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Symmetric Hamilton Cycle Decompositions of Complete Graphs Plus a 1-Factor

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Abstract: Let $n \geq 2$ be an integer. The complete graph K_n with 1-factor I added has a decomposition into Hamilton cycles if and only if n is even. We show that $K_n + I$ has a decomposition into Hamilton cycles which are symmetric with respect to the 1-factor I added. We also show that the complete bipartite graph $K_{n,n}$ plus a 1-factor has a symmetric Hamilton decomposition, where n is odd.

Key Words: Complete graphs, complete bipartite graph, 1-factor, Hamilton cycle decomposition.

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§1. Introduction

By a decomposition of a nonempty graph G is meant a family of subgraphs $G_1, G_2, \cdots, G_k$ of G such that their edge set form a partition of the edge set of G. Any member of the family is called a part (of the decomposition). This decomposition is usually denoted by $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$.

Let $n \geq 2$ be an integer. The complete graph K_n has many Hamilton cycles and since its vertices have degree $n-1$, K_n has a decomposition into Hamilton cycles if and only if n is odd. Suppose that $n = 2m+1$. The familiar Hamilton cycle decomposition of K_n referred to as the Walecki decomposition in [1] is a symmetric decomposition in that each Hamilton cycle H in the decomposition is symmetric in the following sense. Let the vertices of K_n be labeled as $0, 1, 2, \cdots, m, \bar{1}, \bar{2}, \cdots, \bar{m}$. Then each H is invariant under the involution $i \rightarrow \bar{i}$, where $\bar{\bar{i}} = i$; the vertex 0 is a fixed point of this involution. A symmetric Hamilton cycle decomposition of K_n different from Walecki's is constructed in [1].

Let G be a graph, then G[2] is a graph whereby each vertex x is replaced by a pair of two independent vertices $x, \bar{x}$ and each edge xy is replaced by four edges $xy, x\bar{y}, \bar{x}y, \bar{x}\bar{y}$.

Now suppose that n is even. Adding the edges of a 1-factor I to K_n results in a graph

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$K_n + I$ each of whose vertices has even degree n. The graph $K_n + I$ does have a decomposition into Hamilton cycles (see [3]). The complete solution to the problem of decomposing $K_n + I$ into cycles of given uniform length is given in [3].

The degrees of vertices of the complete bipartite graph $K_{n,n}$ equal n, and $K_{n,n}$ has a decomposition into Hamilton cycles if and only if n is even. If n is odd, adding a 1-factor I to $K_{n,n}$ results in a graph $K_{n,n} + I$ with all vertices of even degree $n+1$ and $K_{n,n} + I$ also has a decomposition into Hamilton cycles.

Let $n = 2m$ be an even integer with $m \geq 1$. Consider the complete bipartite graph $K_{n,n}$ with vertex bipartition into sets $\{1, 2, \cdots, n\}$ and $\{\bar{1}, \bar{2}, \cdots, \bar{n}\}$. By a symmetric Hamilton cycle in $K_{n,n}$, we mean a Hamilton cycle such that $i\bar{j}$ is an edge if and only if $\bar{i}j$ is an edge. Thus a Hamilton cycle in $K_{n,n}$ is symmetric if and only if it is invariant under the involution $i \to \bar{i}$.

A symmetric hamilton cycle decomposition of $K_{n,n}$ is a partition of the edges of $K_{n,n}$ into m symmetric Hamilton cycles. Now let $n = 2m+1$ be an odd integer with $m \geq 1$, and consider the 1-factor $I = \{\{1, \bar{n}\}, \{2, \overline{n-1}\}, \cdots, \{n, \bar{1}\}\}$ of $K_{n,n} + I$. A symmetric Hamilton cycle decomposition of $K_{n,n} + I$ is a partition of the edges of $K_{n,n} + I$ into $m+1$ symmetric Hamilton cycles.

Let $m > 1$ be even, consider the vertex set of the complete graph K_{2m} to be $\{1, 2, \cdots, m\} \cup \{\bar{1}, \bar{2}, \cdots, \bar{m}\}$, where $I = \{1\bar{1}, 2\bar{2}, \cdots, m\bar{m}\}$ is a 1-factor of K_{2m}.

The edges of $K_{2m} + I$ are naturally partitioned into edges of K_m on $\{1, 2, \cdots, m\}$, the edges of $K_{m,m} + I$, and the edges of K_m on $\{\bar{1}, \bar{2}, \cdots, \bar{m}\}$. We denote the complete graph on $\{\bar{1}, \bar{2}, \cdots, \bar{m}\}$ by $\bar{K_m}$. We abuse terminology and write this edge partition as:

$$K_{2m} + I = K_m \cup (K_{m,m} + I) \cup \bar{K_m}$$

By a symmetric Hamilton cycle of $K_{2m} + I$ we mean a Hamilton cycle such that

(1) ij is an edge in (K_m) if and only if $(\overline{ij})$ is an edge in $\bar{K_m}$ and

(2) $i\bar{j}$ is an edge in $(K_{m,m} + I)$ if and only if $j\bar{i}$ is an edge in $(K_{m,m} + I)$.

Thus a Hamilton cycle of $K_{2m} + I$ is symmetric if and only if it is invariant under the fixed point free involution ϕ of $K_{2m} + I$, where $\phi(a) = \bar{a}$ for all a in $\{1, 2, \cdots, m\} \cup \{\bar{1}, \bar{2}, \cdots, \bar{m}\}$ and $\bar{\bar{a}} = a$. A symmetric Hamilton cycle decomposition of $K_{2m} + I$ is a decomposition of $K_{2m} + I$ into m symmetric Hamilton cycles. Thus ϕ is a nontrivial automorphism of $K_{2m} + I$, which acts trivially on the cycles in a symmetric Hamilton cycle decomposition of $K_{2m} + I$.

A double cover of K_{2m} by Hamilton cycles is a collection $C_1, C_2, \cdots, C_{2m-1}$ of $2m - 1$ Hamilton cycles such that each edge of K_{2m} occurs as an edge of exactly two of these Hamilton cycles. Note that the sum of the number edges in these Hamilton cycles equals

$$(2m-1)2m = 2\begin{pmatrix} 2m \\ 2 \end{pmatrix},$$

twice the number of edges of K_{2m}, and this also equals half the number of edges of $K_{4m} - I$.

We use $K_n + I$ to denote the multigraph obtained by adding the edges of a 1-factor I to K_n, thus duplicating $\dfrac{n}{2}$ edges.

Let k be a positive integer and $L \subseteq \{1, 2, \cdots, \lfloor\frac{k}{2}\rfloor\}$. A circulant graph $X = X(k; L)$ is a graph with vertex set $V(X) = \{u_1, u_2, \cdots, u_k\}$ and edge set $E(X)$, where $E(X) = \{u_i u_{i+l} : i \in Z_k, l \in L - \{\frac{k}{2}\}\} \cup \{u_i u_{i+k} : i \in \{1, 2, \cdots, \frac{k}{2}\}\}$ if $\frac{k}{2} \in L$, and $E(X) = \{u_i u_{i+l} : i \in Z_k, l \in L\}$ otherwise. An edge $u_i u_{i+l}$, where $l \in L$ is said to be of length l and L is called the edge length set of the circulant X.

Notice that K_n is isomorphic to the circulant $X(n; \{1, 2, \cdots, \lfloor\frac{n}{2}\rfloor\})$. If n is even, $K_n - I$ is isomorphic to $X(n; \{1, 2, \cdots, \frac{n}{2} - 1\})$ and $K_n + I$ is isomorphic to $X(n; \{1, 2, \cdots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2}\})$.

Let $X = X(k; L)$ be a circulant graph with vertex set $\{u_1, u_2, \cdots, u_k\}$. By the rotation ρ we mean the cyclic permutation $\{u_1, u_2, \cdots, u_k\}$.

If $P = x_0 x_1 \cdots x_p$ is a path, $\overleftarrow{P}$ denotes the path $x_p x_{p-1} \cdots x_1 x_0$, the reverse of P.

§2. Proof of the Result

In order that $K_n + I$ have a symmetric Hamilton cycle decomposition, it is necessary that n be even.

Theorem 2.1 *Let $m \geq 2$ be an integer. There is a symmetric Hamilton cycle decomposition of $K_{2m} + I$.*

Proof View the graph $K_{2m} + I$ as the circulant graph $X(2m; \{1, 2, \cdots, m-1, m, m\})$ with vertex set $\{x_1, x_2, \cdots, x_{2m}\}$. Let P be the zig-zag $(m-1)$ path

$$P = x_1 x_{+1} x_{-1} x_{+2} x_{-2} \cdots x_A$$

where $A = 1 - 2 + 3 - \cdots + (-1)^m (m-1)$. Thus P has edge length set $L_p = \{1, 2, \cdots, m-1\}$. It is easy to see that

$$C = P \cup \rho^m(\overleftarrow{P}) x_1$$

is an $2m$-cycle and $\{\rho^i(C) : i = 0, 1, \cdots, m-1\}$ is a Hamilton cycle decomposition of $K_{2m} + I$.

Next relabel the vertices of the graph $K_{2m} + I$ by defining a function f as follows: $f : x_i \to x_i$ for $1 \leq i \leq m$ and $f : x_i \to \bar{x}_{i-m}$ for $m \leq i \leq 2m$. Relabeling of the vertices of each Hamilton cycle C_{2m} with the new labels gives symmetric Hamilton cycle. Hence $K_{2m} + I$ can be decomposed into symmetric Hamilton cycle. □

Lemma 2.2 *Let $m \geq 2$ be an integer, and let C be a symmetric Hamilton cycle of $K_{2m} + I$. Then*

(1) *If x is any vertex of $K_{2m} + I$, the distance between x and $\bar{x}$ in C is odd;*

(2) *C is of the form $x_1, x_2, \cdots, x_m, \bar{x}_m, \bar{x}_{m-1}, \cdots, \bar{x}_2, \bar{x}_1 x_1$ where $x_i \in \{1, 2, \cdots, m, \bar{1}, \bar{2}, \cdots, \bar{m}\}$;*

(3) *The number of edges $x_i \bar{x}_i$ in each symmetric Hamilton cycle is $2, 1 \leq i \leq m$.*

Proof Let x be a vertex of $K_{2m} + I$ and let the distance between x and $\bar{x}$ in C be k. Then there is a path $x = x_1, \cdots, x_{\frac{k+1}{2}}, \bar{x}_{\frac{k+1}{2}}, \cdots, \bar{x}_2 \bar{x}_1 = \bar{x}$ in C. Since for each $x_i, i \in N$ we have

$k, \frac{k+1}{2} \in N$. Suppose k is even, then $\frac{k+1}{2} \notin N$.Therefore k is odd which proves (1).

Assertion (2) is now an immediate consequence. Since the cycle C is given as in (2), we have edges $\{x_1\bar{x}_1\}$ and $\{x_m\bar{x}_m\}$ which proves (3). □

Theorem 2.3 *Let m be an even integer, then the graph $K_m + I[2]$ has a symmetric Hamilton cycle decomposition.*

Proof From the definition of the graph $K_m + I[2]$, each vertex x in $K_m + I$ is replaced by a pair of two independent vertices $x, \bar{x}$ and each edge xy is replaced by four edges $xy, x\bar{y}, \bar{x}y, \bar{x}\bar{y}$. Also note that if the graph H decomposes the graph G, then $H[2]$ decomposes $G[2]$.

By [3], cycle C_m decomposes $K_m + I$, then we have

$$K_m + I[2] = C_m[2] \oplus C_m[2] \oplus \cdots \oplus C_m[2]$$

Now label the vertices of each graph $C_m[2]$ as $x_i\bar{x}_i$, where $i = 1, 2, \cdots, m$. By [2], each graph $C_m[2]$ decomposes into symmetric Hamilton cycle C_{2m}. Therefore $K_m + I[2]$ decomposes into symmetric Hamilton cycles. □

Theorem 2.4 *Let $m \geq 4$ be an even integer. From a symmetric Hamilton cycle decomposition of $K_m + I[2]$ we can construct a double cover of $K_m + I$ by Hamilton cycles.*

Proof By Theorem 2.3, a symmetric Hamilton cycle of $K_m+I[2]$ is of the form $x_1, x_2, \cdots, x_m$, $\bar{x}_m, \bar{x}_{m-1}, \cdots, \bar{x}_2, \bar{x}_1 x_1$ where $x_i \in \{1, 2, \cdots, m, \bar{1}, \bar{2}, \cdots, \bar{m}\}$. Thus $x_1 x_2 ... x_m$ is a path of length $m - 1$ in $K_m + I[2]$ and $\bar{x}_m \bar{x}_{m-1} \cdots \bar{x}_2 \bar{x}_1$ is its mirror image. Let

$$b_i = \begin{cases} x_i & \text{if } x_i \in \{1, 2, \cdots, m\} \\ \bar{x_i} & \text{if } x_i \in \{\bar{1}, \bar{2}, \cdots, \bar{m}\} \end{cases}$$

Then $b_1, b_2, \cdots, b_m, b_1$ is a Hamilton cycle in $K_m + I$, the projection of C on $K_m + I$. Now assume we have a symmetric Hamilton cycle decomposition of $K_m + I[2]$. Then for each edge $x_i x_j$ in $K_m + I$, there are distinct symmetric Hamilton cycles C and C' in our decomposition such that $x_i x_j$ and $\bar{x}_i \bar{x}_j$ are edges of C and $x_i \bar{x}_j$ and $\bar{x}_i x_j$ are edges of C'. Hence from a symmetric Hamilton cycle decomposition of $K_m + I[2]$, we get a double cover of $K_m + I$ from the projections of each symmetric Hamilton cycle. □

Theorem 2.5 *Let $m \geq 4$ be even integer. Then $K_{2m} + I$ has a double cover by Hamilton cycles.*

Proof There is a Hamilton cycle C in $K_{2m} + I$, and there exists disjoint 1-factor I_1 and I_2 whose union is the set of edges of C. The vertices of the graphs $K_{2m} + I_1$ and $K_{2m} - I_2$ have degrees equal to the even number. The graphs $K_{2m} + I_1$ and $K_{2m} - I_2$ have decompositions into Hamilton cycles $C_1, C_2, \cdots, C_m$ and $D_1, D_2, \cdots, D_{m-1}$ respectively. Then $C, C_1, C_2, \cdots, C_m$, $D_1, D_2, . \cdots, D_{m-1}$ is a double cover of $K_{2m} + I$ by Hamilton cycles . □

Theorem 2.6 *For each integer $m \geq 1$, there exist a symmetric Hamilton cycle decomposition of $K_{2m+1,2m+1} + I$.*

Proof Let $n = 2m+1$, we consider the complete bipartite graph $K_{n,n}$ with vertex bipartition $\{1, 2, 3, \cdots, n\}$ and $\{\bar{1}, \bar{2}, \cdots, \bar{n}\}$. Let I be $\{\{1, \bar{n}\}, \{2, \overline{n-1}\}, \{3, \overline{n-2}\}, \cdots, \{n, \bar{1}\}\}$ in $K_{n,n} + I$.

Let the sum of edge $a\bar{b}$ be $a + b \mod n$. Let S_k be the set of edges whose sum is k. Let i be an integer with $1 \leq i \leq m+1$. Consider the union $S_{2i-1} \cup S_{2i}$, $2i$ is calculated modulo n. observe that this collection of edges yields the following symmetric Hamilton cycle of $K_{n,n} + I$;

$$n, \overline{2i-1}, 1, \overline{2i-2}, 2, \overline{2i-3}, 3, \cdots, \overline{2i}, n$$

For each i, let H_i equal $S_{2i-1} \cup S_{2i}$. Then $H_1, H_2, \cdots, H_{m+1}$ is a symmetric Hamilton cycle decomposition of $K_{n,n} + I$. □

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