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On the Forcing Hull and Forcing Monophonic Hull Numbers of Graphs

J.John

(Department of Mathematics , Government College of Engineering, Tirunelveli - 627 007, India)

V.Mary Gleeta

(Department of Mathematics ,Cape Institute of Technology, Levengipuram- 627114, India)

E-mail: johnramesh1971@yahoo.co.in, gleetass@gmail.com

Abstract: For a connected graph $G = (V, E)$, let a set M be a minimum monophonic hull set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum monophonic hull set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic hull number of M , denoted by $f_{mh}(M)$, is the cardinality of a minimum forcing subset of M. The forcing monophonic hull number of G, denoted by $f_{mh}(G)$, is $f_{mh}(G) = min\{f_{mh}(M)\}$, where the minimum is taken over all minimum monophonic hull sets in G. Some general properties satisfied by this concept are studied. Every monophonic set of G is also a monophonic hull set of G and so $mh(G) \leq h(G)$, where $h(G)$ and $mh(G)$ are hull number and monophonic hull number of a connected graph G. However, there is no relationship between $f_h(G)$ and $f_{mh}(G)$, where $f_h(G)$ is the forcing hull number of a connected graph G. We give a series of realization results for various possibilities of these four parameters.

Key Words: hull number, monophonic hull number, forcing hull number, forcing monophonic hull number, Smarandachely geodetic k-set, Smarandachely hull k-set.

AMS(2010): 05C12, 05C05

§1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1,9]. A convexity on a finite set V is a family C of subsets of V, convex sets which is closed under intersection and which contains both V and the empty set. The pair(V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph $G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected subgraph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if

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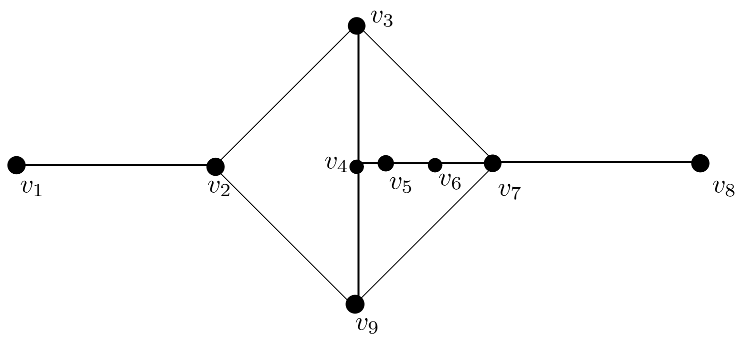
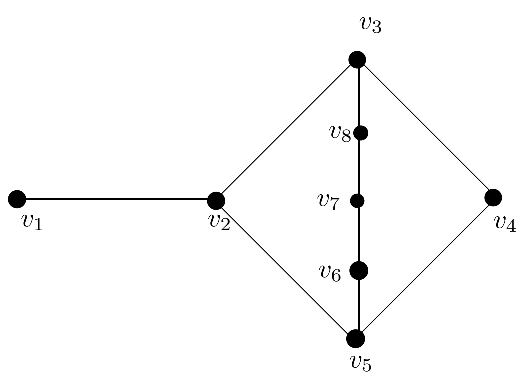
every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W[2,8]. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u,v)$ is called an $u-v$ geodesic. A vertex x is said to lie on a $u-v$ geodesic P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $I[u,v]$ denotes the set of all vertices which lie on $u-v$ geodesic. For a set S of vertices, let $I[S] = \bigcup_{(u,v)\in S} I[u,v]$. The set S is convex if $I[S] = S$. Clearly if $S = \{v\}$or $S = V$, then S is convex. The convexity number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex set containing S is denoted by $I_h(S)$ and called the convex hull of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h[S] \subseteq V$. For an integer $k \geq 0$, a subset $S \subseteq V$ is called a *Smarandachely geodetic k-set* if $I[S \bigcup S^+] = V$ and a *Smarandachely hull k-set* if $I_h(S \bigcup S^+) = V$ for a subset $S^+ \subset V$ with $|S^+| \leq k$. Particularly, if $k = 0$, such Smarandachely geodetic 0-set and Smarandachely hull 0-set are called the *geodetic set* and *hull set*, respectively. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a minimum geodetic set or simply a g- set of G. Similarly, the hull number $h(G)$ of G is the minimum order of its hull sets and any hull set of order $h(G)$ is a minimum hull set or simply a h- set of G. The geodetic number of a graph is studied in [1,4,10] and the hull number of a graph is studied in [1,6].A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum hull set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of M. The forcing hull number of S, denoted by $f_h(S)$, is the cardinality of a minimum forcing subset of S. The forcing hull number of G, denoted by $f_h(G)$,is $f_h(G) = min\,\{f_h(S)\}$, where the minimum is taken over all minimum hull sets S in G. The forcing hull number of a graph is studied in[3,14]. A chord of a path $u_o, u_1, u_2, ..., u_n$ is an edge u_iu_j with $j \geq i + 2 (0 \leq i, j \leq n)$. A $u-v$ path P is called a monophonic path if it is a chordless path. A vertex x is said to lie on a $u-v$ monophonic path P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $J[u,v]$ denotes the set of all vertices which lie on $u-v$ monophonic path. For a set M of vertices, let $J[M] = \cup_{u,v\in M} J[u,v]$. The set M is monophonic convex or m-convex if $J[M] = M$. Clearly if $M = \{v\}$ or $M = V$, then M is m-convex. The m-convexity number, denoted by $C_m(G)$, is the cardinality of a maximum proper m-convex subset of V. The smallest m-convex set containing M is denoted by $J_h(M)$ and called the monophonic convex hull or m-convex hull of M. Since the intersection of two m-convex set is m-convex, the m-convex hull is well defined. Note that $M \subseteq J[M] \subseteq J_h(M) \subseteq V$. A subset $M \subseteq V$ is called a monophonic set if $J[M] = V$ and a m-hull set if $J_h(M) = V$. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a m- set of G. Similarly, the monophonic hull number $mh(G)$ of G is the minimum order of its m-hull sets and any m-hull set of order $mh(G)$ is a minimum monophonic set or simply a mh- set of G. The monophonic number of a graph is studied in [5,7,11,15] and the monophonic hull number of a graph is studied in [12]. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete.Let G be a connected graph and M a minimum monophonic hull set of G. A subset $T \subseteq M$ is called a forcing subset for M

if M is the unique minimum monophonic hull set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic hull number of M, denoted by $f_{mh}(M)$, is the cardinality of a minimum forcing subset of M. The forcing monophonic hull number of G, denoted by $f_{mh}(G)$, is $f_{mh}(G) = min\{f_{mh}(M)\}$, where the minimum is taken over all minimum monophonic hull sets M in G.For the graph G given in Figure 1.1, $M = \{v_1, v_8\}$ is the unique minimum monophonic hull set of G so that $mh(G) = 2$ and $f_{mh}(G) = 0$. Also $S_1 = \{v_1, v_5, v_8\}$ and $S_2 = \{v_1, v_6, v_8\}$ are the only two h-sets of G such that $f_h(S_1) = 1, f_h(S_2) = 1$ so that $f_h(G) = 1$. For the graph G given in Figure 1.2, $M_1 = \{v_1, v_4\}, M_2 = \{v_1, v_6\}, M_3 = \{v_1, v_7\}$ and $M_4 = \{v_1, v_8\}$ are the only four mh-sets of G such that $f_{mh}(M_1) = 1, f_{mh}(M_2) = 1, f_{mh}(M_3) = 1$ and $f_{mh}(M_4) = 1$ so that $f_{mh}(G) = 1$. Also, $S = \{v_1, v_7\}$ is the unique minimum hull set of G so that $h(G) = 2$ and $f_h(G) = 0$. Throughout the following G denotes a connected graph with at least two vertices.



G

Figure 1.1



G

Figure 1.2

The following theorems are used in the sequel

Theorem 1.1 ([6]) *Let G be a connected graph. Then*

a) Each extreme vertex of G belongs to every hull set of G;

(b) $h(G) = p$ *if and only if* $G = K_p$.

Theorem 1.2 ([3]) *Let* G *be a connected graph. Then*

(a) $f_h(G) = 0$ *if and only if* G *has a unique minimum hull set;*

(b) $f_h(G) \leq h(G) - |W|$, *where* W *is the set of all hull vertices of* G.

Theorem 1.3 ([13]) *Let* G *be a connected graph. Then*

(a) Each extreme vertex of G *belongs to every monophonic hull set of* G*;*

(b) $mh(G) = p$ *if and only if* $G = K_p$.

Theorem 1.4 ([12]) *Let* G *be a connected graph. Then*

(a) $f_{mh}(G) = 0$ *if and only if* G *has a unique mh-set;*

(b) $f_{mh}(G) \leq mh(G) - |S|$, *where* S *is the set of all monophonic hull vertices of* G.

Theorem 1.5 ([12]) *For any complete Graph* $G = K_p (p \geq 2), f_{mh}(G) = 0$.

§2. Special Graphs

In this section, we present some graphs from which various graphs arising in theorem are generated using identification.

Let $U_i : \alpha_i, \beta_i, \gamma_i, \delta_i, \alpha_i (1 \leq i \leq a)$ be a copy of cycle C_4. Let V_i be the graph obtained from U_i by adding three new vertices η_i, f_i, g_i and the edges $\beta_i\eta_i, \eta_i f_i, f_i g_i, g_i\delta_i, \eta_i\gamma_i, f_i\gamma_i, g_i\gamma_i (1 \leq i \leq a)$. The graph T_a given in Figure 2.1 is obtained from V_i's by identifying γ_{i-1} of V_{i-1} and α_i of $V_i (2 \leq i \leq a)$.

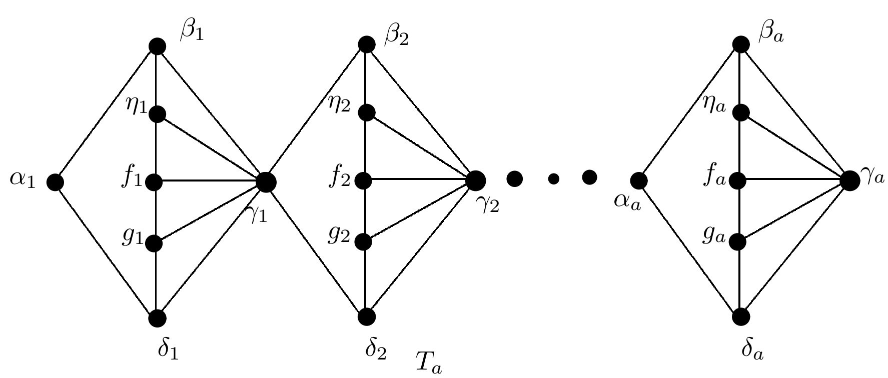


Figure 2.1

Let $P_i : k_i, l_i, m_i, n_i, k_i (1 \leq i \leq b)$ be a copy of cycle C_4. Let Q_i be the graph obtained from P_i by adding three new vertices h_i, p_i and q_i and the edges $l_i h_i, h_i p_i, p_i q_i$, and $q_i m_i (1 \leq i \leq b)$. The graph W_b given in Figure 2.2 is obtained from Q_i's by identifying m_{i-1} of Q_{i-1} and k_i of $Q_i (2 \leq i \leq b)$.

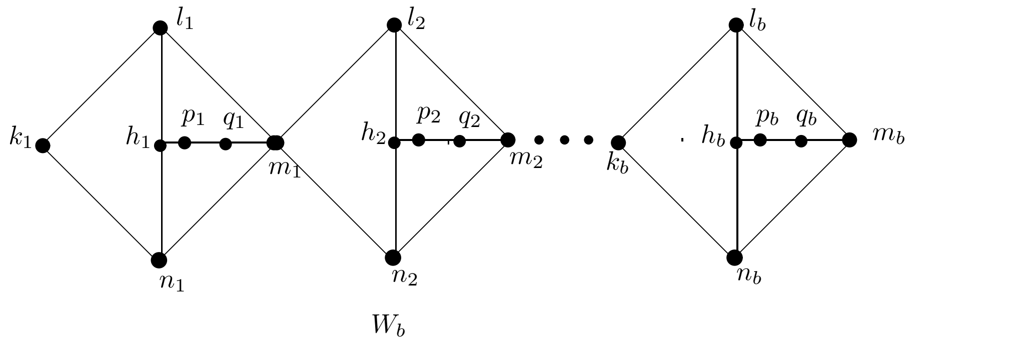


Figure 2.2

The graph Z_b given in Figure 2.3 is obtained from W_b by joining the edge $l_i n_i (1 \leq i \leq b)$.

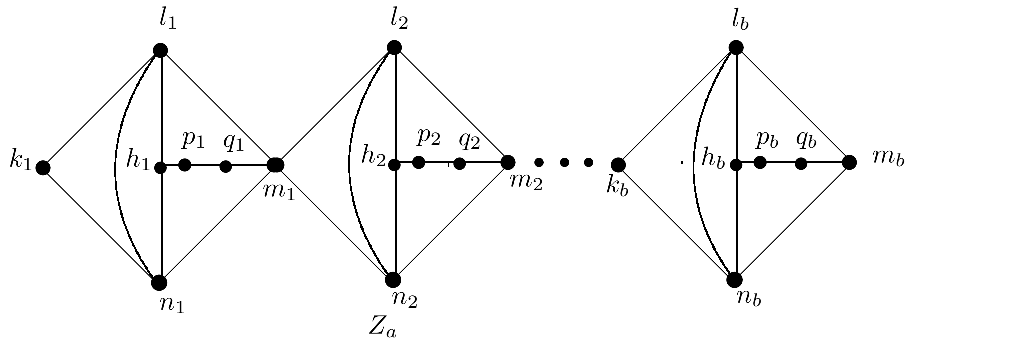


Figure 2.3

Let $F_i : s_i, t_i, x_i, w_i, s_i (1 \leq i \leq c)$ be a copy of cycle C_4. Let R_i be the graph obtained from F_i by adding two new vertices u_i, v_i and joining the edges $t_i u_i, u_i w_i, t_i w_i, u_i v_i$ and $v_i x_i (1 \leq i \leq c)$. The graph H_c given in Figure 2.4 is obtained from R_i's by identifying the vertices x_{i-1} of R_{i-1} and s_i of $R_i (1 \leq i \leq c)$.

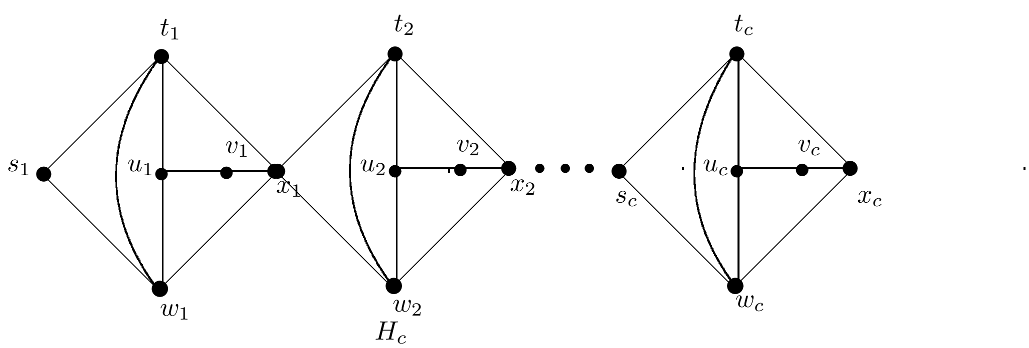


Figure 2.4

Every monophonic set of G is also a monophonic hull set of G and so $mh(G) \leq h(G)$, where $h(G)$ and $mh(G)$ are hull number and monophonic hull number of a connected graph G. However, there is no relationship between $f_h(G)$ and $f_{mh}(G)$, where $f_h(G)$ is the forcing hull number of a connected graph G. We give a series of realization results for various possibilities of these four parameters.

§3. Some Realization Results

Theorem 3.1 *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $f_{mh}(G) = f_h(G) = 0$, $mh(G) = a$ and $h(G) = b$.*

Proof If $a = b$, let $G = K_a$. Then by Theorems1.3(b) and 1.1(b), $mh(G) = h(G) = a$ and by Theorems 1.5 and 1.2(a), $f_{mh}(G) = f_h(G) = 0$. For $a < b$, let G be the graph obtained from T_{b-a} by adding new vertices $x, z_1, z_2, \cdots, z_{a-1}$ and joining the edges $x\alpha_1, \gamma_{b-a}z_1, \gamma_{b-a}z_2, \cdots,$ $\gamma_{b-a}z_{a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{a-1}\}$ be the set of end-vertices of G. Then it is clear that Z is a monophonic hull set of G and so by Theorem 1.3(a), Z is the unique mh-set of G so that $mh(G) = a$ and hence by Theorem 1.4(a), $f_{mh}(G) = 0$. Since $I_h(Z) \neq V, Z$ is not a hull set of G. Now it is easily seen that $W = Z \cup \{f_1, f_2, \cdots, f_{b-a}\}$ is the unique h-set of G and hence by Theorem 1.1(a) and Theorem 1.2(a), $h(G) = b$ and $f_h(G) = 0$. □

Theorem 3.2 *For every integers a, b and c with $0 \leq a < b < c$ and $c > a + b$, there exists a connected graph G such that $f_{mh}(G) = 0, f_h(G) = a, mh(G) = b$ and $h(G) = c$.*

Proof We consider two cases.

Case 1. $a = 0$. Then the graph T_b constructed in Theorem 3.1 satisfies the requirements of the theorem.

Case 2. $a \geq 1$. Let G be the graph obtained from W_a and $T_{c-(a+b)}$ by identifying the vertex m_a of W_a and α_1 of $T_{c-(a+b)}$ and then adding new vertices $x, z_1, z_2, \cdots, z_{b-1}$ and joining the edges $xk_1, \gamma_{c-b-a}z_1, \gamma_{c-b-a}z_2, \cdots, \gamma_{c-b-a}z_{b-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{b-1}\}$. Since $J_h(Z) = V$, Z is a monophonic hull set G and so by Theorem 1.3(a), Z is the unique mh- set of G so that $mh(G) = b$ and hence by Theorem 1.4(a), $f_{mh}(G) = 0$. Next we show that $h(G) = c$. Let S be any hull set of G. Then by Theorem 1.1(a), $Z \subseteq S$. It is clear that Z is not a hull set of G. For $1 \leq i \leq a$, let $H_i = \{p_i, q_i\}$. We observe that every h-set of G must contain at least one vertex from each $H_i (1 \leq i \leq a)$ and each $f_i (1 \leq i \leq c-b-a)$ so that $h(G) \geq b+a+c-a-b = c$. Now, $M = Z \cup \{q_1, q_2, \cdots, q_a\} \cup \{f_1, f_2, \cdots, f_{c-b-a}\}$ is a hull set of G so that $h(G) \leq b+a+c-b-a = c$. Thus $h(G) = c$. Since every h-set contains $S_1 = Z \cup \{f_1, f_2, \cdots, f_{c-b-a}\}$, it follows from Theorem 1.2(b) that $f_h(G) = h(G) - |S_1| = c - (c - a) = a$. Now, since $h(G) = c$ and every h-set of G contains S_1, it is easily seen that every h-set S is of the form $S_1 \cup \{d_1, d_2, \cdots, d_a\}$,where $d_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \phi$, which shows that $f_h(G) = a$. □

Theorem 3.3 *For every integers a, b and c with $0 \leq a < b \leq c$ and $b > a + 1$, there exists a connected graph G such that $f_h(G) = 0, f_{mh}(G) = a, mh(G) = b$ and $h(G) = c$.*

Proof We consider two cases.

Case 1. $a = 0$. Then the graph G constructed in Theorem 3.1 satisfies the requirements of the theorem.

Case 2. $a \geq 1$.

Subcase 2a. $b = c$. Let G be the graph obtained from Z_a by adding new vertices $x, z_1, z_2, \cdots, z_{b-a-1}$ and joining the edges $xk_1, m_az_1, m_az_2, \cdots, m_az_{b-a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{b-a-1}\}$ be the set of end-vertices of G. Let S be any hull set of G. Then by Theorem 1.1(a), $Z \subseteq S$. It is clear that Z is not a hull set of G. For $1 \leq i \leq a$, let $H_i = \{h_i, p_i, q_i\}$. We observe that every h-set of G must contain only the vertex p_i from each H_i so that $h(G) \leq b - a + a = b$. Now $S = Z \cup \{p_1, p_2, p_3, \cdots, p_a\}$ is a hull set of G so that $h(G) \geq b - a + a = b$. Thus $h(G) = b$. Also it is easily seen that S is the unique h-set of G and so by Theorem 1.2(a), $f_h(G) = 0$.Next we show that $mh(G) = b$. Since $J_h(Z) \neq V, Z$ is not a monophonic hull set of G. We observe that every mh-set of G must contain at least one vertex from each H_i so that $mh(G) \geq b-a+a = b$. Now $M_1 = Z \cup \{q_1, q_2, q_3, \cdots, q_a\}$ is a monophonic hull set of G so that $mh(G) \leq b - a + a = b$. Thus $mh(G) = b$. Next we show that $f_{mh}(G) = a$. Since every mh-set contains Z, it follows from Theorem 1.4(b) that $f_{mh}(G) \leq mh(G) - |Z| = b - (b - a) = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set M is of the form $Z \cup \{d_1, d_2, d_3, \cdots, d_a\}$, where $d_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of M with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \phi$, which shows that $f_{mh}(G) = a$.

Subcase 2b. $b < c$. Let G be the graph obtained from Z_a and T_{c-b} by identifying the vertex m_a of Z_a and α_1 of T_{c-b} and then adding the new vertices $x, z_1, z_2, \cdots, z_{b-a-1}$ and joining the edges $x\alpha_1, \gamma_{c-b}z_1, \gamma_{c-b}z_2, \cdots, \gamma_{c-b}z_{b-a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{b-a-1}\}$ be the set of end vertices of G. Let S be any hull set of G. Then by Theorem 1.1(a), $Z \subseteq S$. Since $I_h(Z) \neq V, Z$ is not a hull set of G. For $1 \leq i \leq a$, let $H_i = \{h_i, p_i, q_i\}$. We observe that every h-set of G must contain only the vertex p_i from each H_i and each $f_i (1 \leq i \leq c - b)$ so that $h(G) \geq b - a + a + c - b = c$. Now $S = Z \cup \{p_1, p_2, p_3, \cdots, p_a\} \cup \{f_1, f_2, f_3, \cdots, f_{c-b}\}$is a hull set of G so that $h(G) \leq b - a + a + c - b = c$. Thus $h(G) = c$. Also it is easily seen that S is the unique h-set of G and so by Theorem 1.2(a), $f_h(G) = 0$. Since $J_h(Z) \neq V$, Z is not a monophonic hull set of G. We observe that every mh-set of G must contain at least one vertex from each $H_i (1 \leq i \leq a)$ so that $mh(G) \geq b - a + a = b$. Now, $M_1 = Z \cup \{h_1, h_2, h_3, \cdots, h_a\}$ is a monophonic hull set of G so that $mh(G) \leq b - a + a = b$. Thus $mh(G) = b$. Next we show that $f_{mh}(G) = a$. Since every mh-set contains Z, it follows from Theorem 1.4(b) that $f_{mh}(G) \leq mh(G) - |Z| = b - (b - a) = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set S is of the form $Z \cup \{d_1, d_2, d_3, \cdots, d_a\}$, where $d_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \phi$, which shows that $f_{mh}(G) = a$. □

Theorem 3.4 *For every integers a, b and c with $0 \leq a < b \leq c$ and $b > a + 1$, there exists a connected graph G such that $f_{mh}(G) = f_h(G) = a$, $mh(G) = b$ and $h(G) = c$.*

Proof We consider two cases.

Case 1. $a = 0$, then the graph G constructed in Theorem 3.1 satisfies the requirements of the theorem.

Case 2. $a \geq 1$.

Subcase 2a. $b = c$. Let G be the graph obtained from H_a by adding new vertices $x, z_1, z_2, \cdots, z_{b-a-1}$ and joining the edges $xs_1, x_az_1, x_az_2, \cdots, x_az_{b-a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{b-a-1}\}$ be the set of end-vertices of G. Let M be any monophonic hull set of G. Then by Theorem 1.3(a), $Z \subseteq M$. First we show that $mh(G) = b$. Since $J_h(Z) \neq V, Z$ is not a monophonic hull set of G. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$. We observe that every mh-set of G must contain at least one vertex from each $F_i(1 \leq i \leq a)$. Thus $mh(G) \geq b - a + a = b$. On the other hand since the set $M = Z \cup \{v_1, v_2, v_3, \cdots, v_a\}$ is a monophonic hull set of G, it follows that $mh(G) \leq |M| = b$. Hence $mh(G) = b$. Next we show that $f_{mh}(G) = a$. By Theorem 1.3(a), every monophonic hull set of G contains Z and so it follows from Theorem 1.4(b) that $f_{mh}(G) \leq mh(G) - |Z| = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set M is of the form $Z \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i(1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap F_j = \phi$, which shows that $f_{mh}(G) = a$. By similar way we can prove $h(G) = b$ and $f_h(G) = a$.

Subcase 2b. $b < c$. Let G be the graph obtained from H_a and T_{c-b} by identifying the vertex x_a of H_a and the vertex α_1 of T_{c-b} and then adding the new vertices $x, z_1, z_2, \cdots, z_{b-a-1}$ and joining the edges $xs_1, \gamma_{c-b}z_1, \gamma_{c-b}z_2, \cdots, \gamma_{c-b}z_{b-a-1}$. First we show that $mh(G) = b$. Since $J_h(Z) \neq V, Z$ is not a monophonic hull set of G. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$ .We observe that every mh-set of G must contain at least one vertex from each $F_i(1 \leq i \leq a)$. Thus $mh(G) \geq b - a + a = b$. On the other hand since the set $M = Z \cup \{v_1, v_2, v_3, \cdots, v_a\}$ is a monophonic hull set of G, it follows that $mh(G) \geq |M| = b$. Hence $mh(G) = b$. Next, we show that $f_{mh}(G) = a$. By Theorem 1.3(a), every monophonic hull set of G contains Z and so it follows from Theorem 1.4(b) that $f_{mh}(G) \leq mh(G) - |Z| = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set is of the form $M = Z \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i(1 \leq i \leq a)$. Let T be any proper subset of M with $|T| < a$. Then it is clear that there exists some j such that $T \cup F_j = \phi$, which shows that $f_{mh}(G) = a$. Next we show that $h(G) = c$. Since $I_h(Z) \neq V, Z$ is not a hull set of G. We observe that every h-set of G must contain at least one vertex from each $F_i(1 \leq i \leq a)$ and each $f_i(1 \leq i \leq c-b)$ so that $h(G) \geq b-a+a+c-b = c$. On the other hand, since the set $S_1 = Z \cup \{u_1, u_2, u_3, \cdots, u_a\} \cup \{f_1, f_2, f_3, \cdots, f_{c-b}\}$ is a hull set of G, so that $h(G) \leq |S_1| = c$. Hence $h(G) = c$. Next we show that $f_h(G) = a$. By Theorem 1.1(a), every hull set of G contains $S_2 = Z \cup \{f_1, f_2, f_3, \cdots, f_{c-b}\}$ and so it follows from Theorem 1.2(b) that $f_h(G) \leq h(G) - |S_2| = a$. Now, since $h(G) = c$ and every h-set of G contains S_2, it is easily seen that every h-set S is of the form $S = S_2 \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i(1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap F_j = \phi$, which shows that $f_h(G) = a$. □

Theorem 3.5 *For every integers a, b, c and d with $0 \leq a \leq b < c < d, c > a + 1, d > c - a + b$, there exists a connected graph G such that $f_{mh}(G) = a, f_h(G) = b, mh(G) = c$ and $h(G) = d$.*

Proof We consider four cases.

Case 1. $a = b = 0$. Then the graph G constructed in Theorem 3.1 satisfies the requirements of this theorem.

Case 2. $a = 0, b \geq 1$. Then the graph G constructed in Theorem 3.2 satisfies the requirements of this theorem.

Case 3. $1 \le a = b$. Then the graph G constructed in Theorem 3.4 satisfies the requirements of this theorem.

Case 4. $1 \le a < b$. Let G_1 be the graph obtained from H_a and W_{b-a} by identifying the vertex x_a of H_a and the vertex k_1 of W_{b-a}. Now let G be the graph obtained from G_1 and $T_{d-(c-a+b)}$ by identifying the vertex m_{b-a} of G_1 and the vertex α_1 of $T_{d-(c-a+b)}$ and adding new vertices $x, z_1, z_2, \cdots, z_{c-a-1}$ and joining the edges $xs_1, \gamma_{d-(c-a+b)}z_1, \gamma_{d-(c-a+b)}z_2, \cdots, \gamma_{d-(c-a+b)}z_{c-a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{c-a-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}\,(1 \le i \le a)$. It is clear that any mh-set S is of the form $S = Z \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i(1 \le i \le a)$. Then as in earlier theorems it can be seen that $f_{mh}(G) = a$ and $mh(G) = c$. Let $Q_i = \{p_i, q_i\}$. It is clear that any h-set W is of the form $W = Z \cup \left\{f_1, f_2, f_3, \cdots, f_{d-(c-a+b)}\right\} \cup \{c_1, c_2, c_3, \cdots, c_a\} \cup \{d_1, d_2, d_3, \cdots, d_{b-a}\}$, where $c_i \in F_i(1 \le i \le a)$ and $d_j \in Q_j(1 \le j \le b-a)$. Then as in earlier theorems it can be seen that $f_h(G) = b$ and $h(G) = d$. □

Theorem 3.6 *For every integers a, b, c and d with $a \le b < c \le d$ and $c > b+1$ there exists a connected graph G such that $f_h(G) = a, f_{mh}(G) = b, mh(G) = c$ and $h(G) = d$.*

Proof We consider four cases.

Case 1. $a = b = 0$. Then the graph G constructed in Theorem 3.1 satisfies the requirements of this theorem.

Case 2. $a = 0, b \ge 1$. Then the graph G constructed in Theorem 3.2 satisfies the requirements of this theorem.

Case 3. $1 \le a = b$. Then the graph G constructed in Theorem 3.4 satisfies the requirements of this theorem.

Case 4. $1 \le a < b$.

Subcase 4a. $c = d$. Let G be the graph obtained from H_a and Z_{b-a} by identifying the vertex x_a of H_a and the vertex k_1 of Z_{b-a} and then adding the new vertices $x, z_1, z_2, ..., z_{c-b-1}$ and joining the edges $xs_1, m_{b-a}z_1, m_{b-a}z_2, ..., m_{b-a}z_{c-b-1}$. First we show that $mh(G) = c$. Let $Z = \{x, z_1, z_2, ..., z_{c-b-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}\,(1 \le i \le a)$ and $H_i = \{h_i, p_i, q_i\}\,(1 \le i \le b-a)$. It is clear that any mh-set of G is of the form $S = Z \cup \{c_1, c_2, c_3, ..., c_a\} \cup \{d_1, d_2, d_3, ..., d_{b-a}\}$, where $c_i \in F_i(1 \le i \le a)$ and $d_j \in H_j(1 \le j \le b-a)$. Then as in earlier theorems it can be seen that $f_{mh}(G) = b$ and $mh(G) = c$. It is clear that any h-set W is of the form $W = Z \cup \{p_1, p_2, p_3, ..., p_{b-a}\} \cup \{c_1, c_2, c_3, ..., c_a\}$, where $c_i \in F_i(1 \le i \le a)$. Then as in earlier theorems it can be seen that $f_h(G) = a$ and $h(G) = c$.

Subcase 4b. $c < d$. Let G_1 be the graph obtained from H_a and Z_{b-a} by identifying the vertex x_a of H_a and the vertex k_1 of Z_{b-a}. Now let G be the graph obtained from G_1 and T_{d-c} by identifying the vertex m_{b-a} of G_1 and the vertex α_1 of T_{d-c} and then adding new vertices $x, z_1, z_2, \cdots, z_{c-b-1}$ and joining the edges $xs_1, \gamma_{d-c}z_1, \gamma_{d-c}z_2, \cdots, \gamma_{d-c}z_{c-b-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{c-b-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}\,(1 \le i \le a)$ and $H_i = \{h_i, p_i, q_i\}\,(1 \le i \le b-a)$. It is clear that any mh-set of G is of the form $S = Z \cup$

$\{c_1, c_2, c_3, \cdots, c_a\} \cup \{d_1, d_2, d_3, \cdots, d_{b-a}\}$, where $c_i \in F_i (1 \leq i \leq a)$ and $d_j \in H_j (1 \leq j \leq b-a)$. Then as in earlier theorems it can be seen that $f_{mh}(G) = b$ and $mh(G) = c$. It is clear that any h-set W is of the form $W = Z \cup \{p_1, p_2, p_3, \cdots, p_{b-a}\} \cup \{f_1, f_2, f_3, \cdots, f_{d-c}\} \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i (1 \leq i \leq a)$. Then as in earlier theorems it can be seen that $f_h(G) = a$ and $h(G) = d$. □

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