International J.Math. Combin. Vol.3(2013), 01-15

Modular Equations for Ramanujan's Cubic Continued Fraction And its Evaluations

B.R.Srivatsa Kumar

(Department of Mathematics, Manipal Institute of Technology, Manipal University, Manipal-576104, India)

G.N.Rajappa

(Department of Mathematics, Adichunchanagiri Institute of Technology, Jyothi Nagar, Chikkamagalur-577102, India)

E-mail: sri_vatsabr@yahoo.com

Abstract: In this paper, we establish certain modular equations related to Ramanujan's cubic continued fraction

$$G(q) := \frac{q^{1/3}}{1}_{+}\frac{q+q^2}{1}_{+}\frac{q^2+q^4}{1}_{+\dots}, \qquad |q|<1.$$

and obtain many explicit values of $G(e^{-\pi\sqrt{n}})$, for certain values of n.

Key Words: Ramanujan cubic continued fraction, theta functions, modular equation.

AMS(2010): 33D90, 11A55

§1. Introduction

Let

$$G(q) := \frac{q^{1/3}}{1}_{+}\frac{q+q^2}{1}_{+}\frac{q^2+q^4}{1}_{+\dots}, \tag{1.1}$$

denote the Ramanujan's cubic continued fraction for $|q| < 1$. This continued fraction was recorded by Ramanujan in his second letter to Hardy [12]. Chan [11] and Baruah [5] have proved several elegant theorems for $G(q)$. Berndt, Chan and Zhang [8] have proved some general formulas for $G(e^{-\pi\sqrt{n}})$ and $H(e^{-\pi\sqrt{n}})$ where

$$H(q) := -G(-q)$$

and n is any positive rational, in terms of Ramanujan-Weber class invariant G_n and g_n:

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty$$

and

$$g_n := 2^{-1/4}q^{-1/24}(q;q^2)_\infty, q = e^{-\pi\sqrt{n}}.$$

[1]Received June 24, 2013, Accepted July 25, 2013.

For the wonderful introduction to Ramanujan's continued fraction see [3], [6], [11] and for some beautiful subsequent work on Ramanujan's cubic continued fraction [1], [2], [4], [5], [14] and [15].

In this paper, we establish certain general formulae for evaluating $G(q)$. In section 2 of this paper, we setup some preliminaries which are required to prove the general formulae. In section 3, we establish certain modular equations related to $G(q)$ and in the final section, we deduce the above stated general formulae and obtain many explicit values of $G(q)$. We conclude this introduction by recalling an identity for $G(q)$ stated by Ramanujan.

$$1+\frac{1}{G^3(q)}=\frac{\psi^4(q)}{q\psi^4(q^3)} \tag{1.2}$$

where

$$\psi(q):=\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.3}$$

The proof of (1.2) follows from Entry 1 (ii) and (iii) of Chapter 20 (6, p.345]).

§2. Some Preliminary Results

As usual, for any complex number a,

$$(a;q)_0 := 1$$

and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;x):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n, \qquad |x|<1,$$

with

$$(a)_n := a(a+1)(a+2)...(a+n-1).$$

Then, we say that β is of n^{th} degree over α and call the ratio

$$m:=\frac{z_1}{z_n},$$

the multiplier, where $z_1 ={}_2 F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n ={}_2 F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

Theorem 2.1 *Let $G(q)$ be as defined as in (1.1), then*

$$G(q)+G(-q)+2G^2(-q)G^2(q)=0 \tag{2.1}$$

and

$$G^2(q)+2G^2(q^2)G(q)-G(q^2)=0. \tag{2.2}$$

For a proof of Theorem 2.1, see [11].

Theorem 2.2 *Let β and γ be of the third and ninth degrees, respectively, with respect to α. Let $m = z_1/z_3$ and $m' = z_3/z_9$. Then,*

$$(i) \qquad \left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = \frac{-3m}{m'} \tag{2.3}$$

and

$$(ii) \qquad \left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = \frac{m'}{m}. \tag{2.4}$$

For a proof, see [6], Entry 3 (xii) and $(xiii)$, pp. 352-353.

Theorem 2.3 *Let α, β, γ and δ be of the first, third, fifth and fifteenth degrees respectively. Let m denote the multiplier connecting α and β and let m' be the multiplier relating γ and δ. Then,*

$$(i) \qquad \left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}} \tag{2.5}$$

and

$$(ii) \qquad \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{2.6}$$

For a proof, see [6], Entry 11 $(viii)$ and (ix), p. 383.

Theorem 2.4 *If β, γ and δ are of degrees 3, 7 and 21 respectively, $m = z_1/z_3$ and $m' = z_7/z_{21}$, then*

$$(i) \qquad \left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4} + \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}$$
$$-2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}\left\{1+\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}\right\} = mm' \tag{2.7}$$

and

$$(ii) \qquad \left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/4} + \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4}$$
$$-2\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8}\left\{1+\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}\right\} = \frac{9}{mm'}. \tag{2.8}$$

For a proof, see [6], Entry 13 (v) and (vi), pp. 400-401.

§3. Modular Equations

Theorem 3.1 *Let*

$$R := \frac{\psi(-q^3)\psi(-q^2)}{q^{3/8}\psi(-q)\psi(-q^6)} \qquad \textit{and} \qquad S := \frac{\psi(-q^6)\psi(-q^4)}{q^{3/4}\psi(-q^2)\psi(-q^{12})}$$

then,

$$\left(\sqrt{\frac{R}{S}} + \sqrt{\frac{S}{R}}\right)\left(\sqrt{RS} + \frac{1}{\sqrt{RS}}\right) - 8 = 0. \tag{3.1}$$

Proof From (1.2) and the definition of R and S, it can be seen that

$$B^3(A^3+1)R^4 = A^3(B^3+1) \tag{3.2}$$

and

$$C^3(B^3+1)S^4 = B^3(C^3+1), \tag{3.3}$$

where $A = G(-q), B = G(-q^2)$ and $C = G(-q^4)$.

On changing q to q^2 in (2.1), we have

$$G(q^2) + G(-q^2) + 2G^2(-q^2)G^2(q^2) = 0 \tag{3.4}$$

and also change q to $-q$ in (2.2), we have

$$G^2(-q) + 2G^2(q^2)G(-q) - G(q^2) = 0. \tag{3.5}$$

Eliminating $G(q^2)$ between (3.4) and (3.5) using Maple,

$$2(AB)^4 - 4(AB)^3 + 3(AB)^2 + AB + A^3 + B^3 = 0. \tag{3.6}$$

Now on eliminating A between (3.2) and (3.6) using Maple, we obtain

$$8(BR)^4 - 80(BR)^3 + 63(BR)^2 - 5BR + B^3 - 16B^3R + 72B^3R^2 + 7B^3R^4$$
$$-22B^2R + 2B^2 + 2B^2R^3 - B^2R^4 - 9BR^2 + BR^3 + B + R = 0. \tag{3.7}$$

Changing q to q^2 in (3.6),

$$2(BC)^4 - 4(BC)^3 + 3(BC)^2 + BC + B^3 + C^3 = 0. \tag{3.8}$$

Eliminating C between (3.3) and (3.8) using Maple,

$$8B^4 + 7B^3 - 16S^3B^3 + 72S^2B^3 - 80SB^3 + S^4B^3 + 2B^2S^4 - B^2 + 2B^2S - 22S^3B^2$$
$$+63B^2S^2 - 9BS^2 + SB - 5BS^3 + BS^4 + S^3 = 0. \tag{3.9}$$

Finally on eliminating B between (3.7) and (3.9) using Maple, we have

$$L(R,S)M(R,S) = 0,$$

where,

$$L(R,S) = 15S^3R^6 - 1734R^4S^4 + SR + 49S^2R^2 - S^3 - 137S^4R^2 + 8S^4R + 705S^4R^3$$
$$-137S^2R^4 - 8S^2R - 15S^2R^3 + 8SR^4 - 8SR^2 + 16SR^3 + 705S^3R^4 - 15S^3R^2 + 16S^3R - 327S^3R^3$$
$$-120S^3R^5 + 705R^5S^4 + 15S^2R^5 - SR^5 - S^3R^7 - 137R^6S^4 + 8R^7S^4 - 327R^5S^5 + 49R^6S^6$$
$$+8R^4S^7 - R^5S^8 - 15R^5S^6 - 8R^7S^6 - R^8S^5 - 15R^6S^5 + 16R^7S^5 - 8R^6S^7 + 16R^5S^7 + R^7S^7$$
$$-120S^5R^3 + 15S^5R^2 + 705S^5R^4 - 137S^6R^4 + 15S^6R^3 - S^7R^3 - S^5R - R^3 = 0$$

and

$$M(R,S) = R^2S + RS^2 - 8RS + R + S = 0.$$

Using the series expansion of R and S in the above we find that

$$L(R,S) = 223522 + 8q^{-15/2} - 8q^{-57/8} - 2q^{-55/8} - 56q^{-27/4} + 48q^{-13/2} - 24q^{-49/8} + ...$$

and

$$M(R,S) = q^{-15/8} + q^{-3/2} - 8q^{-9/8} + q^{-7/8} + q^{-3/4} + 2q^{-1/2} + ...,$$

where

$$R = \frac{1}{q^{3/8}} + q^{5/8} + 2q^{29/8} + 2q^{21/8} + 2q^{13/8} + ...$$

and

$$S = \frac{1}{q^{3/4}} + q^{5/4} + 2q^{29/4} + 2q^{21/4} + 2q^{13/4} + ...$$

One can see that $q^{-1}L(R,S)$ does not tend to 0 as $q \to 0$ whereas $q^{-1}M(R,S)$ tends to 0 as $q \to 0$. Hence, $q^{-1}M(R,S) = 0$ in some neighborhood of $q = 0$. By analytic continuation $q^{-1}M(R,S) = 0$ in $|q| < 1$. Thus we have

$$M(R,S) = 0.$$

On dividing throughout by RS we have the result. □

Theorem 3.2 *If*

$$R := \frac{\psi^2(-q^3)}{q^{1/2}\psi(-q)\psi(-q^9)} \qquad \textit{and} \qquad S := \frac{\psi^2(-q^6)}{q\psi(-q^2)\psi(-q^{18})},$$

then

$$\left(\frac{R}{S}\right)^4 + \left(\frac{S}{R}\right)^4 + \left(\frac{R}{S}\right)^2 + \left(\frac{S}{R}\right)^2 - \left(RS - \frac{3}{RS}\right)\left\{\left(\frac{R}{S}\right)^3 + \left(\frac{S}{R}\right)^3\right\}$$
$$-3\left(RS - \frac{3}{RS}\right)\left(\frac{R}{S} + \frac{S}{R}\right) - \left\{(RS)^2 + \frac{9}{(RS)^2}\right\} - 6 = 0. \qquad (3.10)$$

Proof Let

$$P := \frac{\psi^2(q^3)}{q^{1/2}\psi(q)\psi(q^9)} \qquad \text{and} \qquad Q := \frac{\psi^2(q^6)}{q\psi(q^2)\psi(q^{18})}.$$

On using Entry 10 $(ii)$and (iii) of Chapter 17 in [6, p.122] in P and Q, we deduce

$$\frac{P}{Q}=\left(\frac{\alpha\gamma}{\beta^2}\right) \qquad \text{and} \qquad \frac{P^2}{Q}=\left(\frac{z_3^2}{z_1z_9}\right)^{1/2}.$$

Employing these in (2.3) and (2.4) it is easy to see that

$$\left\{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right\}^{1/4}=\frac{Q^2(3+P^2)}{P^2(Q^2-P^2)} \text{ and } \left\{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right\}^{1/4}=\frac{P^2(P^2-1)}{Q^2-P^2}.$$

Multiplying these two, we arrive at

$$P^4-4P^2Q^2+Q^4+3Q^2-P^4Q^2=0. \tag{3.11}$$

Changing q to $-q$ in the above,

$$R^4-4R^2Q^2+Q^4+3Q^2-R^4Q^2=0. \tag{3.12}$$

On eliminating Q between (3.11) and (3.12), we have

$$P^4R^4-5P^4-12P^2+16P^2R^2+4P^2R^4-11R^4-8R^6-R^8+12R^2+4P^4R^2$$
$$=(-4P^2-P^4+4R^2+R^4)\sqrt{6R^4-24R^2+8R^6+R^8+9}$$

On squaring the above and then factorizing, we have

$$P^4-2P^2R^2+R^4-P^4R^2-P^2R^4+3P^2+3R^2=0. \tag{3.13}$$

Changing q to q^2 in (3.13), we have

$$Q^4-2Q^2S^2+S^4-Q^4S^2-Q^2S^4+3Q^2+3S^2=0. \tag{3.14}$$

Eliminating Q between (3.12) and (3.14) and then on dividing throughout by $(RS)^4$ and on simplifying, we obtain the required result.

Theorem 3.3 *If*

$$R:=\frac{\psi(-q^3)\psi(-q^5)}{q^{1/4}\psi(-q)\psi(-q^{15})} \qquad \textit{and} \qquad S:=\frac{\psi(-q^6)\psi(-q^{10})}{q^{1/2}\psi(-q^2)\psi(-q^{30})},$$

then

$$\left(\frac{R^2}{S^2}+\frac{S^2}{R^2}\right)+\left(\frac{R}{S}+\frac{S}{R}\right)-\left(\sqrt{RS}-\frac{1}{\sqrt{RS}}\right)\left\{\sqrt{\frac{S}{R}}+\sqrt{\frac{R}{S}}+\left(\frac{R}{S}\right)^{3/2}+\left(\frac{S}{R}\right)^{3/2}\right\}$$
$$=RS+\frac{1}{RS}. \tag{3.15}$$

Proof Let

$$P:=\frac{\psi(q^3)\psi(q^5)}{q\psi(q)\psi(q^{15})} \qquad \text{and} \qquad Q:=\frac{\psi(q^6)\psi(q^{10})}{q^2\psi(q^2)\psi(q^{30})},$$

On using Entry 11 (*ii*) and (*iii*) of Chapter 17 in [6, p.122] in P and Q we deduce

$$\frac{P}{Q}=\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} \qquad \text{and} \qquad \frac{P^2}{Q}=\left(\frac{m'}{m}\right)^{1/2}.$$

Employing (2.5) and (2.6) in the above, it is easy to check that

$$\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}=\frac{P(P-1)}{Q-P} \qquad \text{and} \qquad \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}=\frac{Q(P+1)}{P(Q-P)}$$

Multiplying these two, we obtain

$$P^2+Q^2-2PQ-P^2Q+Q=0. \tag{3.16}$$

Changing q to $-q$ in the above

$$R^2+Q^2-2RQ-R^2Q+Q=0. \tag{3.17}$$

Eliminating Q between (3.16) and (3.17), we obtain

$$P^2+R^2+(P+R)(1-PR)=0. \tag{3.18}$$

On Changing q to q^2 in the above

$$Q^2+S^2+(Q+S)(1-QS)=0. \tag{3.19}$$

Finally, on eliminating Q between (3.17) and (3.19) and on dividing through out by $(RS)^2$, we have the result. □

Theorem 3.4 *If*

$$R:=q^2\frac{\psi(-q^3)\psi(-q^{21})}{\psi(-q)\psi(-q^7)} \qquad \textit{and} \qquad S:=q^4\frac{\psi(-q^6)\psi(-q^{42})}{\psi(-q^2)\psi(-q^{14})},$$

then

$$\begin{aligned}&y_8-(4+6x_1)y_7+(24+24x_1+9x_2)y_6-(148+12x_1+36x_2)y_5+(145+252x_1)y_4\\&-(648+678x_1-36x_2+54x_3)y_3+(2180+360x_1+441x_2-324x_3)y_2-(1016+2016x_1-396x_2-54x_3)y_1\\&+81x_4-324x_3+1548x_2+1236x_1+5250=0,\end{aligned} \tag{3.20}$$

where

$$x_n=(3RS)^n+\frac{1}{(3RS)^n} \qquad \textit{and} \qquad y_n=\left(\frac{R}{S}\right)^n+\left(\frac{S}{R}\right)^n.$$

Proof Let

$$P:=q^2\frac{\psi(q^3)\psi(q^{21})}{\psi(q)\psi(q^7)} \qquad \text{and} \qquad Q:=q^4\frac{\psi(q^6)\psi(q^{42})}{\psi(q^2)\psi(q^{14})},$$

Using Entry 11 (*ii*) and (*iii*) of Chapter 17 [6, p.122] in P and Q it is easy to deduce

$$\frac{P}{Q}=\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8} \qquad \text{and} \qquad \frac{P^2}{Q}=\frac{1}{\sqrt{mm'}}.$$

Employing (2.5) and (2.6) in the above, it is easy to check that

$$\left\{\frac{(P-Q)^2Px}{Q}-PQ-P^2\right\}^2-4P^3Q-Q^2+2PQ-P^2=0$$

and

$$\left\{\frac{(P-Q)^2}{Px}-P-Q\right\}^2-4PQ-9P^2Q^2+18P^3Q-9P^4=0.$$

where

$$x=\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}.$$

Eliminating x between these two we have

$$\begin{aligned}Q^4+8Q^4P^2-4P^3Q^3-2P^4Q^2-44P^4Q^4+24Q^2P^6-12P^7Q+81P^8Q^4\\ +72P^6Q^4-18Q^2P^8-18Q^6P^4-36Q^5P^5+P^8+Q^8-2Q^6-12P^5Q^3\\ -12P^3Q^5+24Q^6P^2-4Q^5P-36P^7Q^3-12PQ^7=0.\end{aligned} \tag{3.21}$$

On changing q to $-q$ in the above

$$\begin{aligned}Q^4+8Q^4R^2-4R^3Q^3-2R^4Q^2-44R^4Q^4+24Q^2R^6-12R^7Q+81R^8Q^4\\ +72R^6Q^4-18Q^2R^8-18Q^6R^4-36Q^5R^5+R^8+Q^8-2Q^6-12R^5Q^3\\ -12R^3Q^5+24Q^6R^2-4Q^5R-36R^7Q^3-12RQ^7=0.\end{aligned} \tag{3.22}$$

Now on eliminating Q between (3.21) and (3.22),

$$\begin{aligned}R^4-2R^6-18P^8R^2+144P^7R^3-450P^6R^4+504P^5R^5-450P^4R^6-12PR^7\\ -12RP^7+78R^2P^6-228R^3P^5+226R^4P^4-228R^5P^3+78R^6P^2-18R^8P^2\\ +P^4-2P^6+P^8+81P^8R^4+R^8+16RP^5-50R^2P^4\\ +56R^3P^3-50R^4P^2+16R^5P+144P^3R^7-4RP^3+6P^2R^2-4PR^3\\ +486R^6P^6-324R^5P^7-324R^7P^5+81R^8P^4=0.\end{aligned} \tag{3.23}$$

On changing q to q^2 in the above

$$\begin{aligned}Q^4+Q^8-2Q^6+S^4-2S^6+S^8-18Q^8S^2+144Q^7S^3-450Q^6S^4+504Q^5S^5\\ -450Q^4S^6-12QS^7-12SQ^7+78S^2Q^6-228S^3Q^5+226S^4Q^4-228S^5Q^3\\ +78S^6Q^2-18S^8Q^2+81Q^8S^4+16SQ^5-50S^2Q^4+56S^3Q^3\\ -50S^4Q^2+16S^5Q+144Q^3S^7-4SQ^3+6Q^2S^2-4QS^3+486S^6Q^6\\ -324S^5Q^7-324S^7Q^5+81S^8Q^4=0.\end{aligned} \tag{3.24}$$

Finally, on eliminating Q between (3.22) and (3.24), on dividing throughout by $(RS)^8$ and then simplifying we obtain the required result. □

Theorem 3.5 *If*

$$P=\frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \text{and} \qquad Q=\frac{\psi(q^7)}{q^{7/4}\psi(q^{21})}$$

then

$$\left(2(9+(PQ)^4)\right)\left(\left(\frac{P}{Q}\right)^2-\left(\frac{Q}{P}\right)^2\right)+3(PQ)^4+27=15(PQ)^2\left(\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right). \quad (3.25)$$

Proof Let

$$M_n:=\frac{f(-q)}{q^{n/2}f(-q^{3n})}.$$

It is easy to see that

$$P=\frac{M_2^2}{M_1} \qquad \text{and} \qquad Q=\frac{M_{14}^2}{M_7},$$

which implies

$$M_1=\frac{M_2^2}{P} \qquad \text{and} \qquad M_7=\frac{M_{14}^2}{Q}. \quad (3.26)$$

From Entry 51 of Chapter 25 [7, p.204], we have

$$(M_1M_2)^2+\frac{9}{(M_1M_2)^2}=\left(\frac{M_2}{M_1}\right)^6+\left(\frac{M_1}{M_2}\right)^6. \quad (3.27)$$

Using (3.26) in (3.27), we deduce that

$$M_2^{12}=\frac{P^8(P^4-9)}{P^4-1}. \quad (3.28)$$

On changing q to q^7 in (3.28), we have

$$M_{14}^{12}=\frac{Q^8(Q^4-9)}{Q^4-1}.$$

Thus from the above and (3.28)

$$\left(\frac{M_2}{M_{14}}\right)^{12}=\frac{P^8(P^4-9)(Q^4-1)}{Q^8(P^4-1)(Q^4-9)}. \quad (3.29)$$

From Theorem 3.1(*ii*) of [9], we have

$$LM+\frac{1}{LM}=\left(\frac{L}{M}\right)^3+\left(\frac{M}{L}\right)^3+4\left(\frac{L}{M}+\frac{M}{L}\right), \quad (3.30)$$

where

$$L=\frac{M_1}{M_7} \qquad \text{and} \qquad M=\frac{M_2}{M_{14}}.$$

On using (3.26) in L, we obtain

$$L=\left(\frac{M_2}{M_{14}}\right)^2\frac{Q}{P} \qquad \text{and} \qquad M=\frac{M_2}{M_{14}}.$$

Employing this in (3.30) and on dividing throughout by $(PQM_2/M_{14})^3$, we have

$$P^6 - 3\left(\frac{M_2}{M_{14}}\right)^6 P^2Q^4 - 3P^4Q^2 - \left(\frac{M_2}{M_{14}}\right)^6 Q^6 = 0. \tag{3.31}$$

Finally, on eliminating M_2/M_{14} between (3.29) and (3.31) and on dividing throughout by $(PQ)^2$, we have the result. □

§4. Evaluations of Ramanujan's Cubic Continued Fraction

Lemma 4.1 *For $q = e^{-\pi\sqrt{n/3}}$, let*

$$A_n := \frac{1}{\sqrt[4]{3}}\frac{\psi(-q)}{\psi(-q^3)}.$$

Then

$$(i)\quad A_nA_{1/n} = 1, \tag{4.1}$$

$$(ii)\quad A_1 = 1, \tag{4.2}$$

$$(iii)\ H(q) = \frac{1}{\sqrt[3]{3A_n^4+1}}. \tag{4.3}$$

For a proof see [10].

Lemma 4.2

$$3A_n^2A_{9n}^2 + \frac{3}{A_n^2A_{9n}^2} = 3 + 6\frac{A_{9n}^2}{A_n^2} + \frac{A_{9n}^4}{A_n^4}.$$

For a proof, see [10].

Lemma 4.3

$$3(A_nA_{25n})^2 + \frac{3}{(A_nA_{25n})^2} = \left(\frac{A_{25n}}{A_n}\right)^3 - \left(\frac{A_n}{A_{25n}}\right)^3$$
$$+5\left(\frac{A_{25n}}{A_n}\right)^2 + 5\left(\frac{A_n}{A_{25n}}\right)^2 + 5\left(\frac{A_{25n}}{A_n}\right) - 5\left(\frac{A_n}{A_{25n}}\right),$$

For a proof, see [10].

Theorem 4.1 *If A_n is as defined as in Lemma 4.1, then*

$$\left(\sqrt{\frac{A_{4n}^2}{A_nA_{16n}}} + \sqrt{\frac{A_nA_{16n}}{A_{4n}^2}}\right)\left(\sqrt{\frac{A_{16n}}{A_n}} + \sqrt{\frac{A_n}{A_{16n}}}\right) = 8. \tag{4.4}$$

Proof For proof of (4.4), we use Theorem 3.1 with $R(q) = A_{4n}/A_n$ and $S = A_{16n}/A_{4n}$. □

Theorem 4.2 *We have*

$$A_4 = 2 + \sqrt{3}$$

and

$$A_{1/4} = 2 - \sqrt{3}.$$

Proof Put $n = 1/4$ in (4.4) and using (4.1) we obtain the result. □

Corollary 4.1 *We have*

$$H(e^{-\pi\sqrt{4/3}}) = \frac{1}{148}(292 + 168\sqrt{3})^{2/3}(73 - 42\sqrt{3})$$

and

$$H(e^{-\pi\sqrt{1/12}}) = \frac{1}{148}(292 - 168\sqrt{3})^{2/3}(73 + 42\sqrt{3}).$$

Proof On using Theorem 4.2 in (4.3), we have result. □

Theorem 4.3 *If A_n is as defined as in Lemma 4.1, then*

$$\left(\frac{A_{4n}A_{9n}}{A_nA_{36n}}\right)^4 + \left(\frac{A_nA_{36n}}{A_{4n}A_{9n}}\right)^4 + \left(\frac{A_{4n}A_{9n}}{A_nA_{36n}}\right)^2 + \left(\frac{A_nA_{36n}}{A_{4n}A_{9n}}\right)^2 - \left(\frac{A_{9n}A_{36n}}{A_nA_{4n}} - 3\frac{A_nA_{4n}}{A_{9n}A_{36n}}\right)$$
$$\times \left\{\left(\frac{A_{4n}A_{9n}}{A_nA_{36n}}\right)^3 + \left(\frac{A_nA_{36n}}{A_{4n}A_{9n}}\right)^3\right\} - 3\left(\frac{A_{9n}A_{36n}}{A_nA_{4n}} - 3\frac{A_nA_{4n}}{A_{9n}A_{36n}}\right)\left(\frac{A_{4n}A_{9n}}{A_nA_{36n}} + \frac{A_nA_{36n}}{A_{4n}A_{9n}}\right)$$
$$- \left\{\left(\frac{A_{9n}A_{36n}}{A_nA_{4n}}\right)^2 + 9\left(\frac{A_nA_{4n}}{A_{9n}A_{36n}}\right)^2\right\} - 6 = 0. \tag{4.5}$$

Proof The proof is similar to Theorem 4.1 by applying Theorem 3.2. □

Theorem 4.4 *We have*

$$A_6 = \sqrt[4]{6\sqrt{2} - 3\sqrt{3} + 3\sqrt{6} - 6} = A_{1/6}^{-1}$$

and

$$A_{2/3} = \frac{1}{\sqrt{3}}\sqrt[4]{6\sqrt{2} + 3\sqrt{3} + 3\sqrt{6} + 6} = A_{3/2}^{-1}.$$

Proof Setting $n = 1/6$ in (4.5) and upon using (4.1), we find that

$$\left(\frac{A_6}{A_{2/3}}\right)^4 + 9\left(\frac{A_{2/3}}{A_6}\right)^4 + 8\left\{\left(\frac{A_6}{A_{2/3}}\right)^2 - 3\left(\frac{A_{2/3}}{A_6}\right)^2\right\} + 2 = 0.$$

Since A_n is real and increasing in n, we have $A_6/A_{2/3} > 1$. Hence

$$\frac{A_6}{A_{2/3}} = \sqrt{3\sqrt{2} - 3}. \tag{4.6}$$

Again on setting $n = 2/3$ in Lemma 4.2, we have

$$3(A_{2/3}A_6)^2 + \frac{3}{(A_{2/3}A_6)^2} = 3 + 6\frac{A_6^2}{A_{2/3}^2} + \frac{A_6^4}{A_{2/3}^4}.$$

On using (4.6) in this, we obtain

$$A_{2/3}A_6 = \sqrt{2+\sqrt{3}}. \tag{4.7}$$

Finally, on employing (4.6), (4.7) and (4.1) we have the result. □

Corollary 4.2 *We have*

$$H(e^{-\pi\sqrt{2}}) = \frac{1}{(18\sqrt{2}-9\sqrt{3}+9\sqrt{6}-17)^{1/3}}$$

and

$$H(e^{-\pi\sqrt{2/9}}) = \frac{1}{(18\sqrt{2}+9\sqrt{3}+9\sqrt{6}+19)^{1/3}}.$$

Proof On using Theorem 4.4 in (4.3), we have the result. □

Theorem 4.5 *If A_n is as defined as in Lemma 4.1, then*

$$\left(\frac{A_{4n}A_{25n}}{A_nA_{100n}}\right)^2 + \left(\frac{A_nA_{100n}}{A_{4n}A_{25n}}\right)^2 + \left(\frac{A_{4n}A_{25n}}{A_nA_{100n}} + \frac{A_nA_{100n}}{A_{4n}A_{25n}}\right) - \left(\sqrt{\frac{A_{25n}A_{100n}}{A_nA_{4n}}} + \sqrt{\frac{A_nA_{4n}}{A_{25n}A_{100n}}}\right)$$
$$\left(\sqrt{\frac{A_{4n}A_{25n}}{A_nA_{100n}}} + \sqrt{\frac{A_nA_{100n}}{A_{4n}A_{25n}}} + \left(\frac{A_{4n}A_{25n}}{A_nA_{100n}}\right)^{3/2} + \left(\frac{A_nA_{100n}}{A_{4n}A_{25n}}\right)^{3/2}\right) = \frac{A_{25n}A_{100n}}{A_nA_{4n}} + \frac{A_nA_{4n}}{A_{25n}A_{100n}}. \tag{4.8}$$

Proof The proof is similar to Theorem 4.1 by using Theorem 3.3. □

Theorem 4.6 *We have*

$$A_{10} = \sqrt{\frac{2+\sqrt{10}+\sqrt{4\sqrt{10}+10}}{2}}\sqrt[4]{\frac{a-\sqrt{a^2-36}}{6}} = A^{-1}_{1/10}$$

and

$$A_{2/5} = \sqrt{\frac{2+\sqrt{10}-\sqrt{4\sqrt{10}+10}}{2}}\sqrt[4]{\frac{a-\sqrt{a^2-36}}{6}} = A^{-1}_{5/2},$$

where $a = (18+4\sqrt{10})(\sqrt{4\sqrt{10}+10}) + 60 + 20\sqrt{10}$.

Proof Setting $n = 1/10$ in (4.8) and upon using (4.1), we find that

$$x^2 + \frac{1}{x^2} - 4\left(x + \frac{1}{x}\right) - 4 = 0,$$

where $x = A_{10}/A_{2/5}$. Since A_n is real and increasing in n, we have $A_{10}/A_{2/5} > 1$. Hence we choose

$$x + \frac{1}{x} = 2 + \sqrt{10}.$$

On solving

$$\frac{A_{10}}{A_{2/5}} = \frac{1}{2}\left(2+\sqrt{10}+\sqrt{4\sqrt{10}+10}\right). \tag{4.9}$$

Put $n = 2/5$ in Lemma 4.3, we have

$$3(A_{2/5}A_{10})^2 + \frac{3}{(A_{2/5}A_{10})^2} = \left(\frac{A_{10}}{A_{2/5}}\right)^3 - \left(\frac{A_{2/5}}{A_{10}}\right)^3$$

$$+5\left\{\left(\frac{A_{10}}{A_{2/5}}\right)^2 + \left(\frac{A_{2/5}}{A_{10}}\right)^2\right\} + 5\left(\frac{A_{10}}{A_{2/5}} - \frac{A_{2/5}}{A_{10}}\right).$$

On employing (4.9) in this, we obtain

$$A_{2/5}A_{10} = \sqrt{\frac{a - \sqrt{a^2 - 36}}{6}}, \tag{4.10}$$

where $a = (18 + 4\sqrt{10})(\sqrt{4\sqrt{10} + 10}) + 60 + 20\sqrt{10}$. On using (4.9) and (4.10) we have the result. □

Theorem 4.7 *If A_n is as defined as in Lemma 4.1, then*

$$\left(2 + 2(A_nA_{49n})^4\right)\left[\left(\frac{A_n}{A_{49n}}\right)^2 - \left(\frac{A_{49n}}{A_n}\right)^2\right] + 3(A_nA_{49n})^4 + 3$$

$$= 5(A_nA_{49n})^2\left[\left(\frac{A_n}{A_{49n}}\right)^2 + \left(\frac{A_{49n}}{A_n}\right)^2\right]. \tag{4.11}$$

Proof The proof is similar to Theorem 4.1 by applying Theorem 3.5. □

Theorem 4.8 *If A_n is as defined as in Lemma 4.1, then*

$$y_8 - (4 + 6x_1)y_7 + (24 + 24x_1 + 9x_2)y_6 - (148 + 12x_1 + 36x_2)y_5 + (145 + 252x_1)y_4$$

$$-(648+678x_1-36x_2+54x_3)y_3+(2180+360x_1+441x_2-324x_3)y_2-(1016+2016x_1-396x_2-54x_3)y_1$$

$$+81x_4 - 324x_3 + 1548x_2 + 1236x_1 + 5250 = 0, \tag{4.12}$$

where

$$x_m = (3A_nA_{4n}A_{49n}A_{196n})^m + \frac{1}{(3A_nA_{4n}A_{49n}A_{196n})^m}, \qquad m = 1, 2, 3$$

and

$$y_m = \left(\frac{A_{49n}A_{196n}}{A_nA_{4n}}\right)^m + \left(\frac{A_nA_{4n}}{A_{49n}A_{196n}}\right)^m, \qquad m = 1, 2, \cdots, 8$$

Proof The proof is similar to Theorem 4.1 by applying Theorem 3.4. □

Theorem 4.9 *We have*

$$A_{14} = \frac{1}{\sqrt[4]{34}}\{(a + \sqrt{a^2 - 14})(9 + 10\sqrt{2})\}^{1/4} = A_{1/14}^{-1}$$

and

$$A_{2/7} = \left(\frac{2}{17}\frac{9+10\sqrt{2}}{a+\sqrt{a^2-4}}\right)^{1/4} = A_{7/2}^{-1},$$

where

$$a = \frac{1}{3}(197+18\sqrt{113})^{1/3} + \frac{13}{3(197+18\sqrt{113})^{1/3}} + \frac{2}{3}.$$

Proof On setting $n = 1/14$ in (4.12) and upon using (4.1), we find that

$$\begin{aligned}\left(t^8+\frac{1}{t^8}\right) &- 16\left(t^7+\frac{1}{t^7}\right)+90\left(t^6+\frac{1}{t^6}\right)-244\left(t^5+\frac{1}{t^5}\right)+649\left(t^4+\frac{1}{t^4}\right)\\ &- 2040\left(t^3+\frac{1}{t^3}\right)+3134\left(t^2+\frac{1}{t^2}\right)-4148\left(t+\frac{1}{t}\right)+10332=0,\end{aligned}$$

where $t = (A_{2/7}A_{14})^2$. On setting $t + \dfrac{1}{t} = x$, we obtain

$$x^8 - 16x^7 + 82x^6 - 132x^5 + 129x^4 - 1044x^3 + 1332x^2 + 864x + 5184 = 0.$$

On solving this, we obtain

$$x = 6, \quad \frac{1}{3}(197+18\sqrt{113})^{1/3} + \frac{13}{3(197+18\sqrt{113})^{1/3}} + \frac{2}{3}$$

are the double roots and the remaining roots are imaginary. Since A_n is increasing in n, and solving for $(A_{14}/A_{2/7})^2$, it is easy to see that

$$\left(\frac{A_{14}}{A_{2/7}}\right)^2 = \frac{a+\sqrt{a^2-4}}{2},$$

where a is as defined earlier. On setting $n = 2/7$ in (4.11) and on using the above, we have the result. □

Acknowledgement

The authors are thankful to Dr.K.R.Vasuki, Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore for his valuable suggestions to improve the quality of the paper.

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