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Some Prime Labeling Results of H-Class Graphs

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Abstract: Prime labeling originated with Entringer and was introduced by Tout, Dabboucy and Howalla [3]. A Graph $G(V, E)$ is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, 3, \cdots, |V(G)|$ such that for each edge xy the labels assigned to x and y are relatively prime. A graph admits a prime labeling is called a *prime graph*. We investigate the prime labeling of some H-class graphs.

Key Words: labeling, prime labeling, prime graph, H-class graph

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§1. Introduction

A simple graph $G(V, E)$ is said to have a prime labeling (or called prime) if its vertices are labeled with distinct integers $1, 2, 3, \ldots, |V(G)|$, such that for each edge $xy \in E(G)$, the labels assigned to x and y are relatively prime [1].

We begin with listing a few definitions/notations that are used.

(1) A graph $G = (V, E)$ is said to have order $|V|$ and size $|E|$.

(2) A vertex $v \in V(G)$ of degree 1 is called pendant vertex.

(3) P_n is a path of length n.

(4) The H-graph is defined as the union of two paths of length n together with an edge joining the mid points of them. That is, it is obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{(n+1)/2}$ and $u_{(n+1)/2}$ by means of an edge if n is odd and the vertices $v_{(n/2)+1}$ and $u_{n/2}$ if n is even [4].

(5) The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has p_1 points) and p_1 copies of G_2 and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2 [1].

§2. Prime Labeling of H-Class Graphs

Theorem 2.1 *The H-graph of a path of length n is prime.*

Proof Let $G = (V, E)$ be a H-graph of a path of length n. It is obtained from two copies

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of paths of length n. It has $2n$ vertices and $2n-1$ edges.

$$V(G) = \{u_i, v_i / 1 \leq i \leq n\}$$
$$E(G) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{u_{\lceil (n/2) \rceil} v_{\lceil (n/2) \rceil}\}$$

Define $f : V(G) \to \{1, 2, \ldots, 2n\}$ by

$$f(v_{(n+1)/2}) = 1 \text{ if } n \text{ is odd}$$
$$f(v_{n/2}) = 1 \text{ if } n \text{ is even}$$
$$f(u_i) = i+1, 1 \leq i \leq n$$
$$f(v_i) = n+i+1, 1 \leq i \leq (n/2)-1, \text{ when } i \neq (n/2) \text{ if } n \text{ is even}$$
$$f(v_i) = n+i, (n/2)+1 \leq i \leq n, \text{ if } n \text{ is even}$$
$$f(v_i) = n+i+1, 1 \leq i \leq (n-1)/2, \text{ when } i \neq (n+1)/2, \text{ if } n \text{ is odd}$$
$$f(v_i) = n+i, (n+3)/2 \leq i \leq n, \text{ when } i \neq (n+1)/2, \text{ if } n \text{ is odd}$$

Clearly, it is easy to check that $GCD\ (f(u), f(v)) = 1$, for every edge $uv \in E(G)$. Therefore, the H-graph of a path of length n admits prime labeling. □

Example 2.2 The prime labeling for H-graph with $n = 14, 16$ are shown in Fig.1 and 2.

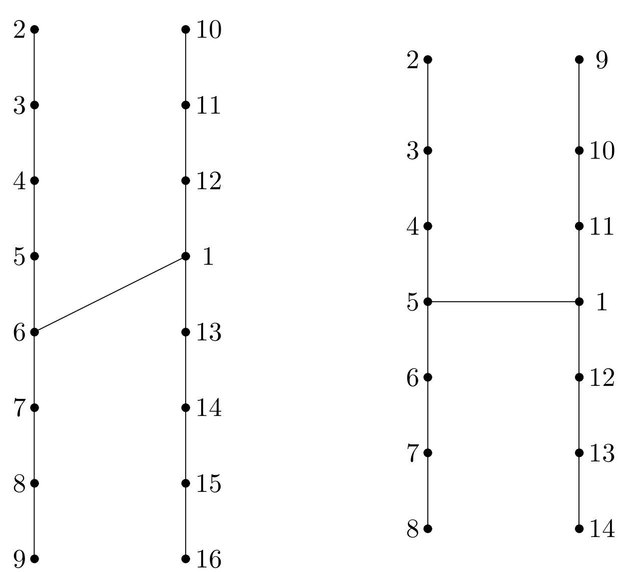

Fig.1 $n \equiv 0(\text{mod}2)$ **Fig.**2 $n \equiv 1(\text{mod}2)$

Theorem 2.3 *The graph $G \odot K_1$ is a prime.*

Proof $G \odot K_1$ is obtained from H-graph by attaching pendant vertices to each of the vertices. The graph has $4n$ vertices and $4n-1$ edges, where $n=|G|$.

$$V(G\odot K_1)=\{u_i,v_i/1\le i\le 2n\}$$
$$E(G\odot K_1)=\{u_iu_{i+1},v_iv_{i+1}/1\le i\le n-1\}\cup\{u_iu_{n+i},v_iv_{n+i}/1\le i\le n\}$$
$$\cup\{u_{(n+1)/2}v_{(n+1)/2}\ n \text{ is odd or } u_{(n/2)+1}v_{n/2}, n \text{ is even.}\}$$

Define $f:V(G\odot K_1)\to\{1,2,\dots,4n\}$ by

$$f(u_i)=2i+1, 1\le i\le n \text{ and } \begin{cases} i\ne (n+1)/2 & \text{if } n \text{ is odd}\\ i\ne n/2 & \text{if } n \text{ is even}\end{cases}$$
$$f(u_{(n+1)/2})=1\ n\text{ odd}$$
$$f(u_{(n/2)+1})=1\ n\text{ even}$$
$$f(u_{n+i})=2i,\ 1\le i\le n$$
$$f(v_i)=2n+2i-1, 1\le i\le n$$
$$f(v_{n+i})=2n+2i, 1\le i\le n$$
$$GCD(f(u_i),f(u_{i+1}))=GCD(2i+1,2i+3)=1, 1\le i\le (n-3)/2, n\text{ odd}$$
$$GCD(f(u_i),f(u_{i+1}))=GCD(2i+1,2i+3)=1, (n+3)/2\le i\le n-1, n\text{ odd}$$
$$GCD(f(u_i),f(u_{i+1}))=GCD(2i+1,2i+3)=1, 1\le i\le (n/2)-1, n\text{ even}$$
$$GCD(f(u_i),f(u_{i+1}))=GCD(2i+1,2i+3)=1, (n/2)+2\le i\le n-1, n\text{ even}$$
$$GCD(f(v_i),f(v_{i+1}))=GCD(2n+2i-1,2n+2i+1)=1, 1\le i\le n$$
$$GCD(f(v_i),f(v_{n+i}))=GCD(2n+2i-1,2n+2i)=1, 1\le i\le n.$$

In this case it can be easily verified that $GCD(f(u),f(v))=1$ for remaining edges $uv\in E(G\odot K_1)$. Therefore, $G\odot K_1$ admits prime labeling. □

Example 2.4 The prime labeling for $G\odot K_1$ and $G\odot K_1$ are shown in Fig.3 and 4.

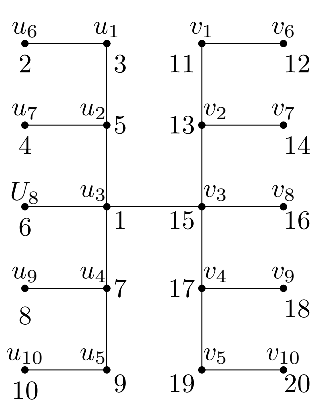

Fig.3 $G\odot K_1$

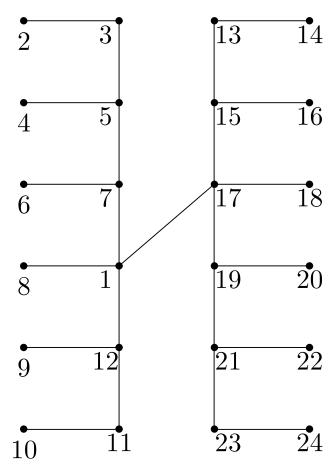

Fig.4 $G\odot K_1$

Theorem 2.5 *The graph $G \odot S_2$ is prime.*

Proof The graph $G \odot S_2$ has $6n$ vertices and $6n-1$ edges, where $n = |G|$.

$$V(G \odot S_2) = \{u_i, v_i / 1 \le i \le n\} \cup \{u_i^{(1)}, u_i^{(2)}, v_i^{(1)}, v_i^{(2)} / 1 \le i \le n\}$$
$$E(G \odot S_2) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \le i \le n-1\} \cup \{u_i u_i^{(1)}, u_i u_i^{(2)}, v_i v_i^{(1)}, v_i v_i^{(2)} / 1 \le i \le n\}$$
$$\cup \{u_{n+1/2} v_{n+1/2} \ n \text{ is odd or } u_{n/2+1} v_{n/2} \ n \text{ is even}.$$

Define $f : V \to \{1, 2, \ldots, 6n\}$ by
$f(u_{n+1/2}) = 1, f(u^{(1)}_{(n+1)/2}) = 6n-1, f(u^{(2)}_{(n+1)/2}) = 6n.$

Case 1 Suppose $n \equiv 1(mod\ 2)$.

Subcase 1.1 $n \equiv 1(mod\ 4)$

$$f(u_{2i-1}) = 6(i-1) + 3, 1 \le i \le (n-1)/4$$
$$f(u_{2i-1}) = f(u_{(n-1)/2}) + 6 + 6[i - ((n-1)/4) + 2],$$
$$((n-1)/4) + 2 \le i \le ((n-1)/2) + 1$$
$$f(u_{2i}) = 6(i-1) + 5, 1 \le i \le (n-1)/4$$
$$f(u_{2i}) = f(u_{(n-1)/2}) + 4 + 6[i - ((n-1)/4) + 1],$$
$$((n-1)/4) + 1 \le i \le (n-1)/2$$
$$f(u^{(1)}_{2i-1}) = f(u_1) - 1 + 6(i-1), 1 \le i \le (n-1)/4$$
$$f(u^{(1)}_{2i-1}) = f(u_{(n-1)/2}) + 7 + 6[i - ((n-1)/4) + 2],$$
$$((n-1)/4) + 2 \le i \le (n-1)/2$$
$$f(u^{(2)}_{2i-1}) = f(u_1) + 1 + 6(i-1), 1 \le i \le (n-1)/4$$
$$f(u^{(2)}_{2i-1}) = f(u_{(n-1)/2}) + 8 + 6[i - ((n-1)/4) + 2],$$
$$((n-1)/4) + 2 \le i \le ((n-1)/2) + 1$$
$$f(u^{(1)}_{2i}) = f(u_2) + 1 + 6(i-1), 1 \le i \le (n-1)/4$$
$$f(u^{(1)}_{2i}) = f(u_{(n-1)/2}) + 3 + 6[i - ((n-1)/4 + 1],$$
$$((n-1)/4) + 1 \le i \le (n-1)/2$$
$$f(u^{(2)}_{2i}) = f(u_2) + 2 + 6(i-1), 1 \le i \le (n-1)/4$$
$$f(u^{(2)}_{2i}) = f(u_{(n-1)/2}) + 5 + 6[i - ((n-1)/4) + 1],$$
$$((n-1)/4) + 1 \le i \le (n-1)/2$$
$$f(v_1) = 3n, f(v_2) = 3n + 2$$
$$f(v_{2i-1}) = f(v_1) + 6(i-1), 2 \le i \le (n+1)/2$$
$$f(v_{2i}) = f(v_2) + 6(i-1), 2 \le i \le (n-1)/2$$
$$f(v^{(1)}_{2i-1}) = f(v_1) - 1 + 6(i-1), 1 \le i \le (n+1)/2$$

$$\begin{aligned}
f(v_{2i-1}^{(2)}) &= f(v_1) + 1 + 6(i-1), 1 \le i \le (n+1)/2 \\
f(v_{2i}^{(1)}) &= f(v_2) + 1 + 6(i-1), 1 \le i \le (n-1)/2 \\
f(v_{2i}^{(2)}) &= f(v_2) + 2 + 6(i-1), 1 \le i \le (n-1)/2 \\
GCD(f(u_{2i-1}), f(u_{2i})) &= GCD(6i-3, 6i-1) = 1, 1 \le i \le (n-1)/4 \\
GCD(f(u_{2i}), f(u_{2i+1})) &= GCD(6i-1, 6i+3) = 1, 1 \le i \le ((n-1)/4) - 1 \\
GCD(f(u_{2i-1}), f(u_{2i})) &= GCD(6i-7, 6i-3) = 1, \\
&\quad ((n-1)/4) + 2 \le i \le (n-1)/2 \\
GCD(f(u_{2i}), f(u_{2i+1})) &= GCD(6i-3, 6i-1) = 1, \\
&\quad ((n-1)/4) + 1 \le i \le (n-1)/2 \\
GCD(f(u_{2i-1}), f(u_{2i-1}^{(2)})) &= GCD(6i-7, 6i-5) = 1, \\
&\quad ((n-1)/4) + 2 \le i \le ((n-1)/2) + 1 \\
GCD(f(u_{2i}), f(u_{2i}^{(2)})) &= GCD(6i-1, 6i+1) = 1, 1 \le i \le (n-1)/4 \\
GCD(f(u_{2i}), f(u_{2i}^{(2)})) &= GCD(6i-3, 6i-2) = 1, \\
&\quad ((n-1)/4) + 1 \le i \le (n-1)/2 \\
GCD(f(v_1), f(v_2)) &= GCD(3n, 3n+2) = 1 \\
GCD(f(v_{2i-1}), f(v_{2i})) &= GCD(3n+6i-6, 3n+6i-4) = 1, \\
&\quad 2 \le i \le (n-1)/2 \\
GCD(f(v_{2i}), f(v_{2i+1})) &= GCD(3n+6i-4, 3n+6i) = 1, \\
&\quad 1 \le i \le (n-1)/2 \\
GCD(f(v_{2i}), f(v_{2i}^{(2)})) &= GCD(3n+6i-4, 3n+6i-2) = 1, \\
&\quad 1 \le i \le (n-1)/2.
\end{aligned}$$

Subcase 1.2 $n \equiv 3(mod\ 4)$

$$\begin{aligned}
f(u_{2i-1}) &= 6(i-1) + 3, 1 \le i \le (n+1)/4 \\
f(u_{2i-1}) &= f(u_{(n-1)/2}) + 2 + 6[i - ((n+1)/4) + 1], \\
&\quad ((n+1)/4 + 1 \le i \le (n+1)/2 \\
f(u_{2i}) &= 6(i-1) + 5, 1 \le i \le ((n+1)/4) - 1 \\
f(u_{2i}) &= f(u_{(n-1)/2}) + 6 + 6[i - ((n+1)/4) + 1], \\
&\quad ((n+1)/4) + 1 \le i \le ((n+1)/2) - 1 \\
f(u_{2i-1}^{(1)}) &= f(u_1) - 1 + 6(i-1), 1 \le i \le (n+1)/4 \\
f(u_{2i-1}^{(1)}) &= f(u_{(n-1)/2}) + 3 + 6[i - ((n+1)/4) + 1], \\
&\quad ((n+1)/4) + 1 \le i \le (n+1)/2 \\
f(u_{2i-1}^{(2)}) &= f(u_1) + 1 + 6(i-1), 1 \le i \le (n+1)/4
\end{aligned}$$

$$f(u^{(2)}_{2i-1}) = f(u_{(n-1)/2}) + 4 + 6[i - ((n+1)/4) + 1],$$
$$((n+1)/4) + 1 \leq i \leq (n+1)/2$$
$$f(u^{(1)}_{2i}) = f(u_2) + 1 + 6(i-1), 1 \leq i \leq ((n+1)/4) - 1$$
$$f(u^{(1)}_{2i}) = f(u_{(n-1)/2}) + 5 + 6[i - ((n+1)/4) + 1],$$
$$((n+1)/4) + 1 \leq i \leq ((n+1)/2) - 1$$
$$f(u^{(2)}_{2i}) = f(u_2) + 2 + 6(i-1), 1 \leq i \leq ((n+1)/4) - 1$$
$$f(u^{(2)}_{2i}) = f(u_{(n-1)/2}) + 7 + 6[i - ((n+1)/4) + 1],$$
$$((n+1)/4) + 1 \leq i \leq ((n+1)/2) - 1$$
$$f(v_1) = 3n, f(v_2) = 3n + 2$$
$$f(v_{2i-1}) = f(v_1) + 6(i-1), 2 \leq i \leq (n+1)/2$$
$$f(v_{2i}) = f(v_2) + 6(i-1), 2 \leq i \leq ((n+1)/2) - 1$$
$$f(v^{(1)}_{2i-1}) = f(v_1) - 1 + 6(i-1), 1 \leq i \leq (n+1)/2$$
$$f(v^{(2)}_{2i-1}) = f(v_1) + 1 + 6(i-1), 1 \leq i \leq (n+1)/2$$
$$f(v^{(1)}_{2i}) = f(v_2) + 1 + 6(i-1), 1 \leq i \leq ((n+1)/2) - 1$$
$$f(v^{(2)}_{2i}) = f(v_2) + 2 + 6(i-1), 1 \leq i \leq ((n+1)/2) - 1.$$

As in the above case it can be verified that $GCD(f(u), f(v)) = 1$ for every edge $uv \in E(G \odot S_2)$.

Case 2. $n \equiv 0 (mod\ 2)$

$$f(u_{(n/2)+1}) = 1, f(u^{(1)}_{(n/2)+1}) = 6n - 1, f(u^{(2)}_{(n/2)+1}) = 6n.$$

Subcase 2.1 $n \equiv 0 (mod\ 4)$

$$f(u_{2i-1}) = 6(i-1) + 3, 1 \leq i \leq n/4$$
$$f(u_{2i-1}) = f(u_{(n/2)}) + 6 + 6[i - (n/4) + 2], (n/4) + 2 \leq i \leq n/2$$
$$f(u_{2i}) = 6(i-1) + 5, 1 \leq i \leq n/4$$
$$f(u_{2i}) = f(u_{(n/2)}) + 4 + 6[i - (n/4) + 1], (n/4) + 1 \leq i \leq n/2$$
$$f(u^{(1)}_{2i-1}) = f(u_1) - 1 + 6(i-1), 1 \leq i \leq n/4$$
$$f(u^{(1)}_{2i-1}) = f(u_{(n/2)}) + 7 + 6[i - (n/4) + 2], (n/4) + 2 \leq i \leq n/2$$
$$f(u^{(2)}_{2i-1}) = f(u_1) + 1 + 6(i-1), 1 \leq i \leq n/4$$
$$f(u^{(2)}_{2i-1}) = f(u_{(n/2)}) + 8 + 6[i - (n/4) + 2], (n/4) + 2 \leq i \leq n/2$$
$$f(u^{(1)}_{2i}) = f(u_2) + 1 + 6(i-1), 1 \leq i \leq n/4$$
$$f(u^{(1)}_{2i}) = f(u_{(n/2)}) + 3 + 6[i - (n/4) + 1], (n/4) + 1 \leq i \leq n/2$$
$$f(u^{(2)}_{2i}) = f(u_2) + 2 + 6(i-1), 1 \leq i \leq n/4$$
$$f(u^{(2)}_{2i}) = f(u_{(n/2)}) + 5 + 6[i - (n/4) + 1], (n/4) + 1 \leq i \leq n/2$$

$$f(v_1) = 3n-1, f(v_2) = 3(n+1)$$
$$f(v_{2i-1}) = f(v_1) + 6(i-1), 2 \leq i \leq n/2$$
$$f(v_{2i}) = f(v_2) + 6(i-1), 2 \leq i \leq n/2$$
$$f(v_{2i-1}^{(1)}) = f(v_1) + 1 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v_{2i-1}^{(2)}) = f(v_1) + 2 + 6 + (i-1), 1 \leq i \leq n/2$$
$$f(v_{2i}^{(1)}) = f(v_2) - 1 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v_{2i}^{(2)}) = f(v_2) + 1 + 6(i-1), 1 \leq i \leq n/2$$

Clearly $GCD(f(u), f(v)) = 1$ for every edge $uv \in E(G \odot S_2)$.

Subcase 2.2 $n \equiv 2(mod\ 4)$

$$f(u_{2i-1}) = 6(i-1) + 3, 1 \leq i \leq (n+2)/4$$
$$f(u_{2i-1}) = f(u_{(n/2)}) + 2 + 6[i - ((n+2)/4) + 1], ((n+2)/4) + 1 \leq i \leq n/2$$
$$f(u_{2i}) = 6(i-1) + 5, 1 \leq i \leq (n-2)/4$$
$$f(u_{2i}) = f(u_{(n/2)}) + 6 + 6]i - ((n-2)/4) + 2], ((n-2)/4) + 2 \leq i \leq n/2$$
$$f(u_{2i-1}^{(1)}) = f(u_1) - 1 + 6(i-1), 1 \leq i \leq (n+2)/4$$
$$f(u_{2i-1}^{(1)}) = f(u_{(n/2)}) + 3 + 6[i - ((n+2)/4) + 1], ((n+2)/4) + 1 \leq i \leq n/2$$
$$f(u_{2i-1}^{(2)}) = f(u_1) + 1 + 6(i-1), 1 \leq i \leq (n+2)/4$$
$$f(u_{2i-1}^{(2)}) = f(u_{(n/2)}) + 4 + 6[i - ((n+2)/4) + 1], ((n+2)/4) + 1 \leq i \leq n/2$$
$$f(u_{2i}^{(1)}) = f(u_2) + 1 + 6(i-1), 1 \leq i \leq (n-2)/4$$
$$f(u_{2i}^{(1)}) = f(u_{(n/2)}) + 5 + 6[i - ((n-2)/4) + 2], ((n-2) + 2 \leq i \leq n/2$$
$$f(u_{2i}^{(2)}) = f(u_2) + 2 + 6(i-1), 1 \leq i \leq (n-2)/4$$
$$f(u_{2i}^{(2)}) = f(u_{(n/2)}) + 7 + 6[i - ((n-2)/4) + 2], ((n-2)/4) + 2 \leq i \leq n/2$$
$$f(v_1) = 3n-1, f(v_2) = 3(n+1)$$
$$f(v_{2i-1}) = f(v_1) + 6(i-1), 2 \leq i \leq n/2$$
$$f(v_{2i}) = f(v_2) + 6(i-1), 2 \leq i \leq n/2$$
$$f(v_{2i-1}^{(1)}) = f(v_1) + 1 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v_{2i-1}^{(2)}) = f(v_1) + 2 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v_{2i}^{(1)}) = f(v_2) - 1 + 6(i-1), 1 \leq i \leq n/2$$
$$f(v_{2i}^{(2)}) = f(v_2) + 1 + 6(i-1), 1 \leq i \leq n/2.$$

In this case also it is easy to check that $GCD(f(u), f(v)) = 1$. Therefore $G \odot S_2$ admits prime labeling. □

Example 2.6 The prime labelings for $G \odot S_2$ with $n \equiv 1(\text{mod}4)$, $n \equiv 2(\text{mod}4)$, $n \equiv 3(\text{mod}4)$, $n \equiv 0(\text{mod}4)$ are respectively shown in Fig.5-8 following.

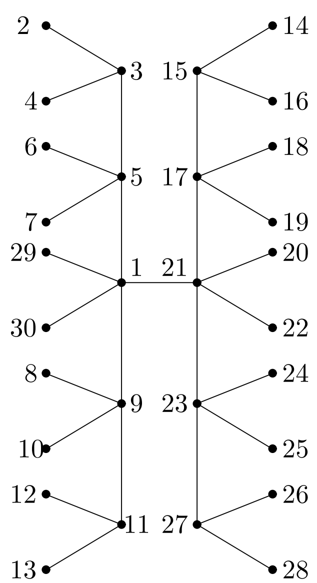

Fig.5 $G \odot S_2$ $n \equiv 1(\text{mod}4)$

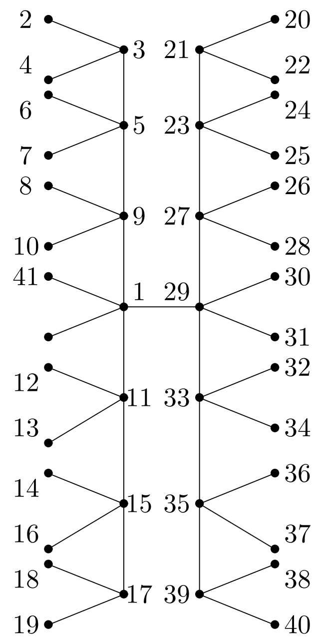

Fig.6 $G \odot S_2$ $n \equiv 3(\text{mod}4)$

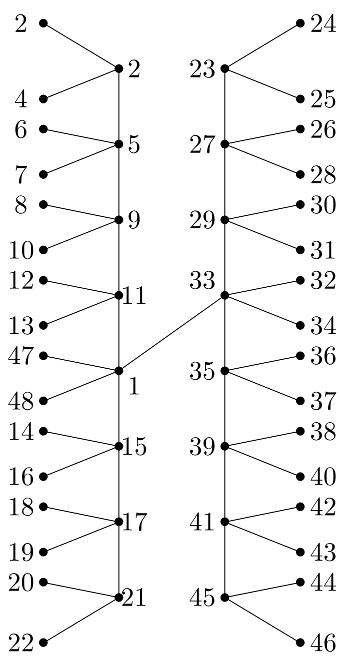

Fig.7 $G \odot S_2$ $n \equiv 0(\text{mod}4)$

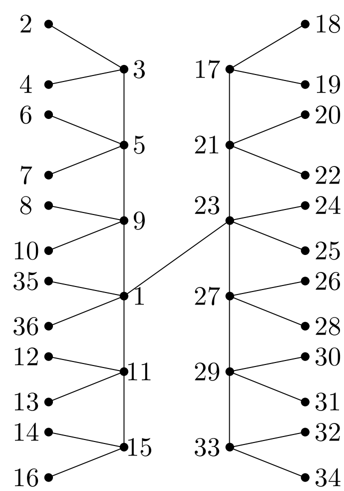

Fig.8 $G \odot S_2$ $n \equiv 2(\text{mod}4)$

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