International J.Math. Combin. Vol.3(2012), 46-53

On the Osculating Spheres of a Real Quaternionic Curve In the Euclidean Space E^4

Derya Sağlam

Derya Sağlam Afyonkarahisar Kocatepe University

Faculty of Art and Sciences, Department of Mathematics, Afyonkarahisar, Turkey

E-mail: dryilmaz@aku.edu.tr

Abstract: In the Euclidean space E^4, there is a unique quaternionic sphere for a real quaternionic curve $\alpha : I \subset \mathbb{R} \to Q_H$ such that it touches α at the fourth order at $\alpha(0)$. In this paper, we studied some characterizations of the osculating sphere of the real quaternionic curves in the four dimensional Euclidean space.

Key Words: Euclidean space, quaternion algebra, osculating spheres.

AMS(2010): 14Q05, 53A17, 53C50

§1. Introduction

The quaternions introduced by Hamilton in 1843 are the number system in four dimensional vector space and an extension of the complex number. There are different types of quaternions, namely: real, complex dual quaternions. A real quaternion is defined as $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is composed of four units $\{1, e_1, e_2, e_3\}$ where e_1, e_2, e_3 are orthogonal unit spatial vectors, q_i $(i = 0, 1, 2, 3)$ are real numbers and this quaternion can be written as a linear combination of a real part (scalar) and vectorial part (a spatial vector) [1,5,8].

The space of quaternions Q are isomorphic to E^4, four dimensional vector space over the real numbers. Then, Clifford generalized the quaternions to bi-quaternions in 1873 [11]. Hence they play an important role in the representation of physical quantities up to four dimensional space. Also they are used in both theoretical and applied mathematics. They are important number systems which use in Newtonian mechanics, quantum physics, robot kinematics, orbital mechanics and three dimensional rotations such as in the three dimensional computer graphics and vision. Real quaternions provide us with a simple and elegant representation for describing finite rotation in space. On the other hand, dual quaternions offer us a better way to express both rotational and translational transformations in a robot kinematic [5].

In 1985, the Serret-Frenet formulas for a quaternionic curve in Euclidean spaces E^3 and E^4 are given by Bharathi and Nagaraj [9]. By using of these formulas Karadağ and Sivridağ gave some characterizations for quaternionic inclined curves in the terms of the harmonic curvatures in Euclidean spaces E^3 and E^4 [10]. Gök et al. defined the real spatial quaternionic b-slant

[1]Received June 30, 2012. Accepted September 4, 2012.

helix and the quaternionic B_2-slant helix in Euclidean spaces E^3 and E^4 respectively and they gave new characterization for them in the terms of the harmonic curvatures [7].

In the Euclidean space E^3, there is a unique sphere for a curve $\alpha : I \subset \mathbb{R} \to E^3$ such that the sphere contacts α at the third order at $\alpha(0)$. The intersection of the sphere with the osculating plane is a circle which contacts α at the second order at $\alpha(0)$ [2,3,6]. In [4], the osculating sphere and the osculating circle of the curve are studied for each of timelike, spacelike and null curves in semi- Euclidean spaces; E_1^3, E_1^4 and E_2^4.

In this paper, we define osculating sphere for a real quaternionic curve $\alpha : I \subset \mathbb{R} \to E^4$ such that it contacts α at the fourth order at $\alpha(0)$. Also some characterizations of the osculating sphere are given in Euclidean space E^4.

§2. Preliminaries

We give basic concepts about the real quaternions. Let Q_H denote a four dimensional vector space over a field H whose characteristic grater than 2. Let e_i $(1 \leq i \leq 4)$ denote a basis for the vector space. Let the rule of multiplication on Q_H be defined on e_i $(1 \leq i \leq 4)$ and extended to the whole of the vector space by distributivity as follows:

A real quaternion is defined by $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + de_4$ where a, b, c, d are ordinary numbers such that

$$\begin{aligned} &\overrightarrow{e_1} \times \overrightarrow{e_2} = \overrightarrow{e_3} = -\overrightarrow{e_2} \times \overrightarrow{e_1}, \\ &\overrightarrow{e_2} \times \overrightarrow{e_3} = \overrightarrow{e_1} = -\overrightarrow{e_3} \times \overrightarrow{e_2}, \\ &\overrightarrow{e_3} \times \overrightarrow{e_1} = \overrightarrow{e_2} = -\overrightarrow{e_1} \times \overrightarrow{e_3}, \\ &\overrightarrow{e_1}^2 = \overrightarrow{e_2}^2 = \overrightarrow{e_3}^2 = -1, \quad e_4^2 = 1. \end{aligned}$$

We can write a real quaternion as a linear combination of scalar part $S_q = d$ and vectorial part $V_q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$. Using these basic products we can now expand the product of two quaternions as

$$p \times q = S_p S_q - \left\langle \overrightarrow{V_p}, \overrightarrow{V_q} \right\rangle + S_p \overrightarrow{V_q} + S_q \overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q} \quad \text{for every} \quad p, q \in Q_H,$$

where $\langle , \rangle$ and $\wedge$ are inner product and cross product on E^3, respectively. There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol γ and defined as follows:

$$\gamma q = -a\overrightarrow{e_1} - b\overrightarrow{e_2} - c\overrightarrow{e_3} + d$$

for every $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + de_4 \in Q_H$ which is called the *Hamiltonian conjugation.* This defines the symmetric, real valued, non-degenerate, bilinear form h as follows:

$$h(p, q) = \frac{1}{2}(p \times \gamma q + q \times \gamma p) \quad \text{for every } p, q \in Q_H.$$

Now we can give the definition of the norm for every quaternion. the norm of any q real quaternion is denoted by

$$\|q\|^2 = h(q, q) = q \times \gamma q = a^2 + b^2 + c^2 + d^2$$

in [5,8].

The four-dimensional Euclidean space E^4 is identified with the space of unit quaternions. A real quaternionic sphere with origin m and radius $R>0$ in E^4 is

$$S^3(m,R)=\{p\in Q_H : h(p-m,p-m)=R^2\}.$$

The Serret-Frenet formulas for real quaternionic curves in E^4 are as follows:

Theorem 2.1([10]) *The four-dimensional Euclidean space E^4 is identified with the space of unit quaternions. Let $I=[0,1]$ denotes the unit interval in the real line $\mathbb{R}$ and $\overrightarrow{e_4}=1$. Let*

$$\begin{aligned}&\alpha : I\subset\mathbb{R}\to Q_H\\ &s\to\alpha(s)=\sum_{i=1}^{4}\alpha_i(s)\overrightarrow{e_i},\end{aligned}$$

be a smooth curve in E^4 with nonzero curvatures $\{K,k,r-K\}$ and the Frenet frame of the curve α is $\{T,N,B_1,B_2\}$. Then Frenet formulas are given by

$$\begin{bmatrix}T'\\ N'\\ B_1'\\ B_2'\end{bmatrix}=\begin{bmatrix}0 & K & 0 & 0\\ -K & 0 & k & 0\\ 0 & -k & 0 & (r-K)\\ 0 & 0 & -(r-K) & 0\end{bmatrix}\begin{bmatrix}T\\ N\\ B_1\\ B_2\end{bmatrix} \tag{2.1}$$

where K is the principal curvature, k is torsion and $(r-K)$ is bitorsion of α.

§3. Osculating Sphere of a Real Quaternionic Curve in E^4

We assume that the real quaternionic curve $\alpha : I\subset\mathbb{R}\to Q_H$ is arc-length parametrized, i.e, $\|\alpha'(s)\|=1$. Then the tangent vector $T(s)=\alpha'(s)=\sum_{i=1}^{4}\alpha_i'(s)\overrightarrow{e_i}$ has unit length. Let (y_1,y_2,y_3,y_4) be a rectangular coordinate system of $\mathbb{R}^4$. We take a real quaternionic sphere $h(y-d,y-d)=R^2$ with origin d and radius R, where $y=(y_1,y_2,y_3,y_4)$. Let $f(s)=h(\alpha(s)-d,\alpha(s)-d)-R^2$. If we have the following equations

$$f(0)=0,\quad f'(0)=0,\quad f''(0)=0,\quad f'''(0)=0,\quad f^{(4)}(0)=0$$

then we say that the sphere contacts at fourth order to the curve α at $\alpha(0)$. The sphere is called osculating sphere.

Theorem 3.1 *Let $\alpha : I\subset\mathbb{R}\to Q_H$ be a real quaternionic curve with nonzero curvatures $K(0),k(0)$ and $(r-K)(0)$ at $\alpha(0)$. Then there exists a sphere which contacts at the fourth order to the curve α at $\alpha(0)$ and the equation of the osculating sphere according to the Frenet frame $\{T_0,N_0,B_{1_0},B_{2_0}\}$ is*

$$x_1^2+(x_2-\rho_0)^2+(x_3-\rho_0'\sigma_0)^2+(x_4-\omega_0((\rho_0'\sigma_0)'+\frac{\rho_0}{\sigma_0}))^2=\rho_0^2+(\rho_0'\sigma_0)^2+\omega_0^2((\rho_0'\sigma_0)'+\frac{\rho_0}{\sigma_0})^2, \tag{3.1}$$

where

$$\rho_0 = \frac{1}{K(0)},\ \sigma_0 = \frac{1}{k(0)},\ \omega_0 = \frac{1}{r(0) - K(0)}.$$

Proof If $f(0) = 0$ then $h(\alpha(0) - d, \alpha(0) - d) = R^2$. Since we have

$$f' = 2h(\alpha', \alpha - d) \text{ and } f'(0) = 0$$

then

$$h(T_0, \alpha(0) - d) = 0. \tag{3.2}$$

Similarly we have

$$f'' = 2[h(\alpha'', \alpha - d) + h(\alpha', \alpha')] \text{ and } f''(0) = 0$$

implies $h(K(0)N_0, \alpha(0) - d) + h(T_0, T_0) = 0$. Since $h(T_0, T_0) = 1$, then

$$h(N_0, \alpha(0) - d) = -\frac{1}{K(0)} = -\rho_0. \tag{3.3}$$

Considering

$$f''' = 2[h(\alpha''', \alpha - d) + 3h(\alpha'', \alpha')] \text{ and } f'''(0) = 0$$

we get

$$h(-K^2(0)T_0 + K'(0)N_0 + K(0)k(0)B_{1_0}, \alpha(0) - d) = 0.$$

From the equations (3.2) and (3.3) we obtain

$$h(B_{1_0}, \alpha(0) - d) = \frac{K'(0)}{K^2(0)k(0)} = -\rho_0'\sigma_0. \tag{3.4}$$

Since

$$f^{(4)} = 2[h(\alpha^{(4)}, \alpha - d) + 4h(\alpha''', \alpha') + 3h(\alpha'', \alpha'')] \text{ and } f^{(4)}(0) = 0,$$

from the equations (2.1), (3.1)-(3.4), we obtain

$$h(B_{2_0}, \alpha(0) - d) = -\frac{1}{r(0) - K(0)}\left[(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0})\right] = -\omega_0\left[(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0})\right]. \tag{3.5}$$

Now we investigate the numbers u_1, u_2 , u_3 and u_4 such that

$$\alpha(0) - d = u_1T_0 + u_2N_0 + u_3B_{1_0} + u_4B_{2_0}.$$

From $h(T_0, \alpha(0) - d) = u_1$ and the equation (3.2), then we find $u_1 = 0$. From $h(N_0, \alpha(0) - d) = u_2$ and the equation (3.3), then we find $u_2 = -\rho_0$. From $h(B_{1_0}, \alpha(0) - d) = u_3$ and the equation (3.4), then we obtain $u_3 = -\rho_0'\sigma_0$. From the equation (3.5), we obtain $u_4 = -\omega_0\left[(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0})\right]$. Also the origin of the sphere that contacts at the fourth order to the curve at the point $\alpha(0)$ is

$$d = \alpha(0) - u_1T_0 - u_2N_0 - u_3B_{1_0} - u_4B_{2_0} \tag{3.6}$$

Given a real quaternionic variable P on the osculating sphere, suppose

$$P = \alpha(0) + x_1T_0 + x_2N_0 + x_3B_{1_0} + x_4B_{2_0}$$

and from the equation (3.6)

$$P-d=x_1T_0+(x_2-\rho_0)N_0+(x_3-\rho_0'\sigma_0)B_{1_0}+(x_4-\omega_0\left[(\rho_0'\sigma_0)'+\frac{\rho_0}{\sigma_0})\right])B_{2_0}.$$

Also

$$h(P-d,P-d)=x_1^2+(x_2-\rho_0)^2+(x_3-\rho_0'\sigma_0)^2+(x_4-\omega_0((\rho_0'\sigma_0)'+\frac{\rho_0}{\sigma_0}))^2$$

and using (3.6), we obtain

$$R^2=h(\alpha(0)-d,\alpha(0)-d)=\rho_0^2+(\rho_0'\sigma_0)^2+\omega_0^2((\rho_0'\sigma_0)'+\frac{\rho_0}{\sigma_0})^2. \qquad \square$$

Definition 3.2 *Let $\alpha: I\subset\mathbb{R}\to Q_H$ be a real quaternionic curve with nonzero curvatures K, k and $r-K$. The functions $m_i: I\to\mathbb{R}$, $1\leq i\leq 4$ such that*

$$\begin{cases} m_1=0, \\ m_2=\dfrac{1}{K}, \\ m_3=\dfrac{m_2'}{k}, \\ m_4=\dfrac{m_3'+km_2}{r-K} \end{cases} \tag{3.7}$$

is called m_i curvature function.

Corollary 3.3 *Let $\alpha: I\subset\mathbb{R}\to Q_H$ be a real quaternionic curve with nonzero curvatures K, k, $r-K$ and the Frenet frame $\{T,N,B_1,B_2\}$. If $d(s)$ is the center of the osculating sphere at $\alpha(s)$, then*

$$d=\alpha(s)+m_2(s)N(s)+m_3(s)B_1(s)+m_4(s)B_2(s). \tag{3.8}$$

Moreover the radius of the osculating sphere at $\alpha(s)$ is

$$R=\sqrt{m_2^2(s)+m_3^2(s)+m_4^2(s)}. \tag{3.9}$$

Let $\alpha: I\subset\mathbb{R}\to Q_H$ be a real quaternionic curve. If $\alpha(I)\subset S^3(m,R)$, then α is called spherical curve. We obtain new characterization for spherical curve α.

Theorem 3.4 *Let $\alpha: I\subset\mathbb{R}\to Q_H$ be a real quaternionic curve and $\alpha(I)\subset S^3(0,R)$. Then*

$$h(\alpha(s),V_j(s))=-m_j(s), \quad 1\leq j\leq 4,$$

where $V_1=T$, $V_2=N$, $V_3=B_1$ and $V_4=B_2$.

Proof Since $\alpha(s)\in S^3(0,R)$ for all $s\in I$, then $h(\alpha(s),\alpha(s))=R^2$. Derivating of this equation with respect to s four times and from the equation (3.7), we get

$$h(V_1(s),\alpha(s))=h(T(s),\alpha(s))=0,$$

$$h(V_2(s),\alpha(s))=h(N(s),\alpha(s))=-\frac{1}{K(s)}=-m_2(s),$$

$$h(V_3(s),\alpha(s)) = h(B_1(s),\alpha(s)) = -\left(\frac{1}{K(s)}\right)'\frac{1}{k(s)} = -\frac{m_2'(s)}{k(s)} = -m_3(s)$$

and

$$\begin{aligned} h(V_4(s),\alpha(s)) &= h(B_2(s),\alpha(s)) \\ &= -\left[\left(\left(\frac{1}{K(s)}\right)'\frac{1}{k(s)}\right)' + \frac{k(s)}{K(s)}\right]\frac{1}{r(s)-K(s)} \\ &= -\frac{m_3'(s)+k(s)m_2(s)}{r(s)-K(s)} \\ &= -m_4(s). \end{aligned}$$ □

Theorem 3.5 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve. If $\alpha(I) \subset S^3(0,R)$, then the osculating sphere at $\alpha(s)$ for each $s \in I$ is $S^3(0,R)$.*

Proof We assume $\alpha(I) \subset S^3(0,R)$. From the equation (3.8), the center of the osculating sphere at $\alpha(s)$ is

$$\begin{aligned} d &= \alpha(s) + m_2(s)N(s) + m_3(s)B_1(s) + m_4(s)B_2(s) \\ &= \alpha(s) + m_2(s)V_2(s) + m_3(s)V_3(s) + m_4(s)V_4(s). \end{aligned}$$

According to Theorem 3.4

$$d = \alpha(s) - \sum_{j=2}^{4} h(\alpha(s),V_j(s))V_j(s). \tag{3.10}$$

On the other hand

$$\alpha(s) = \sum_{j=1}^{4} h(\alpha(s),V_j(s))V_j(s)$$

and since $h(\alpha(s),V_1(s)) = 0$, we have

$$\alpha(s) = \sum_{j=2}^{4} h(\alpha(s),V_j(s))V_j(s). \tag{3.11}$$

From the equations (3.10) and (3.11), we get $d = 0$. In addition we have

$$h(\alpha(s),d) = R.$$ □

In general, above theorem is valid for the sphere $S^3(b,R)$ with the center b. As well as $S^3(0,R)$ isometric to $S^3(b,R)$, the truth can be avowable. Now, we give relationship between center and radius of the osculating sphere following.

Theorem 3.6 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve with nonzero curvatures K, k, $r-K$ and m_4. The radii of the osculating spheres at $\alpha(s)$ for all $s \in I$ is constant iff the centers of the osculating spheres at $\alpha(s)$ are fixed.*

Proof We assume that the radius of the osculating sphere at $\alpha(s)$ for all $s \in I$ is constant. From the equation (3.9)

$$R(s)^2 = m_2^2(s) + m_3^2(s) + m_4^2(s).$$

Derivating of the equation with respect to s, we obtain

$$m_2(s)m_2'(s) + m_3(s)m_3'(s) + m_4(s)m_4'(s) = 0.$$

Since $m_3(s) = \dfrac{m_2'(s)}{k(s)}$ and $m_4(s) = \dfrac{m_3'(s) + k(s)m_2(s)}{r(s) - K(s)}$, then

$$(r(s) - K(s))m_3(s) + m_4'(s) = 0. \tag{3.12}$$

On the other hand derivating of the equation (3.8) with respect to s and from the equations (3.7), (3.12), we get

$$d'(s) = 0.$$

Thus the center $d(s)$ of the osculating sphere at $\alpha(s)$ is fixed.

Conversely, let the center $d(s)$ of the osculating sphere at $\alpha(s)$ for all $s \in I$ be fixed. Since

$$h(d(s) - \alpha(s), d(s) - \alpha(s)) = R^2(s),$$

derivating of the equation with respect to s, we obtain

$$h(T(s), \alpha(s) - d(s)) = R'(s)R(s).$$

Left hand side this equation is zero. Hence $R'(s) = 0$ and than the radius of the osculating sphere at $\alpha(s)$ for all $s \in I$ is constant. □

Theorem 3.7 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve. The curve is spherical iff the centers of the osculating spheres at $\alpha(s)$ are fixed.*

Proof We assume $\alpha(I) \subset S^3(b, R)$. According to Theorem 3.6 the proof is clearly. Conversely, according to Theorem 3.5 if the centers $d(s)$ of the osculating spheres at $\alpha(s)$ for all $s \in I$ are fixed point b, then the radii of the osculating spheres is constant R. Thus $h(\alpha(s), b) = R$ and than α is spherical. □

Now we give a characterization for spherical curve α in terms of its curvatures K, k and $r - K$ in following theorem.

Theorem 3.8 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve with nonzero curvatures K, k, $r - K$ and m_4. The curve α is spherical iff*

$$\frac{r-K}{k}\left(\frac{1}{K}\right)' + \left\{\left[\left(\left(\frac{1}{K}\right)'\frac{1}{k}\right)' + \frac{k}{K}\right]\frac{1}{r-K}\right\}' = 0. \tag{3.13}$$

Proof Let the curve α be spherical. According to Theorem 3.7 the centers $d(s)$ of the osculating spheres at $\alpha(s)$ for all $s \in I$ are fixed. From the equations (3.7) and (3.12) we obtain (3.13).

Conversely we assume

$$\frac{r-K}{k}\left(\frac{1}{K}\right)' + \left\{\left[\left(\left(\frac{1}{K}\right)'\frac{1}{k}\right)' + \frac{k}{K}\right]\frac{1}{r-K}\right\}' = 0$$

From the equation (3.7), we get

$$(r-K)m_3 + m_4' = 0.$$

Derivating equation (3.8) with respect to s and from the last equation and (3.7), we obtain $d'(s) = 0$. Hence $d(s)$ is fixed point. According to Theorem 3.7 the curve α is spherical. □

References

[1] A.C.Çöken A. Tuna, On the quaternionic inclined curves in the semi-Euclidean space E_2^4, *Applied Mathematics and Computation*, 155 (2004), 373-389.

[2] A.Sabuncuoğlu, *Diferensiyel Geometri,* Nobel Press, 2010.

[3] D.J.Struik, *Lectures on Classical Differential geometry,* New-York, 1951.

[4] E.Soytürk, K.İlarslan, D.Sağlam, Osculating spheres and osculating circles of a curve in semi-Riemannian space, *Commun. Fac. Sci. Univ. Ank. Series*, A1, 54 (2)(2005), 39-48.

[5] F.Kahraman, İ. Gök, H. H. Hacısalihoğlu, On the quaternionic B_2 slant helices in the semi-Euclidean space E_2^4. *Applied Mathematics and Computation*, 218(2012), 6391-6400.

[6] H.H.Hacısalihoğlu, *Diferensiyel Geometri,* Faculty of Sciences, University of Ankara Press, 1993.

[7] İ. Gök, O. Z. Okuyucu, F. Kahraman, H. H. Hacısalihoğlu, On the quaternionic B_2 slant helices in the Euclidean space E^4, *Adv Appl. Clifford Algebras*, 21(2011), 707-719.

[8] J.P.Ward, *Quaternions and Cayley Numbers,* Kluwer Academic Publishers, Boston-London, 1997.

[9] K.Bharathi, M.Nagaraj, Quaternion valued function of a real variable Serret-Frenet formulae, *Indian J. Pure Appl. Math.*, 16(1985), 741-756.

[10] M. Karadağ, A. İ. Sivridağ, Characterisations for the quaternionic inclined curves, *Erc. Üniv. Fen Bil. Derg.*, 13, 1-2(1997), 37-53.

[11] W.K.Clifford, Preliminary sketch of biquaternions, *Proc. London Math. Soc.*, 4(1873), 361-395.