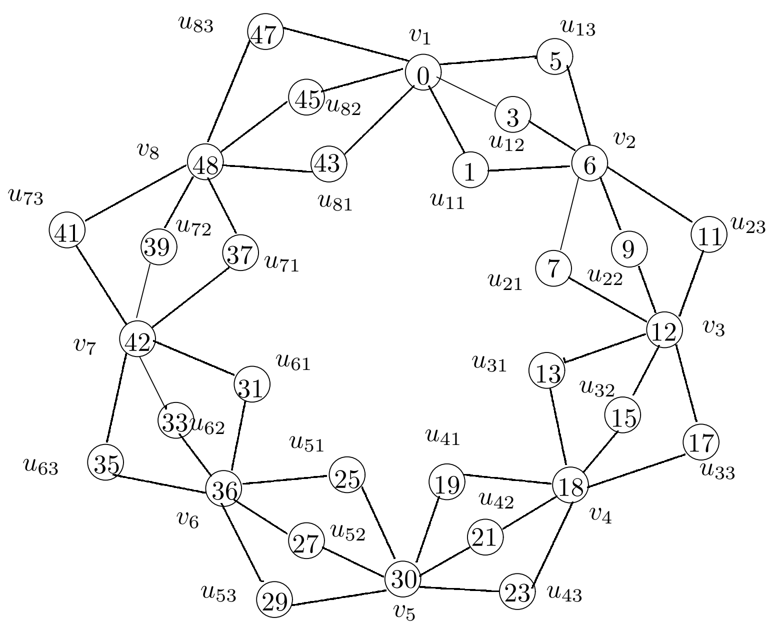

Figure 2

Theorem 2.3 *The graph $SS(C_n)$ where n is even admits odd sequential labeling.*

Proof Let C_n be the cycle containing n vertices $v_1, v_2, \cdots, v_n$, where n is even. Let e_i denotes the edge $v_i v_{i+1}$ in C_n. For $1 \leq i \leq n$ each edge e_i of cycle C_n is replaced by a complete bipartite graph $K_{2,m}$ where m is any positive integer. Let u_{ij} be the vertices of the m vertices part of $K_{2,m}$ where $1 \leq i \leq n$, $1 \leq j \leq m$. Define $f : V(SS(C_n)) \to \{0, 1, \cdots, q\}$ as follows.

Let $f(v_1) = 0$ and $f(v_i) = mi + (i-2)m$ if $2 \leq i \leq \frac{n}{2}$. For $\frac{n}{2} + 1 \leq i \leq n$, let $f(v_i) = 2mi$, $f(u_{1j}) = 2j - 1$ for $1 \leq j \leq m$. For $2 \leq i \leq n$, $1 \leq j \leq m$, let $f(u_{ij}) = mi + (i-2)m + 2j - 1$. Then the above defined labeling pattern f provides odd sequential labeling for $SS(C_n)$ where n is even. That is for even n, $SS(C_n)$ admits odd sequential labeling. □

Illustration 2.4 The Figure 2 shows an odd sequential labeling of $SS(C_8)$ with $K_{2,3}$.

Theorem 2.5 *The split graph of comb is an odd sequential graph.*

Proof Let $\{v_i, 1 \leq i \leq n\}$ and $\{v'_i, 1 \leq i \leq n\}$ be the vertices of comb in which $\{v'_i, 1 \leq i \leq n\}$ are the pendant vertices. Let $\{u_i, 1 \leq i \leq n\}$ and $\{u'_i, 1 \leq i \leq n\}$ be the newly added vertices and let G be the split graph of comb. Define $f : V(G) \to \{0, 1, \cdots, q\}$ as follows.

Let $f(v_i) = 6i - 4$ if i is odd and $6i - 3$ if i is even, and $f(v'_1) = 1$, $f(v'_3) = 15$. Let $f(v'_i) = 6i - 3$ if i is odd, $i \neq 1, 3$, and $6i - 4$ if i is even. Let $f(u_i) = 6i - 6$ if i is odd, and $6i - 7$ if i is even, and $f(u'_1) = 3$, $f(u'_3) = 11$. Let $f(u'_i) = 6i - 7$ if i is odd, $i \neq 1, 3$ and $6i - 6$ if i is even.

Then the above defined function provides an odd sequential labeling for the split graph of comb. That is, split graph of comb is an odd sequential graph. □

Illustration 2.6 The Figure 3 shows an odd sequential labeling of split graph of a comb.

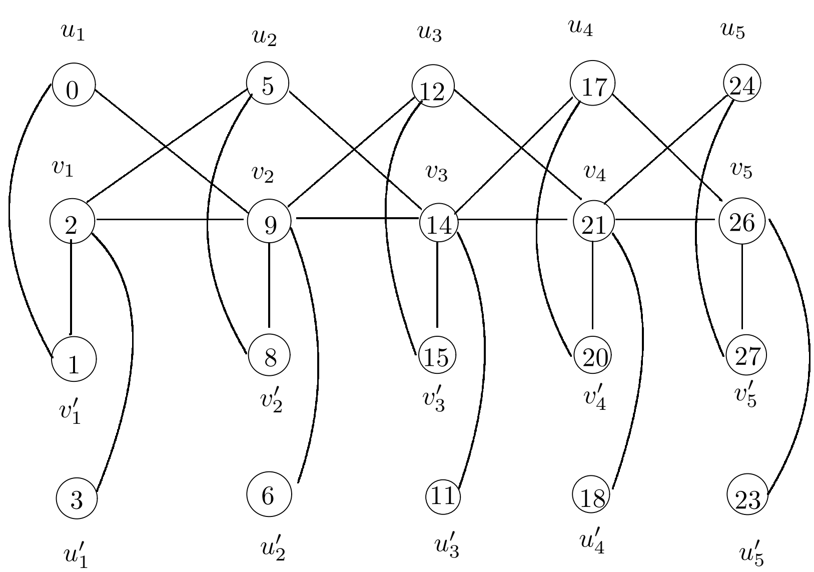

Figure 3

Theorem 2.7 *$D_2(Comb)$ admits sequential labeling.*

Proof Consider two copies of comb G_1 and G_2. Let $\{v_i, 1 \leq i \leq n\}$ and $\{v'_i, 1 \leq i \leq n\}$ be the vertices of comb G_1 and $\{u_i, 1 \leq i \leq n\}$ and $\{u'_i, 1 \leq i \leq n\}$ be the vertices of G_2. Let G be the shadow graph of the comb. Define $f : V(G) \to \{0, 1, \ldots, q\}$ as follows.

For $1 \leq i \leq n$, let $f(v_i) = 8i - 8$ if i is odd and $8i - 7$ if i is even. For $1 \leq i \leq n$, let $f(v'_i) = 8i - 7$ if i is odd, and $8i - 8$ if i is even. For $1 \leq i \leq n$, let $f(u_i) = 8i - 4$ if i is odd and $8i - 5$ if i is even. For $1 \leq i \leq n$, let $f(u'_i) = 8i - 5$ if i is odd and $8i - 4$ if i is even.

In view of the above defined labeling pattern f satisfies the conditions of odd sequential labeling. That is the $D_2(Comb)$ admits odd sequential labeling.

Illustration 2.8 The following Figure 4 shows an odd sequential labeling of $D_2(Comb)$.

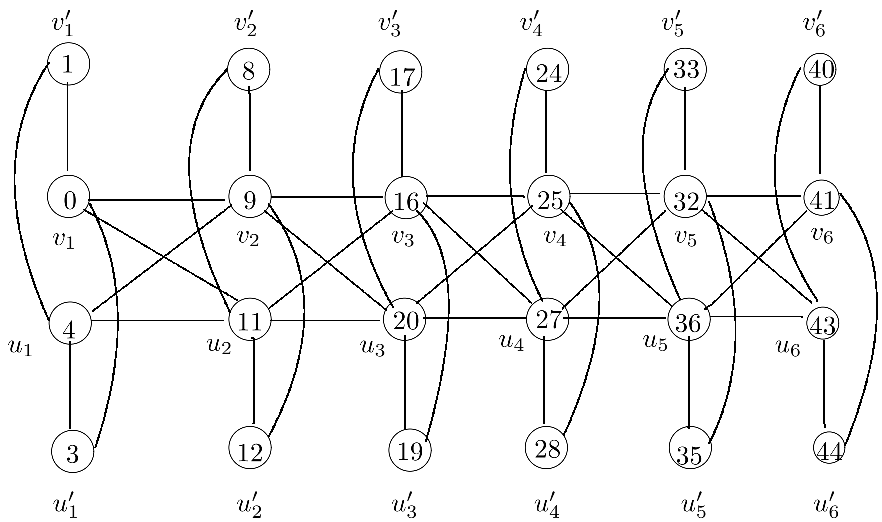

Figure 4

Theorem 2.9 *The graph $D_2(B_{n,n})$ is an odd sequential graph.*

Proof Consider two copies of $B(n, n)$ say $B_1(n, n)$ and $B_2(n, n)$. Let $\{v_1, v_2, v_{1j}, v_{2j}, 1 \leq j \leq n\}$ and $\{u_1, u_2, u_{1j}, u_{2j}, 1 \leq j \leq n\}$ be the vertices of $B_1(n, n)$ and $B_2(n, n)$ where v_1, v_2 and u_1, u_2 are the respective apex vertices. Let $D_2(B_{n,n})$ be the shadow graph of $B_1(n, n)$ and $B_2(n, n)$. Define $f : V(D_2(B_{n,n})) \to \{0, 1, \ldots, q\}$ as follows.

Let $f(v_1) = 0$, $f(v_2) = 8n + 1$, $f(u_1) = 4n$, $f(u_2) = 8n + 3$, $f(v_{1j}) = 4(j - 1) + 1$ if $1 \leq j \leq n$, $f(v_{2j}) = 4j$ if $1 \leq j \leq n - 1$, $f(v_{2n}) = 4(n + 1)$, $f(u_{1j}) = 4j - 1$ if $1 \leq j \leq n$ and $f(u_{2j}) = 4(n + j + 1)$ if $1 \leq j \leq n$.

The above defined function f provides an odd sequential labeling for $D_2(B_{n,n})$. That is $D_2(B_{n,n})$ is an odd sequential graph.

Illustration 2.10 The following *Figure 5* shows an odd sequential labeling of $D_2(B_{4,4})$.

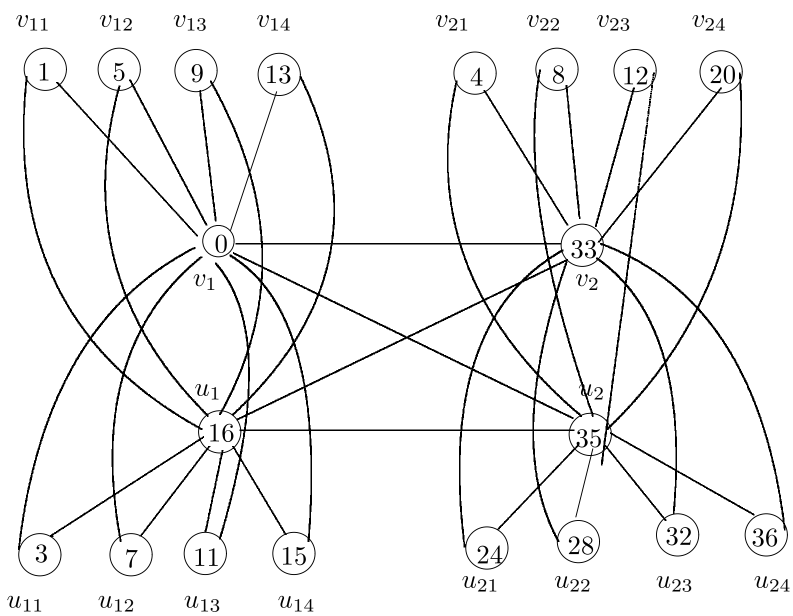

Figure 5

Theorem 2.11 *Path union graph of even cycle C_n is an odd sequential graph.*

Proof Consider k copies of even cycle C_n. Let $v_{i,j}$ be the vertices of C_n where $1 \leq i \leq k$ and $1 \leq j \leq n$. Without loss of generality let each copy of C_n is joined to its succeeding one by the edge $v_{i,1}v_{i+1,1}$; $1 \leq i \leq k-1$. Let G be the path union graph of k copies of even cycle C_n. Define $f : V(G) \to \{0, 1, \cdots, q\}$ as follows.

Case 1. k is odd.

In this case, let $f(v_{i,1}) = (n+1)(i-1)$, $f(v_{i,2}) = (n+1)(i-1)+1$. For $1 \leq i \leq k$ and $1 \leq j \leq \dfrac{n+2}{2}$, let $f(v_{i,j}) = i(n+1) - j + 2$. For $1 \leq i \leq k$ and $\dfrac{n+4}{2} \leq j \leq n$, let $f(v_{i,j}) = i(n+1) - j$ if i is odd, and $i(n+1) - j + 2$ if i is even.

Case 2. k is even.

In this case, for $1 \leq i \leq k$ and $1 \leq j \leq \dfrac{n-2}{2}$,let $f(v_{i,j}) = i(n+1) - j - 2$, for $1 \leq i \leq k$ and $\dfrac{n}{2} \leq j \leq n$, let $f(v_{i,j}) = i(n+1) - j - 2$ if i is odd, and $i(n+1) - j$ if i is even.

The above described function satisfies all the conditions of odd sequential labeling. That is, path union graph of even cycle C_n is an odd sequential graph.

Illustration 2.12 The following Figure 6 shows an odd sequential labeling of 4 copies of cycle C_8.

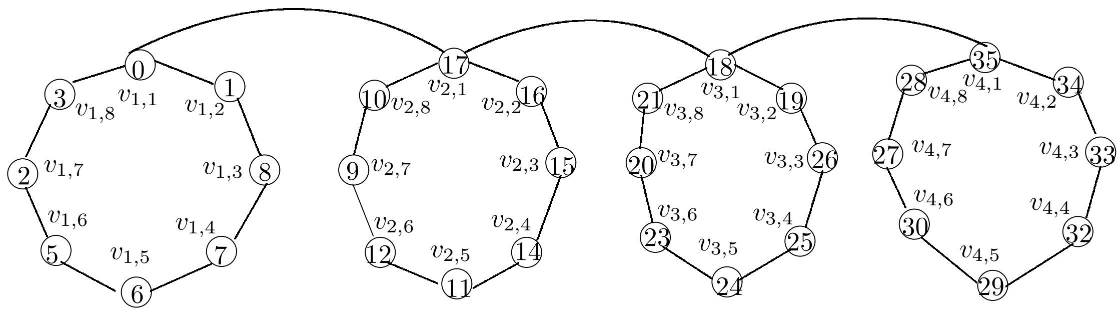

Figure 6

§3. Concluding Remarks

This paper presents 6 families of odd sequential graphs which are generated by some graph operations on some standard graphs. To investigate similar results for other graph families in the context of different labeling techniques is an open area of research.

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Mean Labelings on Product Graphs

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Abstract: Let G be a (p, q) graph and let $f : V(G) \to \{0, 1, \cdots, q\}$ be an injection. Then G is said to have a mean labeling if for each edge uv, there exists an induced injective map $f^* : E(G) \to \{1, 2, \cdots, q\}$ defined by

$$f^*(uv) = \frac{f(u)+f(v)}{2} \text{ if } f(u)+f(v) \text{ is even, and}$$
$$= \frac{f(u)+f(v)+1}{2} \text{ if } f(u)+f(v) \text{ is odd}$$

We extend this notion to *Smarandachely near m-mean labeling* if for each edge $e = uv$ and an integer $m \geq 2$, the induced Smarandachely m-labeling f^* is defined by

$$f^*(e) = \left\lceil \frac{f(u)+f(v)}{m} \right\rceil.$$

A graph that admits a Smarandachely near mean m-labeling is called *Smarandachely near m-mean graph*. The graph G is said to be a near mean graph if the injective map $f : V(G) \to \{1, 2, \cdots, q-1, q+1\}$ induces $f^* : E(G) \to \{1, 2, \cdots, q\}$ which is also injective, defined as above. In this paper we investigate the direct product of paths for their meanness and the Cartesian product of P_n and K_4 for its near-meanness.

Key Words: Smarandachely near m-labeling, Smarandachely near m-mean graph, mean graph, near-mean graph, direct product, Cartesian product.

AMS(2010): 05C78

§1. Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. For all the terminology and notations in graph theory we follow [2] and [5] and for the terminology regarding labeling we follow [1]. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The direct product of G and H is denoted by $G \times H$ and is defined as a graph with vertex set $V(G) \times V(H)$ and edge set

$$\{(g, h), (g', h')/gg' \in E(G) \text{ and } hh' \in E(H)\}.$$

The Cartesian product of G and H is denoted by $G \Box H$ and is defined as a graph with

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vertex set $V(G)\times V(H)$ and edge set $\{(g,h),(g',h')$/either $(g=g'$ and h adj $h')$ or $(g$ adj g' and $h=h')\}$. The concept of mean labeling was introduced in [6] and the notion of near-mean labeling was introduced in [3].

In [4], various product graphs are proved as near-mean graphs.

§2. Direct Product of Graphs

Definition 2.1 *The direct product of G and H is the graph denoted by $G\times H$, whose vertex set is $V(G)\times V(H)$ and for which vertices (g,h) and (g',h') are adjacent precisely if $gg'\in E(G)$ and $hh'\in E(H)$. Thus*

$$\begin{aligned} V(G\times H) &= \{(g,h)/g\in V(G) \text{ and } h\in V(H)\} \\ E(G\times H) &= \{(g,h)(g',h')/gg'\in E(G) \text{ and } hh'\in E(H)\} \end{aligned}$$

Remark 2.1 $P_m\times P_n$ is a disconnected graph with two components. Direct product is both commutative and associative. The maps $(x_1,x_2)\mapsto(x_2,x_1)$ and $((x_1,x_2),x_3)\mapsto(x_1(x_2,x_3))$ give rise to the following isomorphisms

$$G_1\times G_2\cong G_2\times G_1,\qquad (G_1\times G_2)\times G_3\cong G_1\times(G_2\times G_3)$$

Theorem 2.1 *$P_3\times P_m$ is a mean graph when $m\geqslant 3$ and is odd.*

Proof Let u_{ij}; $i=1,2,3; j=1,2,\cdots,m$ be the vertices of $P_3\times P_m$. Note that this graph has $3m$ vertices and $4(m-1)$ edges. Define $f:V(P_3\times P_m)\to\{0,1,\cdots,q\}$ such that

$$\begin{aligned} f(u_{11}) &= 0 \\ f(u_{1j}) &= \begin{cases} 2j-3 & ; j=3,5,\cdots,m \\ 2m & ; j=2 \\ f(u_{1,j-2})+j-k & ; j=4,6,\ldots,m-1; k=1,2,3;1,2,3;1,2,3\cdots \end{cases} \\ f(u_{2j}) &= \begin{cases} 2(j-1) & ; j=2,4,\cdots,m-1 \\ 2(m-1) & ; j=1 \\ f(u_{2,j-2})+4 & ; j=3,5,\cdots,m \end{cases} \\ f(u_{3j}) &= \begin{cases} 2j-1 & ; j=1,3,\cdots,m \\ 2m+1 & ; j=2 \\ f(u_{3,j-2})+4 & ; j=4,6\cdots,m-1 \end{cases} \end{aligned}$$

It can be easily verified that f is one one which induces the edge labels $f^*(E(P_3\times P_m))$. Hence the theorem. $\square$

Example 2.1 The Fig.1 following shows the mean labeling of $P_3\times P_7$.

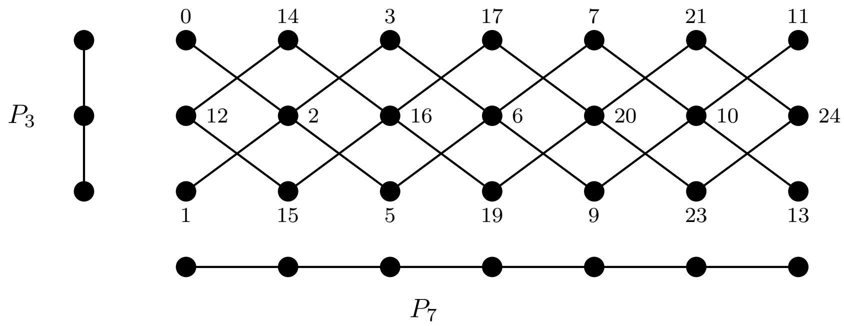

Fig 1

Theorem 2.2 *$P_5 \times P_m$ admits mean labeling when $m \geqslant 7$ and is odd.*

Proof Let $u_{ij},\ i = 1, 2, \cdots, 5$ and $j = 1, 2, \cdots, m$ be the vertices of $P_5 \times P_m$. Consider $f : V(P_5 \times P_m) \to \{0, 1, \cdots, q\}$ which is defined as

$$\begin{aligned}
f(u_{11}) &= 0\\
f(u_{i1}) &= i-2, \quad i = 3, 5\\
f(u_{ij}) &= f(u_{i,j-2}) + 8, \quad i = 1, 3, 5; \ \ j = 3, 5, \cdots, m\\
f(u_{i2}) &= i, \quad i = 2, 4\\
f(u_{ik}) &= f(u_{i,k-2}) + 8, \quad i = 2, 4; k = 4, 6, \cdots, m-1
\end{aligned}$$

And when $i = 1, 3, 5,\ f(u_{i2}) = f(u_{5,m}) + i$; when $i = 2, 4,\ f(u_{i1}) = f(u_{4,m-1}) + i - 1$; when $i = 1, 2, \cdots, 5; l = 3, 4, \cdots, m,\ f(u_{il}) = f(u_{i,l-2}) + 8$.

From the definition of labelings on $V(P_5 \times P_m)$, we can infer that the vertex labels are in an increasing sequence. That is the sequence such as:

For $j = 1, 3, \cdots, m, \langle u_{1j}\rangle, \langle u_{3j}\rangle$ and $\langle u_{5j}\rangle$; for $j = 2, 4, \cdots, m-1,$ $\langle u_{2j}\rangle, \langle u_{4j}\rangle$ and for $k = 2, 4, \cdots, m-1,$ $\langle u_{1k}\rangle, \langle u_{3k}\rangle, \langle u_{5k}\rangle$; for $k = 1, 3, \cdots, m,$ $\langle u_{2k}\rangle$ and $\langle u_{4k}\rangle$, occur as an arithmetic progression.

Also we have

$$\begin{aligned}
f(u_{11}) &= 0, \quad f(u_{31}) = 1\\
f(u_{51}) &= 3, \quad f(u_{22}) = 2\\
f(u_{42}) &= 4
\end{aligned}$$

Hence f_p is one- one with $f_p^* = \{1, 2, \cdots, q\}$. □

Remark 2.2 $P_n \times P_m$ are not mean graphs for all m. Since $P_2 \times P_m$ being a disjoint union of two P_m paths, it has $2(m-1)$ edges on $2m$ vertices. This implies that the number of edges is less than the number of vertices by 2. Hence we cannot label them with $\{0, 1, \cdots, q\}$.

Conjecture 2.1 *For m even $P_3 \times P_m$ and $P_5 \times P_m$ are not mean graphs.*

§3. Cartesian Product of Graphs

Definition 3.1 *Let G and H be graphs with $V(G) = V_1$ and $V(H) = V_2$. The cartesian product of G and H is the graph $G\Box H$ whose vertex set is $V_1 \times V_2$ such that two vertices $u = (x, y)$ and $v = (x', y')$ are adjacent if and only if either $x = x'$ and y is adjacent to y' in H or $y = y'$ and x is adjacent to x' in G. That is u adj v in $G\Box H$ whenever $[x = x'$ and y adj $y']$ or $[y = y'$ and x adj $x']$.*

Definition 3.2 *Let P_n be a path on n vertices and K_4 be the complete graph on 4 vertices. The cartesian product of P_n and K_4 is $P_n\Box K_4$ with $4n$ vertices and $10n - 4$ edges.*

Theorem 3.1 *$P_n\Box K_4$ is a near mean graph.*

Proof Let $G = P_n\Box K_4$ with $V(G) = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}/i = 1, 2, \cdots, n\}$. Define $f : V(G) \to \{0, 1, \cdots, q - 1, q + 1\}$ such that

$$\begin{aligned}
f(u_{11}) &= 0,\quad f(u_{i1}) = 5(2i-1),\ i = 2, 4, \ldots, n \\
&= 5(2i-2),\ i \neq 1,\ \text{odd} \\
f(u_{i2}) &= 10(i-1) + 2 \\
f(u_{i3}) &= 5(2i-1) + (-1)^i 2 \\
f(u_{i4}) &= \begin{cases} 5(2i-1)+3, & i \text{ odd} \\ 5(2i-3)+4 & i \text{ even} \end{cases}
\end{aligned}$$

The edge labels induced by f are as follows:

When i is even,

$$\begin{aligned}
f^*(u_{i1}u_{i2}) &= \frac{1}{2}\Big[f(u_{i1}) + f(u_{i2}) + 1\Big] \\
&= \frac{1}{2}\Big[5(2i-1) + 5(2i-2) + 2 + 1\Big] \\
&= 10i - 6, \quad i = 2, 4, \ldots, n
\end{aligned}$$

When i is odd,

$$\begin{aligned}
f^*(u_{i1}u_{i2}) &= \frac{f(u_{i1}) + f(u_{i2})}{2} \\
&= \frac{5(2i-2) + 5(2i-2) + 2}{2} \\
&= 5(2i-2) + 1, \quad i = 1, 3, 5, \ldots
\end{aligned}$$

Hence the edges $u_{i1}u_{i2}$ carry labels $1, 14, 21, \cdots, 10(n-1)+1$ if n is odd or $1, 14, 21, \cdots, 10n-6$

if n is even.

$$\begin{aligned}
f^*(u_{i1}, u_{i+1,1}) &= \frac{f(u_{i1}) + f(u_{i+1,1}) + 1}{2}, \quad i = 1, 2, \cdots, n-1 \\
&\quad \text{(since } f(u_{i1}) \text{ and } f(u_{i+1,1}) \text{ are of opposite parity)} \\
&= \frac{1}{2}[5(2i-1) + 5(2(i+1) - 2) + 1] \\
&= 10i - 2
\end{aligned}$$

Hence the edges $u_{i1}, u_{i+1,1}$ have labels as $8, 18, 28, \cdots, 10n - 12$.

$$\begin{aligned}
f^*(u_{i2}, u_{i+1,2}) &= \frac{f(u_{i2}) + f(u_{i+1,2})}{2} \\
&\quad \text{(since } f(u_{i2}), f(u_{i+1,2}) \text{ are of same parity)} \\
&= 10i - 3, \ i = 1, 2, \cdots, (n-1)
\end{aligned}$$

The edges $u_{i2}, u_{i+1,2}$ have $7, 17, 27, \cdots, 10n - 13$ as labels.

$$\begin{aligned}
f^*(u_{i3}, u_{i+1,3}) &= \frac{f(u_{i3}) + f(u_{i+1,3})}{2} \\
&= 10i, \ i = 1, 2, \cdots, (n-1)
\end{aligned}$$

Therefore, $u_{i3}u_{i+1,3}$ assume labels $10, 20, 30, \cdots, 10(n-1)$,

$$\begin{aligned}
f^*(u_{i4}, u_{i+1,4}) &= \frac{f(u_{i4}) + f(u_{i+1,4}) + 1}{2} \\
&\quad \text{(since both vertex labels are of opposite parity)} \\
&= \frac{1}{2}[5(2i-1) + 3 + 5(2i-1) + 4 + 1] \\
&= 10i - 1 \\
\text{or} \quad &= \frac{1}{2}[5(2i-3) + 4 + 5(2i+1) + 3 + 1] = 10i - 1
\end{aligned}$$

Therefore $u_{i4}u_{i+1,4}$ have labels as $9, 19, \cdots, 10n - 11$.

When i is odd,

$$\begin{aligned}
f^*(u_{i2}, u_{i4}) &= \frac{f(u_{i2}) + f(u_{i4})}{2} \\
&= \frac{5(2i-2) + 2 + 5(2i-1) + 3}{2} \\
&= 10i - 5
\end{aligned}$$

When i is even,

$$\begin{aligned}
f^*(u_{i2}u_{i4}) &= \frac{f(u_{i2}) + f(u_{i4}) + 1}{2} \\
&= \frac{10(i-1) + 2 + 5(2i-3) + 4 + 1}{2} \\
&= 10i - 9
\end{aligned}$$

Hence $5, 11, 25, \cdots 10n - 9$ if n is even or $5, 11, 25, \cdots, 10n - 5$ if n is odd, correspond to the edges $u_{i2}u_{i4}$

$$f^*(u_{i2}, u_{i3}) = \frac{f(u_{i2}) + f(u_{i3}) + 1}{2} = 10i - 6 + (-1)^i$$

So the edges $u_{i2}u_{i3}$ have labels $3, 15, 23, \cdots, 10n - 6 + (-1)^n$.

$$\begin{aligned} f^*(u_{i3}, u_{i4}) &= \frac{f(u_{i3}) + f(u_{i4})}{2} = 10i - 7 \text{ if } i \text{ is even, or} \\ &= \frac{f(u_{i3}) + f(u_{i4}) + 1}{2} = 10i - 4 \text{ if } i \text{ is odd} \end{aligned}$$

So the values taken by $u_{i3}u_{i4}$ are $6, 13, 26, \cdots 10n - 7$ if n is even or $6, 13, \cdots, 10n - 4$ if n is odd.

If i is odd,

$$f^*(u_{i1}, u_{i3}) = \frac{f(u_{i1})+f(u_{i3})+1}{2} = 10i - 8$$

If i is even,

$$f^*(u_{i1}, u_{i3}) = \frac{f(u_{i1}) + f(u_{i3})}{2} = 10i - 4$$

If i is odd,

$$f^*(u_{i1}, u_{i4}) = \frac{f(u_{i1}) + f(u_{i4})}{2} = 10i - 6$$

If i is even,

$$f^*(u_{i1}, u_{i4}) = \frac{f(u_{i1}) + f(u_{i4})}{2} = 10i - 8$$

Hence the edge values of $u_{i1}u_{ij}$ are $1, 2, 4, \cdots, 10n - 8, 10n - 6, 10n - 4$ if n is even, or $1, 2, \cdots, 10n - 9, 10n - 8, 10n - 6$ if n is odd. Hence the theorem. □

Example 3.1 The Fig.2 following shows the near mean labeling of $P_4 \square K_4$.

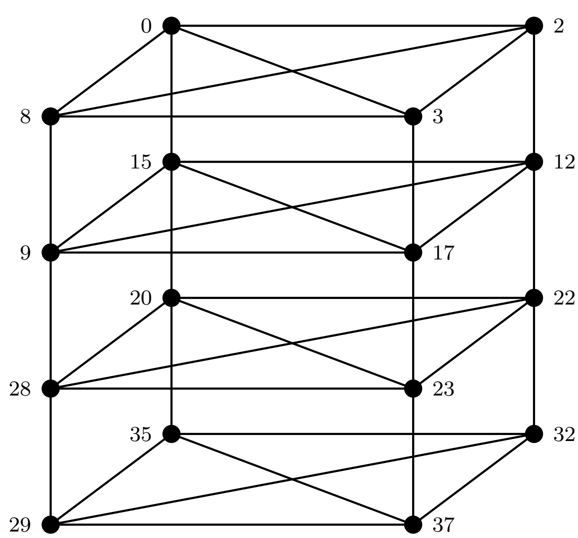

Fig 2

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Total Near Equitable Domination in Graphs

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Abstract: Let $G = (V, E)$ be a graph, $D \subseteq V$ and u be any vertex in D. Then the out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$. A subset $D \subseteq V(G)$ is called a near equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. A near equitable dominating set D is said to be a total near equitable dominating set (tned-set) if every vertex $w \in V$ is adjacent to an element of D. The minimum cardinality of tned-set of G is called the total near equitable domination number of G and is denoted by $\gamma_{tne}(G)$. The maximum order of a partition of V into tned-sets is called the total near equitable domatic number of G and is denoted by $d_{tne}(G)$. In this paper we initiate a study of these parameters.

Key Words: Equitable domination number, near equitable domination number, near equitable domatic number, total near equitable domination Number, total near equitable domatic number, Smarandachely k-dominator coloring.

AMS(2010): 05C22

§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [2].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Let G be a graph without isolated vertices. For an integer $k \geqslant 1$, a Smarandachely k-dominator coloring of G is a proper coloring of G with the extra property that every vertex in G properly dominates a k-color classes. Particularly, a subset S of V is called a dominating set if $N[S] = V$, i.e., a Smarandachely 1-dominator set. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ $(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [5]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [6]. Various types of domination have been defined and

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studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [5]. E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi [3] introduced the concept of total domination in graphs. A dominating set D of a graph G is a total dominating set if every vertex of V is adjacent to some vertex of D. The cardinality of a smallest total dominating set in a graph G is called the total domination number of G and is denoted by $\gamma_t(G)$.

A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

Equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly.

Let $D \subseteq V(G)$ and u be any vertex in D. The out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$. D is called near equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_{ne} and is called the near equitable domination number of G. A partition $P = \{V_1, V_2, \cdots, V_l\}$ of a vertex set $V(G)$ of a graph is called near equitable domatic partition of G if V_i is near equitable dominating set for every $1 \leq i \leq l$. The near equitable domatic number of G is the maximum cardinality of near equitable domatic partition of G and denoted by $d_{ne}(G)$ [7].

For a near equitable dominating set D of G it is natural to look at how total D behaves. For example, for the cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_2, v_3, v_4\}$ are near equitable dominating sets, S_1 is not total and S_2 is total.

In this paper, we introduce the concept of a total near equitable domination to initiate a study of a total near equitable domination number and a total near equitable domatic number.

We need the following to prove main results.

Definition 1.1([7]) *Let $G = (V, E)$ be a graph and D be a near equitable dominating set of G. Then $u \in D$ is a near equitable pendant vertex if $od_D(u) = 1$. A set D is called a near equitable pendant set if every vertex in D is an equitable pendant vertex.*

Theorem 1.2([7]) *Let T be a wounded spider obtained from the star $K_{1,n-1}$, $n \geq 5$ by subdividing m edges exactly once. Then*

$$\gamma_{ne}(T) = \begin{cases} n, & \text{if } m = n-1\ ; \\ n-1, & \text{if } m = n-2; \\ n-2, & \text{if } m \leq n-3. \end{cases}$$

§2. Total Near Equitable Domination in Graphs

A near equitable dominating set D of a graph G is said to be a *total near equitable dominating set* (tned-set) if every vertex $w \in V$ is adjacent to an element of D. The minimum of the cardinality of tned-set of G is called a total near equitable domination number and is denoted by $\gamma_{tne}(G)$. A subset D of V is a minimal tned-set if no proper subset of D is a tned-set.

We note that this parameter is only defined for graphs without isolated vertices and, since each total near equitable dominating set is a near equitable dominating set, we have $\gamma_{ne}(G) \leq \gamma_{tne}(G)$. Since each total near equitable dominating set is a total dominating set, we have $\gamma_t(G) \leq \gamma_{tne}(G)$. The bound is sharp for rK_2, $r \geq 1$. In fact $\gamma_{tne}(G) = \gamma_t(G) = |V|$, for $G = rK_2$, it is easy to see however, that rK_2, $r \geq 1$ is the only graph with this property. Furthermore, the difference $\gamma_{tne}(G) - \gamma_t(G)$ can be arbitrarily large in a graph G. It can be easily checked that $\gamma_t(K_{1,r}) = 2$, while $\gamma_{tne}(K_{1,r}) = n - 2$.

We now proceed to compute $\gamma_{tne}(G)$ for some standard graphs.

1. For any path P_n, $n \geq 4$,

$$\gamma_{tne}(P_n) = \gamma_t(P_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \equiv 2 \ (mod\ 4); \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

where $\lceil x \rceil$ is a least integer not less than x.

2. For any cycle C_n, $n \geq 4$,

$$\gamma_{tne}(C_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \equiv 2 \ (mod\ 4); \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

.

3. For the complete graph K_n, $n \geq 4$ $\gamma_{tne}(K_n) = \gamma_{ne}(K_n) = \lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ is a greatest integer not exceeding x.

4. For the double star $S_{n,m}$,

$$\gamma_{tne}(S_{n,m}) = \gamma_{ne}(S_{n,m}) = \begin{cases} 2, & \text{if } n, m \leq 2 \ ; \\ n + m - 2, & \text{if } n, m \geq 2 \text{ and } n \text{ or } m \geq 3. \end{cases}$$

5. For the complete bipartite graph $K_{n,m}$ with $2 < m \leq n$, we have

$$\gamma_{tne}(K_{n,m}) = \gamma_{ne}(K_{n,m}) = \begin{cases} m - 1, & \text{if } n = m \text{ and } m \geq 3; \\ m, & \text{if } n - m = 1; \\ n - 1, & \text{if } n - m \geq 2. \end{cases}$$

6. For the wheel W_n on n vertices,

$$\gamma_{tne}(W_n) = \gamma_{ne}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1.$$

Theorem 2.1 *Let G be a graph and D be a minimum tned- set of G containing t near equitable pendant vertices. Then $\gamma_{tne}(G) \geq \dfrac{n+t}{3}$.*

Proof Let D be any minimum tned- set of G containing t near equitable pendant vertices . Then $2|D| - t \geq |V - D|$. It follows that, $3|D| - t \geq n$. Hence $\gamma_{tne}(G) \geq \dfrac{n+t}{3}$. □

Theorem 2.2 *Let T be a wounded spider obtained from the star $K_{1,n-1}$, $n \geq 5$ by subdividing m edges exactly once. Then*

$$\gamma_{tne}(T) = \gamma_{ne}(T) = \begin{cases} n, & \text{if } m = n-1\ ; \\ n-1, & \text{if } m = n-2; \\ n-2, & \text{if } m \leq n-3. \end{cases}$$

Proof Proof follows from Theorem 1.2. □

Theorem 2.3 *Let T be a tree of order n, $n \geq 4$ in which every non-pendant vertex is either a support or adjacent to a support and every non- pendant vertex which is support is adjacent to at least two pendant vertices. Then $\gamma_{tne}(T) = \gamma_{ne}(T)$.*

Proof Let D denote set of all non-pendant vertices and all pendant vertices except two for each support of T. Clearly, D is a γ_{ne}-set. Since any support vertex adjacent to at least two pendant vertices, it follows that $\langle D \rangle$ contains no isolate vertex. Hence D is a tned-set and hence $\gamma_{tne}(T) \leq \gamma_{ne}(T)$. Since $\gamma_{ne}(T) \leq \gamma_{tne}(T)$, it follows that $\gamma_{tne}(T) = \gamma_{ne}(T)$. □

Theorem 2.4 *Let G be a connected graph of order n, $n \geq 4$. Then,*

$$\gamma_{tne}(G) \leq n - 2.$$

Proof It is enough to show that for any minimum total near equitable dominating set D of G, $|V - D| \geq 2$. Since G is a connected graph of order n, $n \geq 4$, it follows that $\delta(G) \geq 1$. Suppose $v \in V - D$ and adjacent to $u \in D$. Since $od_{V-D}(v) \geq 1$, then $od_D(u) \geq 2$. □

The star graph $G \cong K_{1,n}$ is an example of a connected graph for which $\gamma_{tne}(G) = 2n - (\Delta(G) + 3)$. The following theorem shows that, this is the best possible upper bound for $\gamma_{tne}(G)$.

Theorem 2.5 *If G is connected of order n, $n \geq 4$, then,*

$$\gamma_{tne}(G) \leq 2n - (\Delta(G) + 3).$$

Proof Let G be a connected graph of order n, $n \geq 4$, then by Theorem 2.4, $\gamma_{tne}(G) \leq n - 2 = 2n - (n - 1 + 3) \leq 2n - (\Delta(G) + 3)$. □

Theorem 2.6 *If G is a graph of order n, $n \geq 4$ and $\Delta(G) \leq n - 2$ such that both G and $\overline{G}$ connected, then*

$$\gamma_{tne}(G) + \gamma_{tne}(\overline{G}) \leq 3n - 6.$$

Proof Let G be a connected graph and $\Delta(G) \leq n - 2$. By Theorem 2.4, $\gamma_{tne}(G) \leq 2n - (\Delta(G)+4) \leq 2n-(\delta(G)+4)$. Since $\overline{G}$ is a connected, by Theorem 2.5, $\gamma_{tne}(\overline{G}) \leq 2n-(\Delta(\overline{G})+3)$, it follows that

$$\begin{aligned}\gamma_{tne}(G)+\gamma_{tne}(\overline{G}) &\leq 2n-(\delta(G)+4)+2n-(\Delta(\overline{G})+3)\\ &= 4n-(\delta(G)+\Delta(\overline{G}))-7\\ &= 3n-6.\end{aligned}$$ □

The bound is sharp for C_4.

Theorem 2.7 *Let G be a graph such that both G and $\overline{G}$ connected. Then,*

$$\gamma_{tne}(G)+\gamma_{tne}(\overline{G}) \leq 2n-4.$$

Proof Since both G and $\overline{G}$ are a connected, it follows by Theorem 2.4 that, $\gamma_{tne}(G)+\gamma_{tne}(\overline{G}) \leq 2n-4$. □

The bound is sharp for P_4. We now proceed to obtain a characterization of minimal tned-sets.

Theorem 2.8 *A tned- set D of a graph G is a minimal tned- set if and only if one of the following holds:*

(i) D is a minimal near equitable dominating set;
(ii) There exist $x, y \in D$ such that $N(y) \cap N(D-\{x\}) = \phi$.

Proof Suppose that D is a minimal tned-set of G. Then for any $u \in D$, $D-\{u\}$ is not tned-set. If D is a minimal near equitable dominating set, then we are done. If not, then there exists a vertex $x \in D$ such that $D-\{x\}$ is a near equitable dominating set, but not a tned- set. Therefore there exists a vertex $y \in D-\{x\}$ such that y is an isolated vertex in $\langle (D-\{x\})\rangle$. Hence $N\{y\} \cap N(D-\{x\}) = \phi$.

Conversely, let D be a tned- set and (i) holds. Suppose D is not a minimal tned-set. Then for every $u \in D$, $D-\{u\}$ is a tned- set. So, D is not a minimal near equitable dominating set, a contradiction. Next, suppose that D is a tned- set and (ii) holds. Then there exist $x, y \in D$ such that $N(y) \cap N(D-\{x\}) = \phi$.
Suppose to the contrary, D is not a minimal tned- set. Then for every $u \in D$, $D-\{u\}$ is a tned- set. So, $\langle (D-\{u\})\rangle$ does not contain any isolated vertex. Therefore for every $x, y \in D$, $N(y) \cap N(D-\{x\}) \neq \phi$, a contradiction. □

Theorem 2.9 *For any positive integer m, there exists a graph G such that $\gamma_{_{tne}}(G)-\left\lfloor \frac{n}{\Delta+1}\right\rfloor =$ m, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.*

Proof For $m=1$, let $G=K_{3,3}$. Then, $\gamma_{_{tne}}(G)-\left\lfloor \frac{n}{\Delta+1}\right\rfloor = 2-1 = 1$.
For $m=2$, let $G=K_{2,4}$. Then, $\gamma_{_{tne}}(G)-\left\lfloor \frac{n}{\Delta+1}\right\rfloor = 3-1 = 2$.
For $m \geq 3$, let $G=S_{r,s}$, where $r+s=m+3$, $s \geq r+3$, $r \geq 2$, $\gamma_{_{tne}}(G) = r+s-2 = m+1$,

$$\left\lfloor \frac{n}{\Delta+1}\right\rfloor = \left\lfloor \frac{r+s+2}{s+2}\right\rfloor = 1$$

and

$$\gamma_{tne}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = r + s - 3 = m. \qquad \square$$

§3. Total Near Equitable Domatic Number

The maximum order of a partition of the vertex set V of a graph G into dominating sets is called the domatic number of G and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [9]. In this section we present few basic results on the total near equitable domatic number of a graph.

Let G be a graph without isolated vertices. A total near equitable domatic partition (tne-domatic partition) of G is a partition $\{V_1, V_2, \cdots, V_k\}$ of $V(G)$ in which each V_i is a tned-set of G. The maximum order of a tne-domatic partition of G is called the *total near equitable domatic number* (tne-domatic number) of G and is denoted by $d_{tne}(G)$.

We now proceed to compute $d_{tne}(G)$ for some standard graphs.

1. For any complete graph K_n, $n \geq 4$, $d_{tne}(K_n) = d_{ne}(K_n) = 2$.

2. For any $n \geq 1$, $d_{tne}(C_{4n}) = 2$.

3. For any star $K_{1,n}$, $n \geq 3$, $d_{tne}(K_{1,n}) = d_{ne}(K_{1,n}) = 1$.

4. For the wheel W_n on n vertices, then $d_{tne}(W_n) = d_{ne}(W_n) = 1$.

5. For the complete bipartite graph $K_{n,m}$, with $2 < m \leq n$

$$d_{tne}(K_{n,m}) = d_{ne}(K_{n,m}) = \begin{cases} 2, & \text{if } |n-m| \leq 2 \text{ ;} \\ 1, & \text{if } |n-m| \geq 3,\ n,m \geq 2. \end{cases}$$

It is obvious that any total near equitable domatic partition of a graph G is also a total domatic partition and any total domatic partition is also a domatic partition, thus we obtain the obvious bound $d_{tne}(G) \leq d_t(G) \leq d(G)$.

Remark 3.1 Let $v \in V(G)$ and $deg(v) = \delta$. Since any tned-set of G must contain either v or a neighbour of v and $d_{tne}(G) \leq d_t(G)$, it follows that $d_{tne}(G) \leq \delta$.

Definition 3.2 *A graph G is called tne-domatically full if $d_{tne}(G) = \delta$.*

For example, a star $K_{1,n}$ is tne-domatically full.

Remark 3.3 Since every member of any tne-domatic partition of a graph G on n vertices has at least $\gamma_{tne}(G)$ vertices, it follows that $d_{tne}(G) \leq \dfrac{n}{\gamma_{tne}(G)}$. This inequality can be strict for rK_2, $r \geq 1$.

Theorem 3.4 *Let G be a graph of order n, $n \geq 4$ with $\Delta(G) \leq 2$ such that both G and $\overline{G}$ are connected. Then $d_{tne}(\overline{G}) \leq 2$.*

proof Since $\Delta(G) \leq 2$, it follows that for any $v \in \overline{G}$, $deg(v) \geq n-3$. Hence $\gamma_{tne}(\overline{G}) \leq \lceil \frac{n}{2} \rceil$. Thus by Remark 3.3, $d_{tne}(G) \leq 2$. □

The bound is sharp for P_n, $n \geq 6$.

Theorem 3.5 *Let G be a graph of order n, $n \geq 4$ with $\Delta(G) \leq 2$ such that both G and $\overline{G}$ are connected. Then $\gamma_{tne}(G) + d_{tne}(\overline{G}) \leq n$.*

Proof Proof follows by Theorem 2.4 and Theorem 3.4. □

theorem 3.6 *For any graph G, $\gamma_{tne}(G) + d_{tne}(G) \leq 2n - 3$.*

proof By Theorem 2.5,

$$\gamma_{tne}(G) \leq 2n - (\Delta(G) + 3) \leq 2n - (\delta(G) + 3) \leq 2n - (d_{tne}(G) + 3).$$

Therefor, $\gamma_{tne}(G) + d_{tne}(G) \leq 2n - 3$. □

The bound is sharp for $2K_2$.

theorem 3.7 *For any graph G, $\gamma_{tne}(G) + d_{tne}(G) \leq n + \delta - 2$.*

Proof Since $d_{tne}(G) \leq d_t(G) \leq \delta(G)$, by Theorem 2.4,

$$\gamma_{tne}(G) + d_{tne}(G) \leq n + \delta - 2.$$ □

The bound is sharp for $K_{1,n}$.

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Give me the greatest pleasure, not knowledge, but continuous learning; not for things, but constantly acquisition; not have reached the heights, but continued to climb.

By Johann Carl Friedrich Gauss, a Germany mathematician.

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[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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