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On Some Characterization of Ruled Surface of a Closed Spacelike Curve with Spacelike Binormal in Dual Lorentzian Space

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Abstract: In this paper, we investigate the relations between the pitch, the angle of pitch and drall of parallel ruled surface of a closed spacelike curve with spacelike binormal in dual Lorentzian space.

Key Words: Spacelike dual curve, ruled surface, Lorentzian space, dual numbers.

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§1. Introduction

Dual numbers were introduced by W.K. Clifford [5] as a tool for his geometrical investigations. After him,e E.Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics [7]. The pitches and the angles of the pitches of the closed ruled surfaces corresponding to the one parameter dual unit spherical curves and oriented lines in $\mathbb{R}^3$ were calculated respectively by Hacısalihoğlu [10] and Gürsoy [8]. Definitions of the parallel ruled surface were presented by Wilhelm Blaschke [6]. The integral invariants of the paralel ruled surfaces in the 3-dimensional Euclidean space $\mathbb{R}^3$ corresponding to the unit dual spherical parallel curves were calculated by Senyurt [14]. The integral invariants of ruled surface of a timelike curve in dual Lorentzian space were calculated by Bektaş and Şenyurt [2]. The integral invariants of ruled surface of a closed spacelike curve with timelike binormal in dual Lorentzian space were calculated by Bektaş and Şenyurt [3].

The set $D = \{\hat{\lambda} = \lambda + \varepsilon\lambda^* | \lambda, \lambda^* \in \mathbb{R}, \varepsilon^2 = 0\}$ is called *dual numbers* set, see [5].On this set, product and addition operations are respectively

$$(\lambda + \varepsilon\lambda^*) + (\beta + \varepsilon\beta^*) = (\lambda + \beta) + \varepsilon(\lambda^* + \beta^*)$$

and

$$(\lambda + \varepsilon\lambda^*)(\beta + \varepsilon\beta^*) = \lambda\beta + \varepsilon(\lambda\beta^* + \lambda^*\beta).$$

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The elements of the set $D^3 = \left\{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \,\middle|\, \overrightarrow{a}, \overrightarrow{a}^* \in \mathbb{R}^3 \right\}$ are called *dual vectors.* On this set addition and scalar product operations are respectively

$$\oplus : D^3 \times D^3 \to D^3$$
$$\left(\overrightarrow{A}, \overrightarrow{B}\right) \quad \to \overrightarrow{A} \oplus \overrightarrow{B} = \overrightarrow{a} + \overrightarrow{b} + \varepsilon \left(\overrightarrow{a}^* + \overrightarrow{b}^*\right),$$

$$\odot : D \times D^3 \to D^3$$
$$\left(\lambda, \overrightarrow{A}\right) \quad \to \lambda \odot \overrightarrow{A} = (\lambda + \varepsilon\lambda^*) \odot (\overrightarrow{a} + \varepsilon \overrightarrow{a}^*) = \lambda \overrightarrow{a} + \varepsilon (\lambda \overrightarrow{a}^* + \lambda^* \overrightarrow{a})$$

The set $\left(D^3, \oplus\right)$ is a module over the ring $(D, +, \cdot)$, called the $D -$ *Modul*.

The Lorentzian inner product of dual vectors $\overrightarrow{A}$, $\overrightarrow{B} \in D^3$ is defined by

$$\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon \left(\left\langle \overrightarrow{a}, \overrightarrow{b}^* \right\rangle + \left\langle \overrightarrow{a}^*, \overrightarrow{b} \right\rangle \right)$$

where $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle$ is the following Lorentzian inner product of vectors $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$,i.e.,

$$\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The set D^3 equipped with the Lorentzian inner product $\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle$ is called 3-dimensional dual Lorentzian space and is denoted in what follows by $D_1^3 = \left\{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \,\middle|\, \overrightarrow{a}, \overrightarrow{a}^* \in \mathbb{R}_1^3 \right\}$ [17].

A dual vector $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in D_1^3$ is called dual space-like vector if $\left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle > 0$ or $\overrightarrow{A} = 0$, a dual time-like vector if $\left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle < 0$, a dual null (light-like) vector if $\left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle = 0$ for $\overrightarrow{A} \neq 0$. For $\overrightarrow{A} \neq 0$, the *norm* $\left\| \overrightarrow{A} \right\|$ of $\overrightarrow{A}$ is defined by

$$\left\| \overrightarrow{A} \right\| = \sqrt{\left| \left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle \right|} = \| \overrightarrow{a} \| + \varepsilon \frac{\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle}{\| \overrightarrow{a} \|}, \quad \| \overrightarrow{a} \| \neq 0.$$

The dual Lorentzian cross-product of $\overrightarrow{A}$, $\overrightarrow{B} \in D^3$ is defined as

$$\overrightarrow{A} \wedge \overrightarrow{B} = \overrightarrow{a} \times \overrightarrow{b} + \varepsilon \left(\overrightarrow{a} \times \overrightarrow{b}^* + \overrightarrow{a}^* \times \overrightarrow{b} \right)$$

where $\overrightarrow{a} \times \overrightarrow{b}$ is the cross-product [14] of $\overrightarrow{a}$, $\overrightarrow{b} \in \mathbb{R}^3$ given by

$$\overrightarrow{a} \times \overrightarrow{b} = (a_3 b_2 - a_2 b_3, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1).$$

Theorem 1.1(E. Study) *The oriented lines in $\mathbb{R}^3$ are in one to one correspondence with the points of the dual unit sphere $\|\overrightarrow{A}\| = (1, 0)$ where $\overrightarrow{A} \neq (\overrightarrow{0}, \overrightarrow{a}) \in$ D-Modul , see [9].*

The dual number $\Phi = \varphi + \varepsilon\varphi^*$ is called dual angle between the unit dual vectors $\overrightarrow{A}$ ve $\overrightarrow{B}$ and keep in mind that

$$\begin{aligned} \sin(\varphi + \varepsilon\varphi^*) &= \sin\varphi + \varepsilon\varphi^* \cos\varphi \, , \\ \cos(\varphi + \varepsilon\varphi^*) &= \cos\varphi - \varepsilon\varphi^* \sin\varphi. \end{aligned}$$

§2. Characterization of Ruled Surface of a Closed Spacelike Curve with Spacelike Binormal in Dual Lorentzian Space $\left(D_1^3\right)$

Let $U : I \to D_1^3,\ t \to \overrightarrow{U}(t) = \overrightarrow{U}_1(t),\ \left\|\overrightarrow{U}(t)\right\| = 1$ be a differentiable spacelike curve with spacelike binormal in the dual unit sphere. Denote by $(\overrightarrow{U})$ the closed ruled generated by this curve.

Let $\left\{\overrightarrow{U_1}, \overrightarrow{U_2}, \overrightarrow{U_3}\right\}$ be the Frenet frame of the curve $\overrightarrow{U} = \overrightarrow{U}_1$ with

$$\overrightarrow{U}_1 = \overrightarrow{U} \quad , \quad \overrightarrow{U}_2 = \overrightarrow{U}' \Big/ \left\|\overrightarrow{U}\right\| \quad , \quad \overrightarrow{U}_3 = \overrightarrow{U}_1 \times \overrightarrow{U}_2$$

Definition 2.1 *The closed ruled surface $(\overrightarrow{U})$ corresponding to the dual spacelike curve $\overrightarrow{U}(t)$ which makes the fixed dual angle $\Phi = \varphi + \varepsilon\varphi^*$ with $\overrightarrow{U}(t)$ determines*

$$\overrightarrow{V} = \cos\Phi\overrightarrow{U_1} + \sin\Phi\overrightarrow{U_3} \tag{2.1}$$

The surface $(\overrightarrow{V})$ corresponding to the dual spacelike vector $\overrightarrow{V}$ is called the *parallel ruled surface* of surface $(\overrightarrow{U})$ in the dual Lorentzian space D_1^3.

Now, take $\overrightarrow{U}(t)$ as a closed spacelike curve with curvature $\kappa = k_1 + \varepsilon k_1^*$ and torsion $\tau = k_2 + \varepsilon k_2^*$. Recall that in the Frenet frames associated to the curve $\overrightarrow{U_1}$ and $\overrightarrow{U_3}$ are spacelike vectors and $\overrightarrow{U_2}$ is timelike vector and we have

$$\overrightarrow{U_1} \times \overrightarrow{U_2} = -\overrightarrow{U_3} \quad , \quad \overrightarrow{U_2} \times \overrightarrow{U_3} = -\overrightarrow{U_1} \quad , \quad \overrightarrow{U_3} \times \overrightarrow{U_1} = \overrightarrow{U_2}. \tag{2.2}$$

Under these conditions, the Frenet formulas are given by ([18])

$$\overrightarrow{U_1}' = \kappa\overrightarrow{U_2} \quad , \quad \overrightarrow{U_2}' = \kappa\overrightarrow{U_1} + \tau\overrightarrow{U_3} \quad , \quad \overrightarrow{U_3}' = \tau\overrightarrow{U_2}. \tag{2.3}$$

The Frenet instantaneous rotation vector (also called instantaneous Darboux vector) for the spacelike curve is given by ([16])

$$\overrightarrow{\Psi} = -\tau\overrightarrow{U_1} + \kappa\overrightarrow{U_3}, \tag{2.4}$$

Let be $\overrightarrow{V}_1 = \overrightarrow{V}$. Differentiating of the vector $\overrightarrow{V_1}$ with respect the parameter t and using the Eq.(2.3) we get

$$\overrightarrow{V_1}' = (\kappa\cos\Phi + \tau\sin\Phi)\overrightarrow{U_2} \tag{2.5}$$

and the norm of that vector denoted by P is

$$P = \kappa\cos\Phi + \tau\sin\Phi. \tag{2.6}$$

Then, substituting the values of (2.5) and (2.6) into Frenet equations gives

$$\overrightarrow{V_2} = \overrightarrow{U_2} \tag{2.7}$$

For the vector $\overrightarrow{V_3}$,we have

$$\overrightarrow{V_3} = \sin\Phi\overrightarrow{U_1} - \cos\Phi\overrightarrow{U_3} \tag{2.8}$$

If Eq.(2.1), (2.7) and (2.8) are written matrix form, we have

$$\begin{bmatrix} \overrightarrow{V_1} \\ \overrightarrow{V_2} \\ \overrightarrow{V_3} \end{bmatrix} = \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & -\cos\Phi \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{U_1} \\ \overrightarrow{U_2} \\ \overrightarrow{U_3} \end{bmatrix}$$

or

$$\begin{bmatrix} \overrightarrow{U_1} \\ \overrightarrow{U_2} \\ \overrightarrow{U_3} \end{bmatrix} = \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & -\cos\Phi \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{V_1} \\ \overrightarrow{V_2} \\ \overrightarrow{V_3} \end{bmatrix}$$

The real and dual parts of $\overrightarrow{U_1}, \overrightarrow{U_2}, \overrightarrow{U_3}$ are

$$\begin{cases} \overrightarrow{u}_1 = \cos\varphi \overrightarrow{v}_1 + \sin\varphi \overrightarrow{v}_3 \\ \overrightarrow{u}_2 = \overrightarrow{v}_2 \\ \overrightarrow{u}_3 = \sin\varphi \overrightarrow{v}_1 - \cos\varphi \overrightarrow{v}_3 \\ \overrightarrow{u}_1^* = \cos\varphi \overrightarrow{v}_1^* + \sin\varphi \overrightarrow{v}_3^* - \varphi^*(\sin\varphi \overrightarrow{v}_1 - \cos\varphi \overrightarrow{v}_3) \\ \overrightarrow{u}_2^* = \overrightarrow{v}_2^* \\ \overrightarrow{u}_3^* = \sin\varphi \overrightarrow{v}_1^* - \cos\varphi \overrightarrow{v}_3^* + \varphi^*(\cos\varphi \overrightarrow{v}_1 - \sin\varphi \overrightarrow{v}_3) \end{cases} \tag{2.9}$$

Let $P = p + \varepsilon p^*$ be the curvature and $Q = q + \varepsilon q^*$ the torsion of curve $\overrightarrow{V}(t)$. Then, the following relating holds between the vectors

$$\overrightarrow{V_1}, \overrightarrow{V_2}, \overrightarrow{V_3} \text{ and } \overrightarrow{V_1}', \overrightarrow{V_2}', \overrightarrow{V_3}' \quad [18]$$

$$\begin{cases} \overrightarrow{V}_1{}' = P\overrightarrow{V}_2, \quad \overrightarrow{V}_2{}' = P\overrightarrow{V}_1 + Q\overrightarrow{V}_3, \quad \overrightarrow{V}_3{}' = Q\overrightarrow{V}_2 \\ P = \sqrt{<\overrightarrow{V}_1{}', \overrightarrow{V}_1{}'>}, \quad Q = \frac{det(\overrightarrow{V}_1, \overrightarrow{V}_1{}', \overrightarrow{V}_1{}'')}{<\overrightarrow{V}_1{}', \overrightarrow{V}_1{}'>}. \end{cases} \tag{2.10}$$

If Eq.(2.10) is separated into its real and dual parts, we get

$$\begin{cases} \overrightarrow{v}_1{}' = p\overrightarrow{v}_2, \quad \overrightarrow{v}_2{}' = p\overrightarrow{v}_1 + q\overrightarrow{v}_3, \quad \overrightarrow{v}_3{}' = q\overrightarrow{v}_2 \\ \overrightarrow{v}'_1{}^* = p\overrightarrow{v}_2^* + p^*\overrightarrow{v}_2, \\ \overrightarrow{v}'_2{}^* = p\overrightarrow{v}_1^* + p^*\overrightarrow{v}_1 + q^*\overrightarrow{v}_3 + q\overrightarrow{v}_3{}^*, \\ \overrightarrow{v}'_3{}^* = q\overrightarrow{v}_2^* + q^*\overrightarrow{v}_2 \end{cases} \tag{2.11}$$

Now, we are ready to calculate the value of Q as function of κ and τ. Differentiating Eq. (2.5) with respect to the curve parameter t we get

$$\begin{aligned} \overrightarrow{V}_1'' =& (+\kappa^2 \cos\Phi + \kappa\tau \sin\Phi)\overrightarrow{U}_1 + \\ &+ (\kappa\cos\Phi + \tau\sin\Phi)'\overrightarrow{U}_2 + (\kappa\tau\cos\Phi + \tau^2\sin\Phi)\overrightarrow{U}_3 \end{aligned} \tag{2.12}$$

Using Eqs.(2.1), (2.5) and (2.12) into Eq.(2.10), we get

$$Q = -\kappa\sin\Phi + \tau\cos\Phi \tag{2.13}$$

and separating Eq.(2.6) and Eq.(2.13) into its dual and real parts gives

$$\begin{cases} p = k_1 \cos\varphi + k_2 \sin\varphi \\ p^* = k_1^* \cos\varphi + k_2^* \sin\varphi - \varphi^*(k_1 \sin\varphi - k_2 \cos\varphi) \\ q = -k_1 \sin\varphi + k_2 \cos\varphi \\ q^* = -k_1^* \sin\varphi + k_2^* \cos\varphi - \varphi^*(k_1 \cos\varphi + k_2 \sin\varphi) \end{cases} \tag{2.14}$$

In its dual unit spherical motion the dual orthonormal system $\left\{\overrightarrow{V_1}, \overrightarrow{V_2}, \overrightarrow{V_3}\right\}$ at any t makes a dual rotation motion around the instantaneous dual Darboux vector. This vector is determined by the following equation ([16]).

$$\overrightarrow{\overrightarrow{\Psi}} = -Q\overrightarrow{V_1} + P\overrightarrow{V_3}. \tag{2.15}$$

For the Steiner vector of the motion, we can write

$$\overrightarrow{\overrightarrow{D}} = \oint \overrightarrow{\overrightarrow{\Psi}} \tag{2.16}$$

or

$$\overrightarrow{\overrightarrow{D}} = -\overrightarrow{V_1} \oint Qdt + \overrightarrow{V_3} \oint Pdt \tag{2.17}$$

Using the values of the vectors $\overrightarrow{U_1}$ and $\overrightarrow{U_3}$ into Eq.(2.4), gives

$$\overrightarrow{\Psi} = -\tau(cos\Phi\overrightarrow{V}_1 + sin\Phi\overrightarrow{V}_3) + \kappa(sin\Phi\overrightarrow{V}_1 - cos\Phi\overrightarrow{V}_3),$$

$$\overrightarrow{\Psi} = -Q\overrightarrow{V}_1 - P\overrightarrow{V}_3 \tag{2.18}$$

Because of the equations $\overrightarrow{D} = \oint \overrightarrow{\Psi}$ for the dual Steiner vector of the motion, we may write

$$\overrightarrow{D} = -\overrightarrow{V_1} \oint Qdt - \overrightarrow{V_3} \oint Pdt \tag{2.19}$$

The real and dual parts of $\overrightarrow{D}$ are

$$\begin{cases} \overrightarrow{d} = -\overrightarrow{v}_1 \oint qdt - \overrightarrow{v}_3 \oint pdt, \\ \overrightarrow{d}^* = -\overrightarrow{v}_1 \oint q^*dt - \overrightarrow{v}_1^* \oint qdt - \overrightarrow{v}_3 \oint p^*dt - \overrightarrow{v}_3^* \oint pdt \end{cases} \tag{2.20}$$

$$\overrightarrow{D} = -\overrightarrow{U_1} \oint \tau dt + \overrightarrow{U_3} \oint \kappa dt \tag{2.21}$$

Eq.(2.21) can be written type of the dual and real part as follow

$$\begin{cases} \overrightarrow{d} = -\overrightarrow{u}_1 \oint k_2dt + \overrightarrow{u}_3 \oint k_1dt, \\ \overrightarrow{d}^* = -\overrightarrow{u}_1^* \oint k_2dt - \overrightarrow{u}_1 \oint k_2^*dt + \overrightarrow{u}_3^* \oint k_1dt + \overrightarrow{u}_3 \oint k_1^*dt \end{cases} \tag{2.22}$$

If the equation (2.3) is separated into the dual and real part, we can obtain

$$\begin{cases} \overrightarrow{u}_1' = k_1\overrightarrow{u}_2, \quad \overrightarrow{u}_2' = k_1\overrightarrow{u}_1 + k_2\overrightarrow{u}_3, \quad \overrightarrow{u}_3' = k_2\overrightarrow{u}_2 \\ \overrightarrow{u}_1'^* = k_1^*\overrightarrow{u}_2 + k_1\overrightarrow{u}_2^*, \\ \overrightarrow{u}_2'^* = k_1^*\overrightarrow{u}_1 + k_2^*\overrightarrow{u}_3 + k_1\overrightarrow{u}_1^* + k_2\overrightarrow{u}_3^* \\ \overrightarrow{u}_3'^* = k_2^*\overrightarrow{u}_2 + k_2\overrightarrow{u}_2^* \end{cases} \tag{2.23}$$

Now, let us calculate the integral invariants of the respective closed ruled surfaces. The pitch of the first closed surface (U_1) is obtained as

$$L_{u_1} = \left\langle \overrightarrow{d}, \overrightarrow{u_1}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{u_1} \right\rangle,$$

$$L_{u_1} = -\oint k_2^* dt. \tag{2.24}$$

The dual angle of the pitch of the closed surface U_1 is

$$\Lambda_{U_1} = -\left\langle \overrightarrow{D}, \overrightarrow{U_1} \right\rangle.$$

and from Eq.(2.21) we obtain

$$\Lambda_{U_1} = \oint \tau dt. \tag{2.25}$$

The real and dual of U_1 are

$$\lambda_{u_1} = \oint k_2 dt \quad , \quad L_{u_1} = -\oint k_2^* dt \tag{2.26}$$

The *drall* of the closed surface (U_1) is

$$P_{U_1} = \frac{\langle d\overrightarrow{u_1}, d\overrightarrow{u_1}^* \rangle}{\langle d\overrightarrow{u_1}, d\overrightarrow{u_1} \rangle}$$

Using the values $d\overrightarrow{u_1}$ and $d\overrightarrow{u_1}^*$ given by Eq.(2.23), we get

$$P_{U_1} = \frac{k_1^*}{k_1} \tag{2.27}$$

The pitch of the closed surface (U_2) is given by

$$L_{u_2} = 0. \tag{2.28}$$

The dual angle of the pitch of the closed surface (U_2) is

$$\Lambda_{U_2} = -\left\langle \overrightarrow{D}, \overrightarrow{U_2} \right\rangle,$$

$$\Lambda_{U_2} = 0. \tag{2.29}$$

The drall of the closed surface (U_2), we may write

$$P_{U_2} = \frac{\langle d\overrightarrow{u_2}, d\overrightarrow{u_2}^* \rangle}{\langle d\overrightarrow{u_2}, d\overrightarrow{u_2} \rangle}$$

Using the values $d\overrightarrow{u_2}$ and $d\overrightarrow{u_2}^*$ given by Eq.(2.23), we get

$$P_{U_2} = \frac{k_1 k_1^* + k_2 k_2^*}{k_1^2 + k_2^2} \tag{2.30}$$

The pitch of the closed surface (U_3) is

$$L_{u_3} = \left\langle \overrightarrow{d}, \overrightarrow{u_3}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{u_3} \right\rangle,$$

$$L_{u_3} = \oint k_1^* dt \tag{2.31}$$

The dual angle of the pitch of the closed surface (U_3) is

$$\Lambda_{U_3} = -\left\langle \overrightarrow{D}, \overrightarrow{U_3} \right\rangle$$

which gives using (2.21)

$$\Lambda_{U_3} = -\oint \kappa dt \tag{2.32}$$

The real and dual parts of Λ_{U_3} are

$$\lambda_{u_3} = -\oint k_1 dt \quad , \quad L_{u_3} = \oint k_1^* dt \tag{2.33}$$

The drall of the closed surface (U_3) is

$$P_{U_3} = \frac{\langle d\overrightarrow{u_3}, d\overrightarrow{u_3}^* \rangle}{\langle d\overrightarrow{u_3}, d\overrightarrow{u_3} \rangle}$$

Using the values of $d\overrightarrow{u_3}$ and $d\overrightarrow{u_3}^*$ given in Eq.(2.23) gives

$$P_{U_3} = \frac{k_2^*}{k_2}. \tag{2.34}$$

Let $\Omega(t) = \omega(t) + \varepsilon\omega^*(t)$ be the Lorentzian timelike angle between the instantaneous dual Pfaffion vector $\overrightarrow{\Psi}$ and the vector $\overrightarrow{U_3}$. In this case dual Pfaffion vector $\overrightarrow{\Psi}$ is spacelike vector and so,

$$\kappa = \left\| \overrightarrow{\Psi} \right\| \cos\Omega \quad , \quad \tau = \left\| \overrightarrow{\Psi} \right\| \sin\Omega$$

then $\overrightarrow{C} = \overrightarrow{c} + \varepsilon \overrightarrow{c}^*$, the unit vector in the $\overrightarrow{\Psi}$ direction is

$$\overrightarrow{C} = -\sin\Omega \overrightarrow{U_1} + \cos\Omega \overrightarrow{U_3} \tag{2.35}$$

and the real and dual parts of $\overrightarrow{C}$ are

$$\begin{cases} \overrightarrow{c} = -\sin\omega \overrightarrow{u_1} + \cos\omega \overrightarrow{u_3} \\ \overrightarrow{c}^* = -\sin\omega \overrightarrow{u_1}^* + \cos\omega \overrightarrow{u_3}^* - \omega^* \cos\omega \overrightarrow{u_1} - \omega^* \sin\omega \overrightarrow{u_3} \end{cases} \tag{2.36}$$

The pitch of the closed surface $(\overrightarrow{C})$ generated by $\overrightarrow{C}$ is given by

$$L_C = < \overrightarrow{d}, \overrightarrow{c}^* > + < \overrightarrow{d}^*, \overrightarrow{c} >$$

$$L_C = \cos\omega \oint k_1^* dt + \sin\omega \oint k_2^* dt - \omega^* (\sin\omega \oint k_1 dt - \cos\omega \oint k_2 dt) \tag{2.37}$$

If we use Eq.(2.26) and Eq.(2.33) into Eq.(2.37) we get

$$L_C = -\sin\omega L_{u_1} + \cos\omega L_{u_3} + \omega^* (\cos\omega \lambda_{u_1} + \sin\omega \lambda_{u_3}) \tag{2.38}$$

The dual angle of the pitch of that closed ruled surface $(\overrightarrow{C})$, we have

$$\Lambda_C = -\left\langle \overrightarrow{D}, \overrightarrow{C} \right\rangle$$

and from Eq.(2.21) and (2.35) it follows that

$$\Lambda_{U_3} = - < \overrightarrow{U_1} \oint \tau dt + \overrightarrow{U_3} \oint \kappa dt, - \sin \Omega \overrightarrow{U_1} + \cos \Omega \overrightarrow{U_3} >,$$

$$\Lambda_C = - \sin \Omega \oint \tau dt - \cos \Omega \oint \kappa dt \tag{2.39}$$

Using Eg.(2.25) and (2.32) gives

$$\Lambda_C = - \sin \Omega \Lambda_{U_1} + \cos \Omega \Lambda_{U_3} \tag{2.40}$$

The drall of the closed surface $(\overrightarrow{C})$ is

$$P_C = \frac{\langle d\overrightarrow{c}, d\overrightarrow{c}^* \rangle}{\langle d\overrightarrow{c}, d\overrightarrow{c} \rangle}$$

$$P_C = \frac{\omega^{'} \omega^{*'} - (k_2 \cos \omega - k_1 \sin \omega) \left[(k_2^* - k_1 \omega^*) \cos \omega - (k_2 \omega^* + k_1^*) \sin \omega \right]}{\omega'^2 - (k_2 \cos \omega - k_1 \sin \omega)^2} \tag{2.41}$$

Now, let us calculate the integral invariants of the respective closed ruled surfaces. The pitch of the closed (V_1) surface is given by

$$L_{V_1} = \left\langle \overrightarrow{d}, \overrightarrow{v_1}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{v_1} \right\rangle,$$

$$L_{V_1} = - \oint q^* dt. \tag{2.42}$$

Substituting by the value q^* into Eq.(2.42)

$$L_{V_1} = \sin \varphi \oint k_1^* dt - \cos \varphi \oint k_2^* dt + \varphi^* (\cos \varphi \oint k_1 dt + \sin \varphi \oint k_2 dt) \tag{2.43}$$

or

$$L_{V_1} = \cos \varphi L_{u_1} + \sin \varphi L_{u_3} + \varphi^* \left(\sin \varphi \lambda_{u_1} - \cos \varphi \lambda_{u_3} \right). \tag{2.44}$$

The dual angle of the pitch of the closed ruled surface (V_1) , we have

$$\Lambda_{V_1} = - \left\langle \overrightarrow{D}, \overrightarrow{V_1} \right\rangle$$

and using Eq.(2.19) we obtain

$$\Lambda_{V_1} = - < - \overrightarrow{V_1} \oint Q dt - \overrightarrow{V_3} \oint P dt, \overrightarrow{V_1} >,$$

$$\Lambda_{V_1} = \oint Q dt. \tag{2.45}$$

Using Eq.(2.13) into the last equation, we get

$$\Lambda_{V_1} = - \sin \Phi \oint \kappa dt + \cos \Phi \oint \tau dt$$

or

$$\Lambda_{V_1} = \cos \Phi \Lambda_{U_1} + \sin \Phi \Lambda_{U_3} \tag{2.46}$$

Separating Eq.(2.46) into its real and dual parts gives

$$\begin{cases} \lambda_{V_1} = \cos\varphi\lambda_{u_1} + \sin\varphi\lambda_{u_3} \\ L_{V_1} = \cos\varphi L_{u_1} + \sin\varphi L_{u_3} + \varphi^* \left(\sin\varphi\lambda_{u_1} - \cos\varphi\lambda_{u_3}\right) \end{cases} \tag{2.47}$$

The drall of the closed surface (V_1) is

$$P_{V_1} = \frac{\langle d\overrightarrow{v_1}, d\overrightarrow{v_1}^* \rangle}{\langle d\overrightarrow{v_1}, d\overrightarrow{v_1} \rangle}$$

which gives using the values of $d\overrightarrow{v_1}$ and $d\overrightarrow{v_1}^*$ in Eq.(2.11)

$$P_{V_1} = \frac{p^*}{p} \tag{2.48}$$

and using the values of p and p^* given by Eq.(2.14) gives

$$P_{V_1} = \frac{k_1^* \cos\varphi + k_2^* \sin\varphi}{k_1 \cos\varphi + k_2 \sin\varphi} - \varphi^* \frac{k_1 \sin\varphi - k_2 \cos\varphi}{k_1 \cos\varphi + k_2 \sin\varphi} \tag{2.49}$$

Theorem 2.1 *Let (V_1) be the parallel surface of the surface (U_1). The pitch, drall and the dual of the pitch of the ruled surface (V_1) are*

$$1-)L_{V_1} = -\oint q^* dt \qquad 2-)\Lambda_{V_1} = \oint Q dt \qquad 3-)P_{V_1} = \frac{p^*}{p}$$

Corollary 2.1 *Let (V_1) be the parallel surface of the surface (U_1). The pitch and the dual of the pitch of the ruled surface (V_1) related to the invariants of the surface (U_1) are written as follow*

$1^-)$ $L_{V_1} = \cos\varphi L_{u_1} + \sin\varphi L_{u_3} + \varphi^* \left(\sin\varphi\lambda_{u_1} - \cos\varphi\lambda_{u_3}\right)$;
$2^-)$ $\Lambda_{V_1} = \cos\Phi\Lambda_{U_1} + \sin\Phi\Lambda_{U_3}$

The pitch of the closed surface (V_2) is given by

$$L_{V_2} = \left\langle \overrightarrow{d}, \overrightarrow{v_2}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{v_2} \right\rangle$$

$$L_{V_2} = 0 \tag{2.50}$$

The dual angle of the pitch of the closed ruled surface (V_2) is

$$\Lambda_{V_2} = -\left\langle \overrightarrow{D}, \overrightarrow{V_2} \right\rangle$$

Using Eq.(2.19) we get

$$\Lambda_{V_2} = 0 \tag{2.51}$$

The drall of the closed surface (V_2) is

$$P_{V_2} = \frac{\langle d\overrightarrow{v_2}, d\overrightarrow{v_2}^* \rangle}{\langle d\overrightarrow{v_2}, d\overrightarrow{v_2} \rangle}$$

Using the values of $d\overrightarrow{v_2}$ and $d\overrightarrow{v_2}^*$ given by Eq.(2.11) gives

$$P_{V_2} = \frac{pp^* + qq^*}{p^2 + q^2} \tag{2.52}$$

and with the values of p, p^*, q and q^* given by Eq.(2.14) we get

$$P_{V_2} = \frac{k_1 k_1^* + k_2 k_2^*}{k_1^2 + k_2^2} \tag{2.53}$$

Theorem 2.2 *Let (V_1) be the parallel surface of the surface (U_1). The pitch, drall and the dual of the pitch of the ruled surface (V_2) are*

$$1-) L_{V_2} = 0 \qquad 2-) \Lambda_{V_2} = 0 \qquad 3-) P_{V_2} = \frac{pp^* + qq^*}{p^2 + q^2}$$

The pitch of the closed surface (V_3) is given by

$$L_{V_3} = \left\langle \overrightarrow{d}, \overrightarrow{v_3}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{v_3} \right\rangle,$$

$$L_{V_3} = -\oint p^* dt \tag{2.54}$$

and using Eq.(2.54)

$$L_{V_3} = -\cos\varphi \oint k_1^* dt - \sin\varphi \oint k_2^* dt + \varphi^* (\sin\varphi \oint k_1 dt - \cos\varphi \oint k_2 dt) \tag{2.55}$$

or

$$L_{V_3} = \sin\varphi L_{u_1} - \cos\varphi L_{u_3} - \varphi^* \left(\cos\varphi \lambda_{u_1} + \sin\varphi \lambda_{u_3}\right) \tag{2.56}$$

The dual angle of the pitch of the closed ruled surface (V_3) is

$$\Lambda_{V_3} = -\left\langle \overrightarrow{D}, \overrightarrow{V_3} \right\rangle$$

Due to Eq.(2.19) we have

$$\Lambda_{V_3} = - < \overrightarrow{V_1} \oint Qdt - \overrightarrow{V_3} \oint Pdt, \overrightarrow{V_3} >,$$

$$\Lambda_{V_3} = \oint Pdt. \tag{2.57}$$

and using Eq.(2.6) into the last equation gives

$$\Lambda_{V_3} = \cos\Phi \oint \kappa dt + \sin\Phi \oint \tau dt$$

or

$$\Lambda_{V_3} = \sin\Phi \Lambda_{U_1} - \cos\Phi \Lambda_{U_3} \tag{2.58}$$

Separating Eq.(2.58) into its real and dual parts give

$$\begin{cases} \lambda_{v_3} = \sin\varphi \lambda_{u_1} - \cos\varphi \lambda_{u_3} \\ L_{v_3} = \sin\varphi L_{u_1} - \cos\varphi L_{u_3} - \varphi^* \left(\cos\varphi \lambda_{u_1} + \sin\varphi \lambda_{u_3}\right) \end{cases} \tag{2.59}$$

The drall of the closed surface (V_3) is

$$P_{V_3} = \frac{\langle d\overrightarrow{v_3}, d\overrightarrow{v_3}^* \rangle}{\langle d\overrightarrow{v_3}, d\overrightarrow{v_3} \rangle}$$

Using the values of $d\overrightarrow{v_3}$ and $d\overrightarrow{v_3}^*$ given by Eq.(2.11) gives

$$P_{V_3} = \frac{q^*}{q} \tag{2.60}$$

and using the values of q and q^* given by Eq.(2.14) into the last equations, we get

$$P_{V_3} = \frac{-k_1^* \sin\varphi - k_2^* \cos\varphi}{-k_1 \sin\varphi + k_2 \cos\varphi} - \varphi^* \left(\frac{k_1 \cos\varphi + k_2 \sin\varphi}{-k_1 \sin\varphi + k_2 \cos\varphi} \right) \tag{2.61}$$

Theorem 2.3 *Let (V_1) be the parallel surface of the surface (U_1). The pitch, drall and the dual of the pitch of the ruled surface (V_3) are*

$$1-) L_{V_3} = -\oint p^* dt \qquad 2-) \Lambda_{V_3} = \oint P dt \qquad 3-) P_{V_3} = \frac{q^*}{q}.$$

Corollary 2.2 *Let (V_1) be the parallel surface of the surface (U_1). The pitch and the dual of the pitch of the ruled surface (V_3) related to the invariants of the surface (U_1) are written as follow*

$1^-)$ $L_{V_3} = \sin\varphi L_{u_1} - \cos\varphi L_{u_3} - \varphi^* (\cos\varphi \lambda_{u_1} + \sin\varphi \lambda_{u_3})$;
$2^-)$ $\Lambda_{V_3} = \sin\Phi \Lambda_{U_1} - \cos\Phi \Lambda_{U_3}$.

Let $\Theta(t) = \theta(t) + \varepsilon\theta^*(t)$ be the Lorentzian timelike angle between the instantaneous dual Pfaffion vector $\overrightarrow{\overline{\Psi}}$ and the vector $\overrightarrow{V_3}$.

In this case dual Pfaffion vector $\overrightarrow{\overline{\Psi}}$ is spacelike vector,

$$P = \left\| \overrightarrow{\overline{\Psi}} \right\| \cos\Theta, \qquad Q = \left\| \overrightarrow{\overline{\Psi}} \right\| \sin\Theta$$

The unit vector $\overrightarrow{\overline{C}} = \overrightarrow{c} + \varepsilon \overrightarrow{c}^*$, in the $\overrightarrow{\overline{\Psi}}$ direction is

$$\overrightarrow{\overline{C}} = -\sin\Theta \overrightarrow{V_1} + \cos\Theta \overrightarrow{V_3} \tag{2.62}$$

Using the values of the vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_3}$ given by Eq.(2.9) into Eq.(2.62), we get

$$\overrightarrow{\overline{C}} = -\sin\Theta \left(\cos\Phi \overrightarrow{U_1} + \sin\Phi \overrightarrow{U_3} \right) + \cos\Theta \left(\sin\Phi \overrightarrow{U_1} - \cos\Phi \overrightarrow{U_3} \right)$$

$$\overrightarrow{\overline{C}} = \sin(\Theta - \Phi) \overrightarrow{U_1} - \cos(\Theta - \Phi) \overrightarrow{U_3} \tag{2.63}$$

The real and dual parts of $\overrightarrow{\overline{C}}$ are

$$\begin{cases} \overrightarrow{c} = -\sin\theta \overrightarrow{v_1} + \cos\theta \overrightarrow{v_3} \\ \overrightarrow{c}^* = -\sin\theta \overrightarrow{v_1}^* + \cos\theta \overrightarrow{v_3}^* - \theta^* \cos\theta \overrightarrow{v_1} - \theta^* \sin\theta \overrightarrow{v_3} \end{cases} \tag{2.64}$$

The pitch of the closed surface $(\overrightarrow{\overline{C}})$ is given by

$$L_{\overline{C}} = \left\langle \overrightarrow{d}, \overrightarrow{c}^* \right\rangle + \left\langle \overrightarrow{d}^*, \overrightarrow{c} \right\rangle$$

and using the values of $\overrightarrow{d}$ and $\overrightarrow{d}^*$ given by Eq.(2.22) into the last equation we get

$$L_{\overline{C}} = -\cos\theta \oint p^* dt + \sin\theta \oint q^* dt + \theta^* \left(\cos\theta \oint q dt + \sin\theta \oint p dt \right) \tag{2.65}$$

or

$$L_{\overline{C}} = -\sin\theta L_{V_1} + \cos\theta L_{V_3} + \theta^* \left(\cos\theta \lambda_{V_1} + \sin\theta \lambda_{V_3} \right) \tag{2.66}$$

Finally if we use Eq.(2.47) and Eq.(2.59) into Eq.(2.66), we get

$$\begin{aligned} L_{\overline{C}} = & \sin(\varphi - \theta) L_{U_1} - \cos(\varphi - \theta) L_{U_3} + \\ & (\varphi^* - \theta^*) (\cos(\varphi - \theta)\lambda_{U_1}) + \sin(\varphi - \theta) \lambda_{U_3} \end{aligned} \tag{2.67}$$

The dual angle of the pitch of the closed ruled surface $(\overrightarrow{\overline{C}})$, we may write

$$\Lambda_{\overline{C}} = -\left\langle \overrightarrow{D}, \overrightarrow{\overline{C}} \right\rangle$$

and using Eq.(2.21) and Eq.(2.62) we get

$$\Lambda_{\overline{C}} = - < -\overrightarrow{V_1} \oint Q dt - \overrightarrow{V_3} \oint P dt, -\sin\Theta V_1 + \cos\Theta V_3 >,$$

$$\Lambda_{\overline{C}} = -\sin\Theta \oint Q dt + \cos\Theta \oint P dt \tag{2.68}$$

If we use the Eqs.(2.45) and (2.57) into the last equation, we get

$$\Lambda_{\overline{C}} = -\sin\Theta \Lambda_{V_1} + \cos\Theta \Lambda_{V_3} \tag{2.69}$$

If we use Eq.(2.46), we get

$$\Lambda_{\overline{C}} = -\sin(\Theta - \Phi)\Lambda_{U_1} - \cos(\Theta - \Phi)\Lambda_{U_3} \tag{2.70}$$

The drall of the closed surface $(\overrightarrow{\overline{C}})$, we may write

$$P_{\overline{C}} = \frac{\left\langle d\overrightarrow{c}, d\overrightarrow{c}^* \right\rangle}{\left\langle d\overrightarrow{c}, d\overrightarrow{c} \right\rangle}$$

$$P_{\overline{C}} = \frac{\theta' \theta^{*'} - (q\cos\theta - p\sin\theta) \left[(q^* - p\theta^*)\cos\theta - (q\theta^* + p^*)\sin\theta \right]}{\theta'^2 - (q\cos\theta - p\sin\theta)^2} \tag{2.71}$$

References

[1] Ayyildiz N., Coken A. C. and Yucesan A., On The dual Darboux rotation axis of the spacelike dual space curve, *Demonstratio Mathematica*, 37(1), 197-202 (2004).

[2] Bektas Ö. and Senyurt Ş., On some characterizations of ruled surface of a closed timelike curve in dual Lorentzian space, *Advances in Applied Clifford Algebras*, Vol. 4, Issue 22(2012), pp.939-953.

[3] Bektas Ö., Senyurt Ş., On some characterizations of ruled surface of a closed spacelike curve with timelike binormal in dual Lorentzian space V, *Ankara Mathematic Days Congress*, 03 - 04 June (2010).

[4] Birman G.S. and Nomizu K., Trigonometry in Lorentzian geometry, *Am. Math. Mont.*, 91(1984), 543-549.

[5] Clifford W.K., Preliminary sketch of biquaternions, *Proceedings of London Math. Soc.*, 4(1873), 361-395.

[6] Erim K., *Lessons of Differential Geometry*, Istanbul, 1949.

[7] Study E., Study E., *Geometrie der Dynamen*, Leibzig, 1903.

[8] Gursoy O., The dual angle of pitch of a closed ruled surface, *Mech. and Mach. Theory*, Vol.25, No.2(1990), pp. 131-140.

[9] Hacısalihoglu H.H., *Geometry of Motion and Theory of Quaternions*, Publishings of Science and Art Faculty of Gazi University, Math. No.2, Ankara, 338p (1983).

[10] Hacısalihoglu, H.H., On the pitch of a closed ruled surface, *Mech. and Mach. Theory*, Vol.7, pp. 291-305 (Pargaman Press. Printed in Great Britian) (1972).

[11] Hacısalihoglu, H.H., *Differential Geometry*, Publishings of Science and Art Faculty of Inonu University, Math. No.2, Malatya, 894pp (1983).

[12] Muller H. R., *Lessons of Kinematic*, Publishings of Science Faculty of Ankara University, Mat.27, Ankara, 292 p (1963).

[13] Ratcliffe J. G., *Foundations of Hyperbolic Manifolds*, Springer-Verlag New York, Inc., New York, 736p. (1994).

[14] Senyurt S., *Paralel Regle Yüzeyler ve Bazı Karakteristik Özellikleri*, Institute of Science of Ondokuz Mayıs University Ph. D. Thesis (1999).

[15] Turgut A., *3-Boyutlu Minkowski Uzayında Spacelike ve Timelike Regle Yüzeyler*, Institute of Science of Ankara University, Ankara Ph. D. Thesis, 96p (1995).

[16] Ugurlu H.H., On The geometry of timelike surfaces, *Commun. Fac. Sci. Ank.* Series A1 V.46.pp. 211-223 (1997).

[17] Ugurlu H.H. and Calıskan A., The study mapping for directed spacelike and timelike lines in Minkowski 3-space, *Math. Comp. Appl.*, 1(2), 142-148 (1996).

[18] Woestijne V.D.I., Minimal surfaces of the 3-dimensional Minkowski space, *Proc. Congres Géométrie différentielle et Applications*, Avignon (30 May 1988), Word Scientific Publishing. Singapore, 344-369 (1990).

[19] Yucesan A., Coken. A. C. and Ayyildiz N., On the dual Darboux rotation axis of the timelike dual space curve, *Balkan Journal of Geometry and Its App.*, Vol.7, No.2(2002), 137-142.