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More on p^* Graceful Graphs

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Abstract: A p^* graceful labeling of a graph G is an assignment f_p of labels to the vertices of G, that induces for each edge uv, a label $f_p^* = |f_p(u) - f_p(v)|$ so that the resulting edge labels are distinct pentagonal numbers. In this paper, we investigate the p^* graceful nature of some graphs based on some graph theoretic operations.

Key Words: Pentagonal numbers, p^*-graceful graphs, comb graph, twig graph, banana trees

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§1. Introduction

Unless otherwise mentioned, a graph in this paper means a simple graph without isolated vertices. For all the terminology and notations in graph theory, we follow [1] and [2] and for the definition regarding p^* graceful graphs, we follow [4].

A labeling f of a graph G is one-one mapping from the vertex set of G into the set of integers. Consider a graph G with q edges. Let $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$ such that $f_p^*(uv) = |f_p(u) - f_p(v)|$. If f_p^* is a sequence of distinct consecutive pentagonal numbers, then the function f_p is said to be p^* graceful labeling and the graph which admits the p^* graceful labeling is called p^* graceful graph. Here $\omega^p(q) = \dfrac{q(3q-1)}{2}$ is the q^{th} pentagonal number.

In [4], we proved that the paths, star graphs, comb graphs and twig graphs are p^* graceful. In this paper, we are having some generalizations on p^* graceful graphs.

Theorem 1.1 *$S(n, 1, n)$ is p^* graceful.*

Proof Let $G = S(n, 1, n)$. Let u_1, u_2, u_3 be the vertices of P_3 and $u_{1i}, u_{21}, u_{3i}, i = 1, 2, \cdots n$ be the pendant vertices attached with the vertices of P_3. Define $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$ such that $f_p(u_1) = 0$

$f_p(u_{1i}) = \omega^p(i),\ i = 1, 2, \cdots, n;$
$f_p(u_2) = \omega^p(q),\ f_p(u_{21}) = f_p(u_2) - \omega^p(q-1);$
$f_p(u_3) = f_p(u_2) - \omega^p(q-2),\ f_p(u_{3i}) = f_p(u_3) + \omega^p(q-2-i), \quad i = 1, 2, \ldots, n.$

Then we can easily verify that f_p generates f_p^* as required. Hence the result. □

[1]Received May 13, 2013, Accepted August 30, 2013.

Theorem 1.2 *The union of two p^* graceful trees is p^* graceful.*

Proof Let G_1 and G_2 be two p^* graceful trees. Let n_1 be the number of edges of G_1 and n_2 be the number of edges of G_2 such that $n_1 + n_2 = q$, the number of edges of $G_1 \cup G_2$. The p^* graceful labeling of $G_1 \cup G_2$ can be obtained as by assigning the vertices in the first copy of $G_1 \cup G_2$ i.e, G_1 in such a way as to get the edge labels $\{\omega^p(q), \ldots, \omega^p(q-(n_1-1))\}$ and then by assigning the first vertex of G_2 by $\omega^p(q-(n_1-1))-1$. The remaining vertices of G_2 are labeled so as to get $\{\omega^p(q-n_1), \cdots, \omega^p(1)\}$ as edge labels. □

Corollary 1.1 *The union of n, p^* graceful graphs is p^* graceful.*

Definition 1.1 *Let S_n be a star with n pendant vertices. Take m isomorphic copies of S_n. Let u_i and u_{ij}, $j = 1, 2, \cdots, n$ for $i = 1, 2, \cdots, m$ be the vertices of the i^{th} copy of S_n. Join u_1 to $u_{1+i,1}$ for $i = 1, 2, \cdots, m-1$. The resultant graph is denoted by S_n^m. Note that S_n^m has $mn+n$ vertices and $m(n+1)-1$ edges.*

Theorem 1.3 *The graph S_n^m exhibits p^* gracefulness.*

Proof Let the vertex set of S_n^m be $\{u_i u_{ij}/i = 1, 2, \cdots, m, j = 1, 2, \ldots n\}$. Define $f_p : V(S_n^m) \to \{0, 1, \cdots, \omega^p(q)\}$ such that $f_p(u_1) = \omega^p(q), f_p(u_{11}) = 0$;

$f_p(u_{1i}) = f_p(u_1) - \omega^p(q-(i-1)),\ i = 2, 3, \cdots, n;$
$f_p(u_{k1}) = |f_p(u_1) - \omega^p(q-(k-1)n-(k-2))|,\ k = 2, 3, \cdots, m;$
$f_p(u_k) = |f_p(u_{k1}) - \omega^p(q-(k-1)n-(k-2)-1)|,\ k = 2, 3, \cdots, m;$
$f_p(u_{ki}) = |f_p(u_k) - \omega^p(q-(k-1)n-(k-2)-i)|,\ i = 2, 3, \cdots, n.$

If the vertex labeling is less than the corresponding $\omega^p(n)$, instead of subtraction, addition may be done. Clearly f_p defined in this manner generates f_p^* as required. □

For example, the p^* graceful labeling of ${S_4}^5$ is shown in Figure 1.

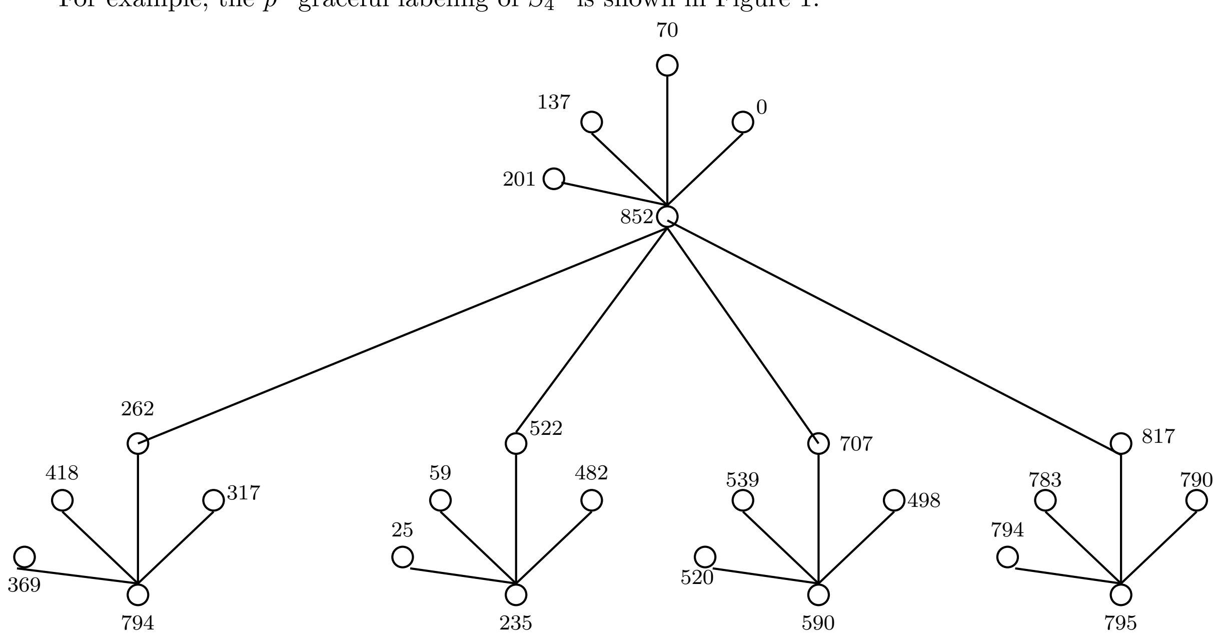

Figure 1

§2. On Cycles and Related Graphs

Theorem 2.1 *Cycles are p^* graceful graphs for some $n \geqslant 6$.*

Proof Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle.

Case 1 $n \equiv 0 (\bmod\ 4)$

Let $n = 4k$ for some k. Define $f_p : V(C_n) \to \{0, 1, \cdots, \omega^p(q)\}$ as follows.

$f_p(u_1) = 0,\ f_p(u_2) = \omega^p(q);$
$f_p(u_i) = f_p(u_{i-1}) + (-1)^i \omega^p(q - 2i + 3),\ 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 2;$
$f_p(u_{q-i}) = f_p(u_{q-i+1}) + (-1)^i \omega^p(q - 2i),\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 4$ and $f_p(u_q) = \omega^p(q - 1)$.

As we reach $u_{\lfloor \frac{n}{2} \rfloor - 1}$ and $u_{q - \lfloor \frac{n}{2} \rfloor + 3}$, a stage may be reached when the vertex label is big enough to accommodate two or more consecutive $\omega^p(i)$. Hence or otherwise we can complete the proof in Case 1, by allotting all pentagonal numbers from $\omega^p(1)$ to $\omega^p(q)$. For example, p^* graceful labeling of C_{16} is shown in Figure 2.

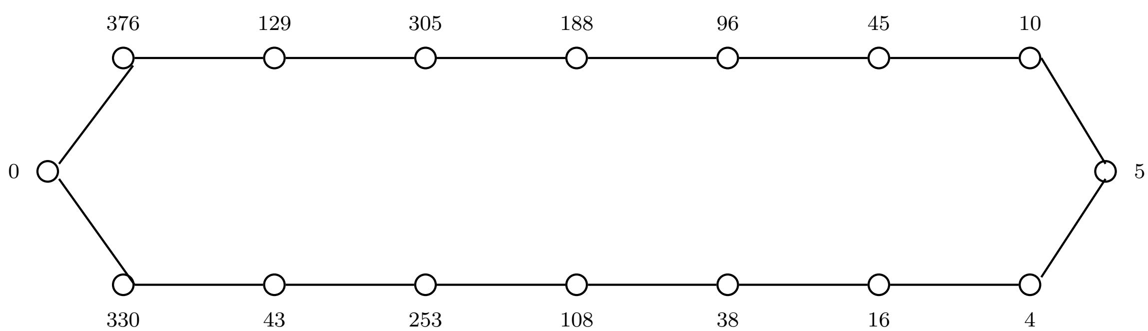

Figure 2

Case 2 $n \equiv 2 (\bmod\ 4)$

Let $n = 4k + 2$ for some k. Define $f_p : V(C_n) \to \{0, 1, \cdots, \omega^p(q)\}$ such that

$f_p(u_1) = 0, \quad f_p(u_2) = \omega^p(q);$
$f_p(u_i) = f_p(u_{i-1}) + (-1)^i \omega^p(q - 2i + 4),\ 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 2;$
$f_p(u_{q-i}) = f_p(u_{q-i+1}) + (-1)^i \omega^p(q - 2i - 1),\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 4$ and $f_p(u_q) = \omega^p(q - 1)$.

As discussed in the earlier case, after the above defined stages we may make suitable increments or decrements depending upon the size of vertex labels, to get the remaining $\omega^p(i)$. As an example consider the labeling of C_{14} in Figure 3.

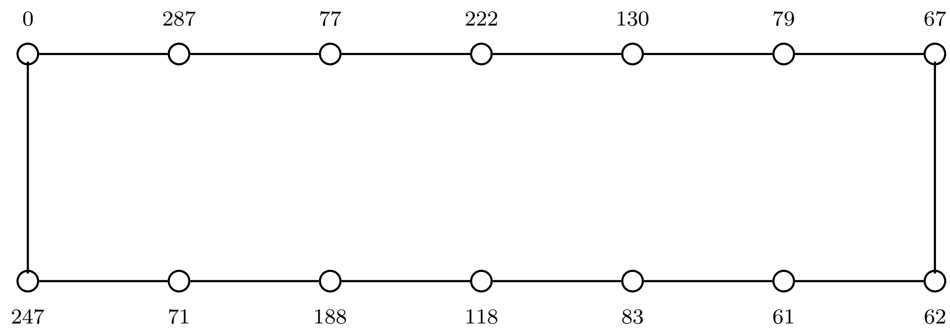

Figure 3

Case 3 $n \equiv 3 (\bmod 4)$

Let $n = 4k - 1$ for some k. Here we define f_p on $V(C_n)$ as follows:

$f_p(u_1) = 0,\ f_p(u_2) = \omega^p(q);$
$f_p(u_i) = f_p(u_{i-1}) + (-1)^i \omega^p(q - 2i + 4),\ 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 1;$
$f_p(u_{q-i}) = f_p(u_{q-i+1}) + (-1)^i \omega^p(q - 2i - 1),\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 3$ and $f_p(u_q) = \omega^p(q - 1)$.

As we reach the vertex at $\lfloor \frac{n}{2} \rfloor$ ie, $u_{\lfloor \frac{n}{2} \rfloor}$ and the vertex $u_{q - \lfloor \frac{n}{2} \rfloor + 2}$ a stage will be reached where the vertex labels is big enough to accommodate two or more consecutive $\omega^p(i)$. Hence or otherwise we can complete the labeling in the required manner. For example, consider the p^* graceful labeling of C_{15} in Figure 4.

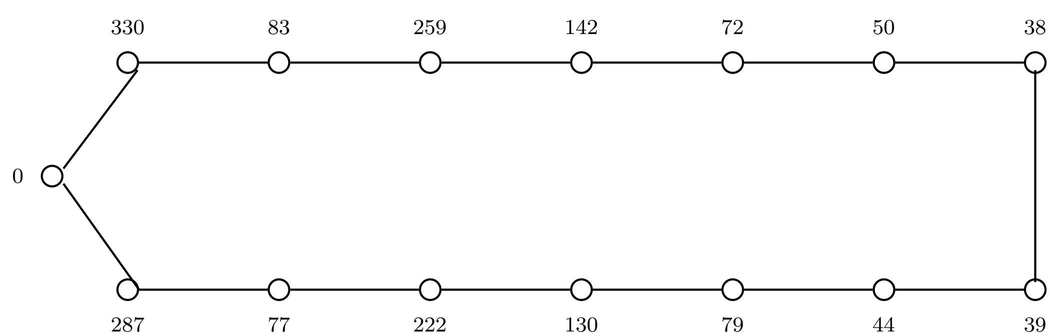

Figure 4

Definition 2.1 *The armed crown is a graph obtained from cycle C_n by attaching a path P_m at each vertex of C_n and is denoted by $C_n \Theta P_m$.*

Definition 2.2 *Biarmed crown $C_n \Theta 2P_n$ is a graph obtained from C_n by identifying the pendant vertices of two vertex disjoint paths of same length $m - 1$ at each vertex of the cycle.*

Corollary 2.1 *The armed crown $C_n \Theta P_m$ and bi-armed crown $C_n \Theta 2P_m$ are p^* graceful for some n and m.*

§3. p^* Gracefulness of Some Duplicate Graphs

Definition 3.1 *Let G be a graph with $V(G)$ as vertex set. Let V' be the set of vertices $|V'| = |V|$ where each $a \in V$ is associated with a unique $a' \in V'$. The duplicate graph of G, denoted by $D(G)$ has the vertex set $V \cup V'$ and $E(D(G))$ defined as,*

$$E(D(G)) = \{ab' \text{ and } a'b : ab \in E(G)\} \text{ (see [2])}$$

For example, $D(C_3) = C_6$.

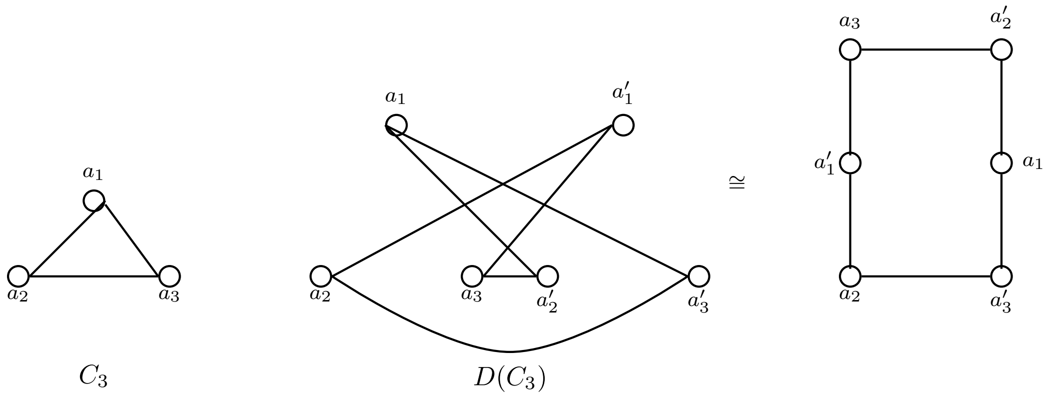

Figure 5

Theorem 3.1 *The duplicate graph of a path is p^* graceful.*

Proof Let P_n be a path.

$$D(P_n) = P_n \cup P_n$$

By Theorem 1.2, $D(P_n)$ is p^* graceful. □

Theorem 3.2 *The duplicate graph of a star S_n is p^* graceful.*

Proof Let $S_n = K_{1,n}$ be a star.

$$D(S_n) = S_n \cup S_n$$

By Theorem 1.2, $D(S_n)$ is p^* graceful. □

Theorem 3.3 *The duplicate graph of H graph admits p^* graceful labeling.*

Proof Let G be an H-graph on $2n$ vertices. $D(G) = G \cup G$. Again by the same theorem mentioned above, we have the result. □

Theorem 3.4 *The duplicate graph $C_3 \hat{o} K_{1,n}$ $n \geqslant 5$ admits p^* graceful labeling.*

Proof $D(C_3 \hat{o} K_{1,n}) = C_6 \hat{o} 2K_{1,n}$. Let $u_i, i = 1, 2, \ldots, 6$ be the vertices of C_6 and u_{1i} and u_{4i}; $i = 1, 2, \ldots, n$ be the pendant vertices attached with u_1 and u_4 respectively.

Consider the mapping f_p on the vertices of $G = C_6\hat{o}2K_{1,n}$ as $f_p : V(G) \to \{0, 1, \ldots, \omega^p(q)\}$ such that

$$f_p(u_1) = 0, \quad f_p(u_2) = \omega^p(6);$$
$$f_p(u_3) = 29, \quad f_p(u_4) = 24, \quad f_p(u_5) = 23, \quad f_p(u_6) = 35;$$
$$f_p(u_{1i}) = \omega^p(6 + n + i), \qquad i = 1, 2, \ldots, n;$$
$$f_p(u_{4i}) = f_p(u_4) + \omega^p(7 + i - 1), \qquad i = 1, 2, \cdots, n.$$

Obviously f_p defined as above give rise to f_p^* as required. Hence the result. □

In general $D(C_m\hat{o}K_{1,n})$ is p^* graceful for some m.

Remark 3.1 *$D(C_{2n}) = C_{2n} \cup C_{2n}$ for all n is not p^* graceful.*

But $D(C_{2n+1}) = C_{2(2n+1)}$ is p^* graceful, if C_{2n+1} is so.

Conjecture *All trees are p^* graceful.*

References

[1] F.Harary, *Graph Theory*, Addison-Wesley Publishing Company, 1969.

[2] Joseph A. Gallian, A dynamic survey of Graph Labeling, *The Electronic Journal of Combinatorics*, DS6 (2009), Ninth version.

[3] Mathew Varkey T.K, *Some Graph Theoretic Operations Associated with Graph Labelings*, Ph.D Thesis (2000), University of Kerala, Kerala, India.

[4] Teena Liza John, Mathew Varkey T.K, p^*-graceful graphs, *International J.Math.Combin.*, Vol.4, 2012, 97-102.