

Vertex Graceful Labeling-Some Path Related Graphs

P.Selvaraju¹, P.Balaganesan² and J.Renuka³

¹Department of Mathematics, Vel Tech Engineering College, Avadi, Chennai- 600 062, Tamil Nadu, India

²Department of Mathematics, School of Engineering, Saveetha University, Chennai- 602 105, Tamil Nadu, India

³ Departments of Mathematics, Sai Ram College of Engineering, Chennai - 600 044, India

E-mail: pselvarr@gmail.com, balki2507@yahoo.co.in

Abstract: In this article, we show that an algorithm for VG of a caterpillar and proved that $A(m_j, n)$ is vertex graceful if m_j is monotonically increasing, $2 \leq j \leq n$, when n is odd, $1 \leq m_2 \leq 3$ and $m_1 < m_2$, $(m_j, n) \cup P_3$ is vertex graceful if m_j is monotonically increasing, $2 \leq j \leq n$, when n is odd, $1 \leq m_2 \leq 3$, $m_1 < m_2$ and $C_n \cup C_{n+1}$ is vertex graceful if and only if $n \geq 4$.

Key Words: Vertex graceful graphs, vertex graceful labeling, caterpillar, actinia graphs, Smarandachely vertex m -labeling.

AMS(2010): 05C78

§1. Introduction

A graph G with p vertices and q edges is said to be vertex graceful if a labeling $f : V(G) \rightarrow \{1, 2, 3 \dots p\}$ exists in such a way that the induced labeling $f^+ : E(G) \rightarrow Z_q$ defined by $f^+((u, v)) = f(u) + f(v) \pmod{q}$ is a bisection. The concept of vertex graceful (VG) was introduced by Lee, Pan and Tsai in 2005. Generally, if replacing q by an integer m and $f^S : E(G) \rightarrow Z_m$ also is a bijection, such a labeling is called a *Smarandachely vertex m -labeling*. Thus a vertex graceful labeling is in fact a Smarandachely vertex q -labeling.

All graphs in this paper are finite simple graphs with no loops or multiple edges. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G . The cardinality of the vertex set is called the order of G . The cardinality of the edge set is called the size of G . A graph with p vertices and q edges is called a (p, q) graph.

§2. Main Results

Algorithm 2.1

1. Let $v_1, v_2 \dots v_n$ be the vertices of a path in the caterpillar. (refer Figure 1).
2. Let v_{ij} be the vertices, which are adjacent to v_i for $1 \leq i \leq n$ and for any j .
3. Draw the caterpillar as a bipartite graph in two partite sets denoted as Left (L) which

¹Received April 10, 2013, Accepted August 15, 2013.

contains $v_1, v_{2j}, v_3, v_{4j}, \dots$ and for any j and Right (R) which contains $v_{1j}, v_2, v_{3j}, v_4, \dots$ and for any j . (refer Figure 2).

4. Let the number of vertices in L be x .
5. Number the vertices in L starting from top down to bottom consecutively as $1, 2, \dots, x$.
6. Number the vertices in R starting from top down to bottom consecutively as $(x + 1), \dots, q$. Note that these numbers are the vertex labels.
7. Compute the edge labels by adding them modulo q .
8. The resulting labeling is vertex graceful labeling.

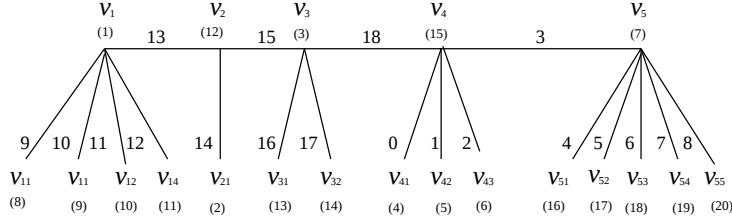


Figure 1: A caterpillar

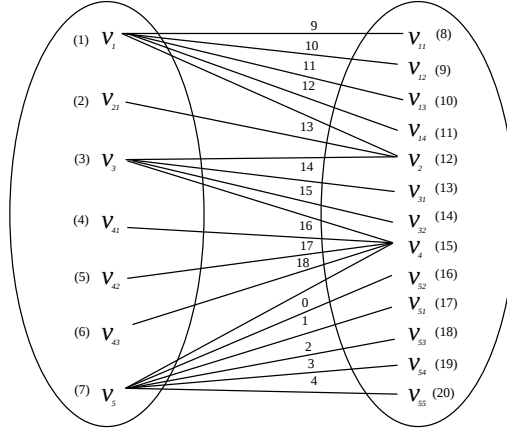


Figure 2: A caterpillar as bipartite graph

Definition 2.2 The graph $A(m, n)$ obtained by attaching m pendent edges to the vertices of the cycle C_n is called Actinia graph.

Theorem 2.3 A graph $A(m_j, n)$, m_j is monotonically increasing with difference one, $2 \leq j \leq n$ is vertex graceful, $1 \leq m_2 \leq 3$ when n is odd.

Proof Let the graph $G = A(m_j, n)$, m_j be monotonically increasing with difference one, $2 \leq j \leq n$, n be odd with $p = n + m_n(\frac{m_n+1}{2}) - m_1(\frac{m_1+1}{2})$, $m_1 = m_2 - 1$ vertices and $q = p$ edges. Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of the cycle C_n . Let v_{ij} ($j = 1, 2, 3, \dots, n$) denote the vertices which are adjacent to v_i . By definition of vertex graceful labeling, the required

vertices labeling are

$$v_i = \begin{cases} \frac{(i-1)}{2} \left(m_2 + \frac{(i+1)}{2} \right) + 1, & 1 \leq i \leq n, i \text{ is odd}, \\ (m_2 + 1) \frac{(n+1)}{2} + \left(\frac{n-1}{2} \right)^2 + \frac{(i-2)}{2} \left(m_2 + \frac{i}{2} \right) + \frac{i}{2}, & 1 \leq i \leq n, i \text{ is even}. \end{cases}$$

$$v_{ij} = \begin{cases} \frac{(n-1)}{2} \left(m_2 + \frac{(n+1)}{2} \right) + \frac{i-1}{2} \left(m_2 + \frac{i-3}{2} \right) + \frac{i+1}{2} + j, & 1 \leq j \leq m_2 + i - 1, i \text{ is odd}; \\ \frac{(i-2)}{2} \left(m_2 + \frac{i-2}{2} \right) + \frac{i}{2} + j, & 1 \leq j \leq m_2 + i - 1, i \text{ is even}. \end{cases}$$

The corresponding edge set labels are as follows:

Let $A = \{e_i = v_i v_{i+1} / 1 \leq i \leq n-1 \cup e_n = v_n v_1\}$, where

$$e_i = \left[\frac{(m_2 + 1)(n + 1)}{2} + \left(\frac{n-1}{2} \right)^2 + m_2(i-1) + \frac{i(i+1)}{2} + 1 \right] \pmod{q}$$

for $1 \leq i \leq n$. $B = \{e_{ij} = v_i v_{ij} / 1 \leq i \leq n\}$, where

$$e_{ij} = \left[\frac{(n-1)}{2} \left(m_2 + \frac{(n+1)}{2} \right) + (i-1) \left(m_2 + \frac{i-1}{2} \right) + \frac{(i+1)}{2} + j + 1 \right] \pmod{q}$$

for $1 \leq i \leq n$ and i is odd, $j = 1, 2, \dots, m_2 + i - 1$. $C = \{e_{ij} = v_i v_{ij} / 1 \leq i \leq n\}$, where

$$e_{ij} = \left[(m_2 + 1) \frac{(n+1)}{2} + \left(\frac{n-1}{2} \right)^2 + \frac{i-2}{2} (2m_2 + i - 1) + i + j \right] \pmod{q}$$

for $1 \leq i \leq n$ and i is even, $j = 1, 2, \dots, m_2 + i - 1$.

Hence, the induced edge labels of G are q distinct integers. Therefore, the graph $G = A(m_j, n)$ is vertex graceful for n is odd, and $m \geq 1$. $\square$

Theorem 2.4 *A graph $A(m_j, n) \cup P_3, m_j$ be monotonically increasing, $2 \leq j \leq n$ is vertex graceful, $1 \leq m_2 \leq 3, n$ is odd.*

Proof Let the graph $G = A(m_j, n) \cup P_3, m_j$ be monotonically increasing, $2 \leq j \leq n$, n is odd with $p = n + 3 + m_n \frac{(m_n+1)}{2} - m_1 \frac{(m_1+1)}{2}$, $m_1 < m_2$ vertices and $q = p - 1$ edges. Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of the cycle C_n . Let $v_{ij} (j = 1, 2, 3, \dots, n)$ denote the vertices which are adjacent to v_i . Let u_1, u_2, u_3 be the vertices of the path P_3 . By definition of vertex graceful labeling, the required vertices labeling are

$$v_i = \begin{cases} \frac{i-1}{2} \left(m_2 + \frac{i+1}{2} \right) + 1; & 1 \leq i \leq n, i \text{ is odd}; \\ (m_2 + 1) \frac{(n+1)}{2} + \left(\frac{n-1}{2} \right)^2 + \frac{(i-2)}{2} \left(m_2 + \frac{i}{2} \right) + \frac{i}{2} + 2; & 1 \leq i \leq n, i \text{ is even}. \end{cases}$$

$$v_{ij} = \begin{cases} \frac{n-1}{2} \left(m_2 + \frac{n+1}{2} \right) + \frac{i-1}{2} \left(m_2 + \frac{i-3}{2} \right) + \frac{i+1}{2} + j + 2; & 1 \leq i \leq n, i \text{ is odd}, \\ \frac{i-2}{2} \left(m_2 + \frac{i-2}{2} \right) + \frac{i}{2} + j + 2; & 1 \leq i \leq n, i \text{ is even}. \end{cases}$$

$$u_i = \frac{n-1}{2} \left(m_2 + \frac{n+1}{2} \right) + \frac{i+1}{2} \text{ for } i = 1, 3 \text{ and } u_2 = p.$$

The corresponding edge labels are as follows:

Let $A = \{e_i = v_i v_{i+1} / 1 \leq i \leq n-1 \cup e_n = v_n v_1\}$, where

$$e_i = \left[\frac{(m_2 + 1)(n + 1)}{2} + \left(\frac{n-1}{2} \right)^2 + m_2(i-1) + \frac{i(i+1)}{2} + 3 \right] \pmod{q}$$

for $1 \leq i \leq n$. $B = \{e_{ij} = v_i v_{ij}/1 \leq i \leq n\}$, where

$$e_{ij} = \left[\frac{(n-1)}{2} \left(m_2 + \frac{(n+1)}{2} \right) + (i-1) \left(m_2 + \frac{i-1}{2} \right) + \frac{(i+1)}{2} + j + 3 \right] \pmod{q}$$

for $1 \leq i \leq n$ and i is odd, $j = 1, 2, \dots, m_2 + i - 1$. $C = \{e_{ij} = v_i v_{ij}/1 \leq i \leq n\}$, where

$$e_{ij} = \left[(m_2 + 1) \frac{(n+1)}{2} + \left(\frac{n-1}{2} \right)^2 + \frac{i-2}{2} (2m_2 + i - 1) + i + j + 2 \right] \pmod{q}$$

for $1 \leq i \leq n$ and i is even, $j = 1, 2, \dots, m_2 + i - 1$. $D = \{e_i = u_i u_{i+1} \text{ for } i = 1, 2\}$, where

$$e_i = \left[\frac{n-1}{2} (m_2 + \frac{n+1}{2} + i + 1) \right] \pmod{q}$$

for $i = 1, 2$. Hence, the induced edge labels of G are q distinct integers. Therefore, the graph $G = A(m_j, n) \cup P_3$ is vertex graceful for n is odd. $\square$

Definition 2.5 A regular lobster is defined by each vertex in a path is adjacent to the path P_2 .

Theorem 2.6 A regular lobster is vertex graceful.

Proof Let G be a 1- regular lobster with $3n$ vertices and $q = 3n - 1$ edges. Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of a path P_n . Let v_i be the vertices, which are adjacent to v_{i1} and v_{i2} adjacent to v_{i2} for $1 \leq i \leq n$ and n is even. The theorem is proved by two cases. By definition of Vertex graceful labeling, the required vertices labeling are

Case 1 n is even

$$\begin{aligned} v_i &= \begin{cases} \frac{3i-1}{2}; 1 \leq i \leq n, i \text{ is odd,} \\ \frac{3(n+i)}{2}; 1 \leq i \leq n, i \text{ is even.} \end{cases} \\ v_{i1} &= \begin{cases} \frac{3(n+i)-1}{2} / 1 \leq i \leq n, i \text{ is odd} \\ \frac{3i-2}{2} + 3 / 1 \leq i \leq n, i \text{ is even.} \end{cases} \\ v_{i2} &= \begin{cases} \frac{3(i-1)}{2} + 2; 1 \leq i \leq n, i \text{ is odd,} \\ \frac{3(n+i)}{2} - 1; 1 \leq i \leq n, i \text{ is even.} \end{cases} \end{aligned}$$

The corresponding edge labels are as follows:

$$\begin{aligned} \text{Let } A &= \{e_i = v_i v_{i+1}/1 \leq i \leq n-1\}, \text{ where } e_i = \left(\frac{3(n+2i)}{2} + 1 \right) \pmod{q} \text{ for } 1 \leq i \leq n-1, \\ B &= \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}, \text{ where } e_{i1} = \left(\frac{3(n+2i)}{2} - 1 \right) \pmod{q} \text{ for } 1 \leq i \leq n \text{ and } i \text{ is odd,} \\ C &= \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}, \text{ where } e_{i1} = \left(\frac{3(n+2i)}{2} \right) \pmod{q} \text{ for } 1 \leq i \leq n \text{ and } i \text{ is even,} \\ D &= \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}, \text{ where } e_{i2} = \left(\frac{3(n+2i)}{2} \right) \pmod{q} \text{ for } 1 \leq i \leq n \text{ and } i \text{ is odd,} \\ E &= \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}, \text{ where } e_{i2} = \left(\frac{3(n+2i)}{2} - 1 \right) \pmod{q} \text{ for } 1 \leq i \leq n \text{ and } i \text{ is even.} \end{aligned}$$

Case 2 n is odd

$$v_i = \begin{cases} \frac{3i-1}{2}; & 1 \leq i \leq n, i \text{ is odd,} \\ \frac{3(n+i)+1}{2}; & 1 \leq i \leq n, i \text{ is even,} \end{cases}$$

$$v_{i1} = \begin{cases} \frac{3(n+i)}{2}; & 1 \leq i \leq n, i \text{ is odd,} \\ \frac{3(i-2)}{2} + 3; & 1 \leq i \leq n, i \text{ is even,} \end{cases}$$

$$v_{i2} = \begin{cases} \frac{3(i-1)}{2} + 2; & 1 \leq i \leq n, i \text{ is odd,} \\ \frac{3(n+i-1)}{2} + 1; & 1 \leq i \leq n, i \text{ is even.} \end{cases}$$

The corresponding edge labels are determined by $A = \{e_i = v_i v_{i+1}/1 \leq i \leq n-1\}$, where $e_i = \left(\frac{3(n+2i+1)}{2}\right) \pmod{q}$ for $1 \leq i \leq n-1$, $B = \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}$, where $e_{i1} = \left(\frac{3(n+2i)-1}{2}\right) \pmod{q}$ for $1 \leq i \leq n$ and i is odd, $C = \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}$, where $e_{i1} = \left(\frac{3(n+2i)+1}{2}\right) \pmod{q}$ for $1 \leq i \leq n$ and i is even, $D = \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}$, where $e_{i2} = \left(\frac{3(n+2i)+1}{2}\right) \pmod{q}$ for $1 \leq i \leq n$ and i is odd, $E = \{e_{i2} = v_i v_{i2}/1 \leq i \leq n\}$, where $e_{i2} = \left(\frac{3(n+2i)-1}{2}\right) \pmod{q}$ for $1 \leq i \leq n$ and i is even. Hence the induced edge labels of G are q distinct edges. Therefore, the graph G is vertex graceful. $\square$

Theorem 2.7 $C_n \cup C_{n+1}$ is vertex graceful if and only if $n \geq 4$.

Proof Let $G = C_n \cup C_{n+1}$ with $p = 2n + 1$ vertices and $q = 2n + 1$ edges. Suppose that the vertices of the cycle C_n run consecutively $u_1, u_2, \dots, u_n$ with u_n joined to u_1 and that the vertices of the cycle C_{n+1} run consecutively $v_1, v_2, \dots, v_{n+1}$ with v_{n+1} joined to v_1 .

By definition of vertex graceful labeling

(a) $u_1 = 1, u_n = 2, u_i = 2i$ for $i = 2, 3, \dots, \lfloor (n+1)/2 \rfloor, u_j = 2(n-j) + 3$ for $j = \lfloor (n+3)/2 \rfloor, \dots, n-1$.

(b) $v_1 = 2, v_2 = 2n-1$ and

(i) $v_{3s+t} = 2n-4t-6s+7, t=0, 1, 2, s=1, 2, \dots, \lfloor (n+1-3t)/6 \rfloor$ if $s = \lfloor \frac{n+1-3t}{6} \rfloor < 1$ then no s .

(ii) Write $\alpha(0) = 0, \alpha(1) = 4, \alpha(2) = 2, \beta(0) = 0, \beta(1) = 3 = \beta(2)$
 $v_{n+1-3s-t} = 2n-6s-\alpha(t), t=0, 1, 2, s=0, 1, \dots, \lfloor \frac{n-5-\beta(t)}{6} \rfloor$. If $s = \lfloor \frac{n-5-\beta(t)}{6} \rfloor < 0$ then no s value exists.

(iii) We consider as that v_i to $f(i)$; and suppose that $n-2 = \theta \pmod{3}, 0 \leq \theta \leq 2$. There are $2 + \theta$ vertices as yet unlabeled. These middle vertices are labeled according to congruence class of modulo 6.

Congruence class	
$n \equiv 0 \pmod{6}$	$f((n+2)/2) = n+2, f((n+4)/2) = n+3,$ $f((n+6)/2) = n+4$
$n \equiv 1 \pmod{6}$	$f((n+1)/2) = n+2, f((n+3)/2) = n+3,$ $f((n+5)/2) = n+4, f((n+7)/2) = n+5$
$n \equiv 2 \pmod{6}$	$f((n+2)/2) = n+2, f((n+4)/2) = n+3$
$n \equiv 2 \pmod{6}$	$f((n+1)/2) = n+4, f((n+3)/2) = n+3,$ $f((n+5)/2) = n+2$
$n \equiv 4 \pmod{6}$	$f((n+2)/2) = n+5, f((n+3)/2) = n+4,$ $f((n+4)/2) = n+3, f((n+5)/2) = n+2$
$n \equiv 4 \pmod{6}$	$f((n+3)/2) = n+3, f((n+5)/2) = n+2$

To check that f is vertex graceful is very tedious. But we can give basic idea. The C_n cycle has edges with labels $\{2k+2/k = 4, 5, \dots, n-1\} \cup \{0, 3, 5, 7\}$. In this case all the labeling of the edges of the cycle C_{n+1} run consecutively v_1v_2 as follows:

$1, (2n-1, 2n-3), (2n-11, 2n-13, 2n-15), \dots, (2n+1-12k, 2n-1-12k, 2n-3-12k), \dots,$
middle labels, $\dots, (2n+3-12k, (2n+5-12k, (2n+7-12k), \dots, (2n-21, 2n-19, 2n-17), (2n-9, 2n-7, 2n-5), 2$. The middle labels depend on the congruence class modulo and are best summarized in the following table. If n is small the terms in brackets alone occur.

Congruence class	
$n \equiv 0 \pmod{6}$	$\dots (11, 9), 6, 4, 7, (13, 15, 17) \dots$
$n \equiv 1 \pmod{6}$	$\dots (13, 11), 6, 4, 7, (13, 15, 17) \dots$
$n \equiv 2 \pmod{6}$	$\dots (11), 6, 4, 7, (9) \dots$
$n \equiv 2 \pmod{6}$	$\dots (13), 7, 4, 6, (9, 11) \dots$
$n \equiv 4 \pmod{6}$	$\dots (15, 9), 6, 4, 7, (11, 13) \dots$
$n \equiv 4 \pmod{6}$	$\dots (9), 7, 6, 4, (11, 13, 15) \dots$

Thus, all these edge labelings are distinct. $\square$

References

- [1] J.A.Gallian, A Dynamic Survey of graph labeling, *The Electronic journal of Coimbinatorics*, **18** (2011), #DS6.
- [2] Harary F., *Graph Theory*, Addison Wesley, Mass Reading, 1972.
- [3] Sin-Min Lee, Y.C.Pan and Ming-Chen Tsai, On vertex- graceful $(p, p+1)$ Graphs, *Congressus Numerantium*, **172** (2005), 65-78.
- [4] M.A Seoud and A.E.I Abd el Maqsood, Harmonious graphs, *Utilitas Mathematica*, **47** (1995), pp. 225-233.
- [5] P.Balaganesan, P.Selvaraju, J.Renuka,V.Balaji, On vertex graceful labeling, *Bulletin of Kerala Mathematics Association*, Vol.9,(June 2012), 179-184.