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Simplicial Branched Coverings of the 3-Sphere

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Abstract: In this article we give, for each $d > 1$, a simplicial branched covering map $\lambda_d : S^3_{3(d+1)} \to S^3_6$ of degree d. And by using the simplicial map λ_2 we demonstrate a well known topological fact that the space obtained by identifying diagonally opposite points of the 3-sphere is homeomorphic to the 3-sphere.

Key Words: Branched Covering, simplicial map, triangulation of map.

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§1. Introduction

In articles [5] and [6] we have given *simplicial branched coverings* of the Real Projective Plane and the 2-Sphere respectively. The present article is in continuation of these articles. Here we give, for each $d > 1$, a simplicial branched covering map, $\lambda_d : S^3_{3(d+1)} \to S^3_6$, from a 3(d+1) vertex triangulation of the 3-sphere onto a 6 vertex triangulation of the 3-sphere. For $d = 2$, we show that the simplicial branched covering map $\lambda_2 : S^3_9 \to S^3_6$ is a minimal triangulation of the well known two fold branched covering map $S^3 \to S^3/(x, y) \sim (y, x)$. Moreover the simplicial map λ_2 verifies a familiar topological fact that after identifying diagonally symmetric points of the 3-sphere we get a homeomorphic copy of the 3-sphere. Branched coverings of the low dimensional manifolds have been discussed extensively (e.g. see [1], [3] and [4]) but the explicit constructions, which we are giving here are missing. The purpose here is to give some concrete examples, which are not at all trivial but explain some important topological facts.

§2. Preliminary Notes

Definition 2.1 *An abstract simplicial complex K on a finite set V is a collection of subsets of V, which is closed under inclusion i.e. if $s \in K$ and $s' \subset s$ then $s' \in K$. The elements of K are called simplices and in particular a set $\gamma \in K$ of cardinality n+1 is called an n-simplex; 0-simplices are called vertices, 1-simplices are called edges and so on.*

A geometric n-simplex is the convex hull of $n + 1$ affinely independent points of $\mathbb{R}^N$(see [2]). A geometric simplicial complex is a collection of geometric simplices such that all faces of

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these simplices are also in the collection and intersection of any two of these simplices is either empty or a common face of both of these simplices. It is easy to see that corresponding to each geometric simplicial complex there is an abstract simplicial complex. Converse is also true i. e. corresponding to any abstract simplicial complex K there is a topological space $|K| \subset R^N$, made up of geometric simplices, called its geometric realization (see [2], [7], [8]). If K is an abstract simplicial complex and M is a subspace of $\mathbb{R}^N$ such that there is a homeomorphism $h : |K| \to M$ then we say $(|K|, h)$ is a triangulation of M or K triangulates the topological space M.

Definition 2.2 *A map $f : K \to L$, between two abstract simplicial complexes K and L, is called a simplicial map if image, $f(\sigma) = \{f(v_0), f(v_1), ..., f(v_k)\}$, of any simplex $\sigma = \{v_0, v_1, ..., v_k\}$ of K is a simplex of L. Further if $|K|$ and $|L|$ are geometric realizations of K and L respectively then there is a piecewise-linear continuous map $|f| : |K| \to |L|$ defined as follows. As each point x of $|K|$ is an interior point of exactly one simplex (say $\sigma = \{v_0, .., v_k\}$) of $|K|$, so for each $x \in \sigma$ we have $x = \sum_{i=1}^{k} \lambda_i v_i$ where $\lambda_i \geq 0$, $\sum \lambda_i = 1$. Therefore we may define $|f|(x) = \sum_{i=1}^{k} \lambda_i f(v_i)$.*

Definition 2.3 *A simplicial branched covering map between two triangulated n-manifolds K and L is defined by a dimension preserving piecewise linear map $p : |K| \to |L|$, which is an ordinary covering over the complement of some specific co-dimension 2 sub-complex L' of L (for more detailed definition see [1], [3], [4], [5]). The sub-complex L' is called branch set of the branched covering map and a point $x \in p^{-1}(L')$ is called a singular point if p fails to be a local homeomorphism at x.*

§3. Main Results

3.1 Simplicial branched covering map $\lambda_d : S^3_{3(d+1)} \to S^3_6$

We first define a simplicial branched covering map $\lambda_2 : S^3_9 \to S^3_6$ of degree 2 and then show that the same method gives, for each $d > 2$, a simplicial branched covering map $\lambda_d : S^3_{3(d+1)} \to S^3_6$ of degree d.

Since the join of two 1-spheres is a 3-sphere, so in order to get the desired 9 vertex 3-sphere S^3_9, we take join of a three vertex 1-sphere $S^1_3 = \{A_0, E_0, F_0, A_0E_0, E_0F_0, F_0A_0\}$ with the six vertex 1-sphere $S^1_6 = \{B_0, C_0, D_0, B_1, C_1, D_1, C_0B_0, B_0D_0, D_0C_1, C_1B_1, B_1D_1, D_1C_0\}$. The 3-sipmlices of $S^3_9 = S^1_3 * S^1_6$ are shown in Figure 1.

We define a map on the vertex set of S^3_9, as $A_0 \to A, E_0 \to E, F_0 \to F, X_i \to X$ for each $X \in \{B, C, D\}, i \in \{0, 1\}$ and extend it linearly on the 3-simplices of S^3_9. The image of this map is a simplicial complex whose 3-simplices are ABCE, ACDE, ABDE, EDBF, EBCF, EDCF, CDAF, DBAF and CBAF. This simplicial complex triangulates the 3-sphere because its geometric realization is homeomorphic to the 3-sphere as it is a disjoint union of two 3-balls having a common boundary S^2 (see figure 2 below). We denote this simplicial complex by S^3_6 and the map just defined is the simplicial map $\lambda_2 : S^3_9 \to S^3_6$. Notice that the map $\lambda_2 : S^3_9 \to S^3_6$ is a 2-fold simplicial branched covering map because pre-image of each 3-simplex of S^3_6 consists of exactly two 3-simplices of S^3_9; each is being mapped, under the map λ_2, with the same orientation. Branching set and the singular set of the map are $AE + EF + FA$ and $A_0E_0 + E_0F_0 + F_0A_0$ respectively.

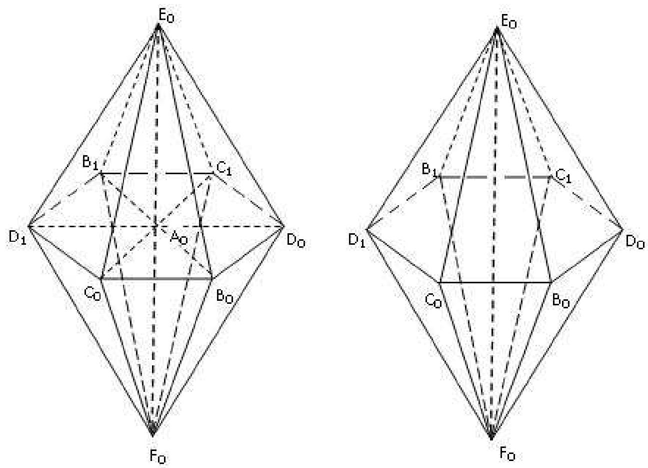

Figure 1

In order to get a simplicial branched covering map, $\lambda_d : S^3_{3(d+1)} \to S^3_6$, of degree d (for each $d > 2$) we consider the join of a 3 vertex 1-sphere with the 3d vertex 1-sphere. i.e. $S^3_{3(d+1)} = S^1_3 * S^1_{3d} = \{A_0, E_0, F_0, A_0E_0, E_0F_0, F_0A_0\} * \{B_i, C_i, D_i, C_iB_i, B_iD_i, D_iC_{i+1} : i \in Z_d\}$.

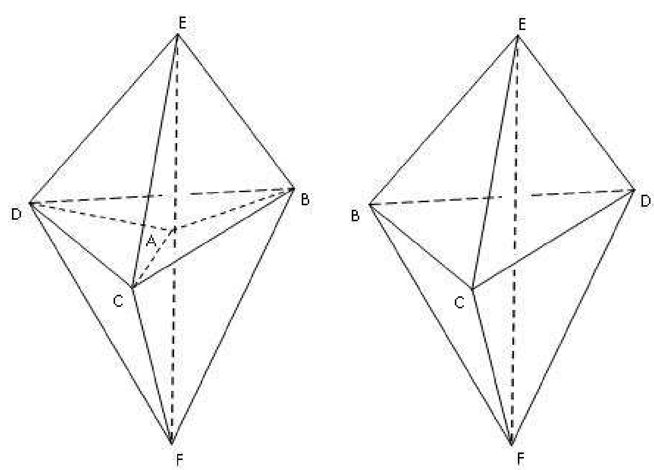

Figure 2

The 3-simplices of $S^3_{3(d+1)}$ are $\{A_0E_0C_iB_i, A_0E_0B_iD_i, A_0E_0D_iC_{i+1}, E_0F_0C_iB_i, E_0F_0B_iD_i,$ $E_0F_0D_iC_{i+1}, F_0A_0C_iB_i, F_0A_0B_iD_i, F_0A_0D_iC_{i+1} : i \in Z_d$, addition in the subscripts is mod d$\}$. Notice that a map, defined on the vertices of the simplicial complex $S^3_{3(d+1)}$, as $A_0 \to A, E_0 \to$

$E, F_0 \to F, X_i \to X$ for each $X \in \{B, C, D\}$ and for $i \in \{0, 1, , d-1\}$ is a simplicial branched covering map $\lambda_d : S^3_{3(d+1)} \to S^3_6$ of degree d.

Remark 3.1.1 We shall now show that the simplicial map $\lambda_2 : S^3_9 \to S^3_6$ triangulates the 2-fold branched covering map $S^3 \to S^3/(x, y) \sim (y, x)$ but before that we prove the following theorem.

Theorem 3.1.1 *The simplicial map $\lambda_2 : S^3_9 \to S^3_6$ is a minimal triangulation of the 2-fold branched covering map q: $S^3 \to S^3/(x, y) \sim (y, x)$.*

Proof Notice that branching of the map q occurs along the diagonal circle of the quotient space and pre-image of the branching circle is the diagonal circle of the domain of the map q. Let $\lambda_2 : S^3_{\alpha_0} \to S^3_{\beta_0}$ be a minimal triangulation of the map q, so the branching circle and the singular circle are at least triangles. Since the polygonal link of any singular 1-simplex of $S^3_{\alpha_0}$, is to be mapped with degree 2 by the map λ_2 so the link will have at least 6-vertices. The image of this link will be a circle with at least 3 vertices, which are different from the vertices of the branching circle. This implies that the domain 3-sphere of the map λ_2 will have at least 9 vertices and its image will have at least 6 vertices i.e. $\alpha_0 \geq 9$ and $\beta_0 \geq 6$. □

Note 3.1.1 Following description of the simplicial complex S^3_9 enables us to show that the simplicial map $\lambda_2 : S^3_9 \to S^3_6$ triangulates the 2-fold branched covering map q: $S^3 \to S^3/(x, y) \sim (y, x)$. It also leads to a combinatorial proof of the fact that after identification of diagonally symmetric points of the 3-sphere we get the 3-sphere again.

3.2 Diagonally Symmetric Triangulation of the 3-Sphere

The diagonal of the standard 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ is the subspace $\Delta = \{(z_1, z_2) \in S^3 : z_1 = z_2\}$. A triangulation of S^3 will be called diagonally symmetric if whenever there is a vertex at a point (z_1, z_2) there is a vertex at the point (z_2, z_1) and whenever there is a 3-simplex on the vertices $(z_{i_1}, z_{i_2}), (z_{i_3}, z_{i_4}), (z_{i_5}, z_{i_6}), (z_{i_7}, z_{i_8})$, there is a 3-simplex on the vertices $(z_{i_2}, z_{i_1}), (z_{i_4}, z_{i_3}), z_{i_6}, z_{i_5}), (z_{i_8}, z_{i_7})$. We show that the simplicial complex S^3_9 obtained above is a diagonally symmetric triangulation of the 3-sphere and the simplicial branched covering map $\lambda_2 : S^3_9 \to S^3_6$ is equivalent to the map q: $S^3 \to S^3/(x, y) \sim (y, x)$. In order to show this we consider the following description of the 3-sphere:

$$S^3 = T_1 \bigcup T_2,$$

where $T_1 = \{(z_1, z_2) \in S^3 : |z_1| \leq |z_2|\}$, $T_2 = \{(z_1, z_2) \in S^3 : |z_1| \geq |z_2|\}$ and

$$T = T_1 \bigcap T_2 = \{(z_1, z_2) \in S^3 : |z_1| = |z_2| = 1/\sqrt{2}\} \cong S^1 \times S^1.$$

A map $\theta : S^3 \to S^3$ defined as $(z_1, z_2) \to (z_2, z_1)$ swaps the interiors of the solid tori T_1 and T_2 homeomorphically. We triangulate T_1 and T_2 in such a way that the homeomorphism θ induces a simplicial isomorphism between the triangulations of T_1 and T_2. The triangulations of T, T_1 and T_2 are described as follows.

In Figure 3 below we give a triangulated 2-torus T, which is the common boundary of both of the solid tori T_1 and T_2. The vertices X_0, X_1 for each $X \in \{B, C, D\}$ are symmetric about the diagonal Δ and the vertices A_0, E_0, F_0 triangulate the diagonal.

Since there are precisely two ways to fold a square to get a torus, viz (i) first identify vertical boundaries and then identify horizontal boundaries of the square (ii) first identify horizontal boundaries and then vertical boundaries of the square, so we use this fact to obtain the solid tori T_1 and T_2.

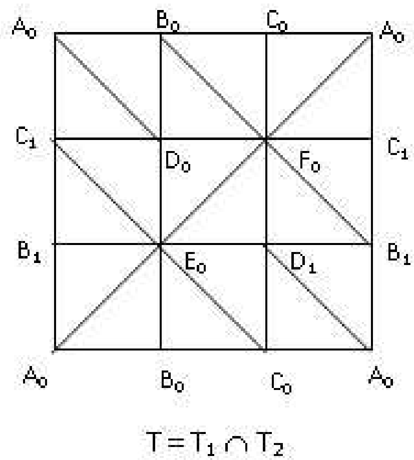

Figure 3

The solid torus T_1 has been obtained by first identifying the vertical edges, of the square of Figure 3, and then top and bottom edges (see Figure 4 below). Its three, of the total nine, 3-simplices are $A_0B_0C_0E_0, A_0C_0E_0D_1$ and $A_0D_1E_0B_1$ and remaining six 3-simplices can be obtained from an automorphism defined by $A_0 \to E_0 \to F_0 \to A_0, B_0 \to D_1 \to C_1 \to B_0$ and $C_0 \to B_1 \to D_0 \to C_0$.

The solid torus T_2 has been obtained by first identifying the horizontal edges, of the square of Figure 3, and then the other two sides as shown in Figure 4. The nine 3-simplices of T_2 are $\{A_0D_0E_0B_0, E_0A_0C_1D_0, A_0B_1C_1E_0, E_0C_0F_0D_1, F_0E_0B_0C_0, E_0D_0B_0F_0, F_0B_1A_0C_1, A_0F_0D_1B_1, F_0C_0D_1A_0\}$. These simplices can also be obtained from the 3-simplices of T_1 by using the permutation $\rho = (B_0B_1)(C_0C_1)(D_0D_1)$, which is equivalent to the Z_2-action defined by the map $\theta : (x, y) \to (y, x)$ on S^3.

The nine 3-simplices of T_1 together with the nine 3-simplices of T_2 constitute a diagonally symmetric triangulation of the 3-sphere with 9 vertices. And since the list of 3-simplices of $S^1_3 * S^1_6$ is same as that of the 3-simplices of the simplicial complex obtained now, so the two simplicial complexes are isomorphic.

Notice that the identification of diagonally symmetric vertices / simplices of the 3-sphere (obtained now) is equivalent to the identifications provided by the simplicial map λ_2. This equivalence implies that the identification of diagonally symmetric points of the 3-sphere gives a 3-sphere.

Remark 3.2.1 In Figure 3 (triangulation of T) if we replace the edges A_0D_0 and A_0D_1 by the edges B_0C_1 and B_1C_0 respectively then we get another triangulation of T, which is also symmetric about the diagonal. But this triangulation under the diagonal action does not give a simplicial branched covering map.

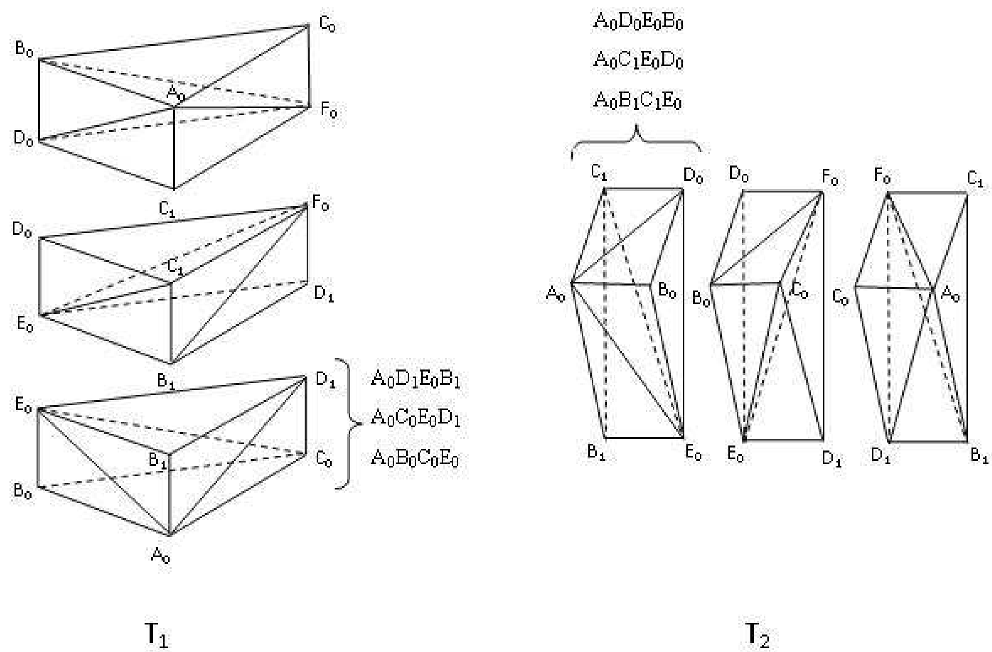

Figure 4

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