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Graphs and Cellular Foldings of 2-Manifolds

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Abstract: In this paper we considered the set of regular CW-complexes or simply complexes. We obtained the necessary and sufficient condition for the composition of cellular maps to be a cellular folding. Also the necessary and sufficient condition for the composition of a cellular folding with a cellular map to be a cellular folding is declared. Then we proved that the Cartesian product of two cellular maps is a cellular folding iff each map is a cellular folding. By using these results we proved some other results. Once again we generalized the first three results and in each case we obtained the folding graph of the new map in terms of the original ones.

Key Words: Graph, cellular folding, 2-manifold, Cartesian product.

AMS(2010): 57M10, 57M20

§1. Introduction

A *cellular folding* is a folding defined on regular CW-complexes first defined by E. El-Kholy and H. Al-Khursani [1], and various properties of this type of folding are also studied by them. By a cellular folding of regular CW-complexes, it is meant a cellular map $f : K \to L$ which maps i-cells of K to i- cells of L and such that $f|_{e^i}$ for each i-cells e is a homeomorphism onto its image.

The set of regular CW-complexes together with cellular foldings form a category denoted by $C(K, L)$. If $f \in C(K, L)$, then $x \in K$ is said to be a *singularity* of f iff f is not a local homeomorphism at x. The set of all singularities of f is denoted by $\sum f$. This set corresponds to the folds of map. It is noticed that for a cellular f, the set $\sum f$ of singularities of f is a proper subset of the union of cells of dimension$\leq n-1$. Thus, when we consider any $f \in C(K, L)$, where K and L are connected regular CW-complexes of dimension 2, the set $\sum f$ will consists of 0- cells, 1-cells, and each 0-cell (vertex) has an even valency [2]. Of course, $\sum f$ need not be connected. Thus in this case $\sum f$ has the structure of a locally finite graph Γ_f embedded in K, for which every vertex has an even valency. Note that if K is compact, then Γ_f is finite, also any

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compact connected 2-manifold without boundary (surface) K with a finite cell decomposition is a regular CW-complex, then the 0-and 1-cells of the decomposition K from a finite graph Γ_f without loops and f folds K along the edges or 1-cells of Γ_f. Let K and L be complexes of the same dimension n. A neat cellular folding $f: K \to L$ is a cellular folding such that $L^n - L^{n-1}$ consists of a single n-cell, Int$L$ that is f satisfies the following:

(i) f maps i-cells to i-cells;

(ii) for each $\overline{e}$ which contains n vertices, $\overline{f(e)}$ is mapped on the single n- cell, $\overline{\text{Int}L}$, [3].

The set of regular CW-complexes together with neat cellular foldings form a category which is denoted by $NC(K, L)$. This category is a subcategory of cellular foldings $C(K, L)$. From now we mean by a complex a regular CW-complex in this paper.

§2. Main Results

Theorem 2.1 *Let M, N and L be complexes of the same dimension 2 such that $L \subset N \subset M$. Let $f: M \to N$, $g: N \to L$ be cellular maps such that $f(M) = N$, $g(N) = L$. Then $g \circ f$ is a cellular folding iff f and g are cellular foldings. In this case, $\Gamma_{g\circ f} = \Gamma_f \bigcup f^{-1}(\Gamma_g)$.*

Proof Let M, N and L be complexes of the same dimension 2, let $f: M \to N$ be a cellular folding such that $\sum f \neq \emptyset$, i.e., $f(M) = N \neq M$. Then $\sum f$ form a graph Γ_f embedded in M. Let $g: N \to N$ be a cellular folding such that $g(N) = L \neq N$, $\sum g = \Gamma_g$ is embedded in N. Now, let $\sigma \in M^{(i)}$, $i = 0, 1, 2$ be an arbitrary i-cell in M such that $\overline{\sigma}$ has S distinct vertices then $(g \circ f)(\sigma) = g(f(\sigma)) = g(\sigma')$, where $\sigma' \in N^{(i)}$ such that $\overline{\sigma}$ has S distinct vertices since f is a cellular folding. Also $g(\sigma') \in L^{(i)}$ such that $\overline{g(\sigma')}$ has S distinct vertices since g is a cellular folding. Thus $g \circ f$ is a cellular folding. In this case $\sum g \circ f$ is $\sum f \bigcup f^{-1}(\sum g)$. In other words, $\Gamma_{f\circ g} = \Gamma_f \bigcup f^{-1}(\Gamma_g)$.

Conversely, suppose $f: M \to N$ and $g: N \to L$ are cellular maps such that $g \circ f: M \to L$ is a cellular folding. Now, let $\sigma \in M^{(i)}$ be an i-cell in M. Suppose $f(\sigma) = \sigma'$ is a j-cell in N, such that $j \neq i$. Then since f is a cellular map, then $j \leq i$. But $j \neq i$, thus $j < i$. Since $f(\sigma) = \sigma'$, then $(g \circ f)(\sigma) = g(f(\sigma)) = g(\sigma')$. But $g \circ f$ is a cellular folding, thus $(g \circ f)(\sigma)$ is an i- cell in L and so is $g(\sigma')$. Since σ' is a j- cell in N and g is a cellular map, then i must be less than j and this contradicts the assumption that $j < i$. Hence the only possibly is that $i = j$. Note that the above theorem is true if we consider f and g are neat cellular foldings instead of cellular folding. □

Example 2.2 Consider a complex on $M = S^2$ with cellular subdivision consists of six-vertices, twelve 1-cells and eight 2-cells. Let $f: M \to N$ be a cellular folding given by:

$$f(e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_6^0) = (e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_1^0),$$
$$f(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1, e_9^1, e_{10}^1, e_{11}^1, e_{12}^1) = (e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1, e_3^1, e_4^1, e_1^1, e_2^1),$$
$$f(e_1^2, e_2^2, e_3^2, e_4^2, e_5^2, e_6^2, e_7^2, e_8^2) = (e_1^2, e_2^2, e_3^2, e_4^2, e_1^2, e_2^2).$$

In this case $f(M) = N$ is a complex with five vertices, eight 1-cells and four 2-cells, see Fig.1(a). The folding graph Γ_f is shown Fig.1(b).

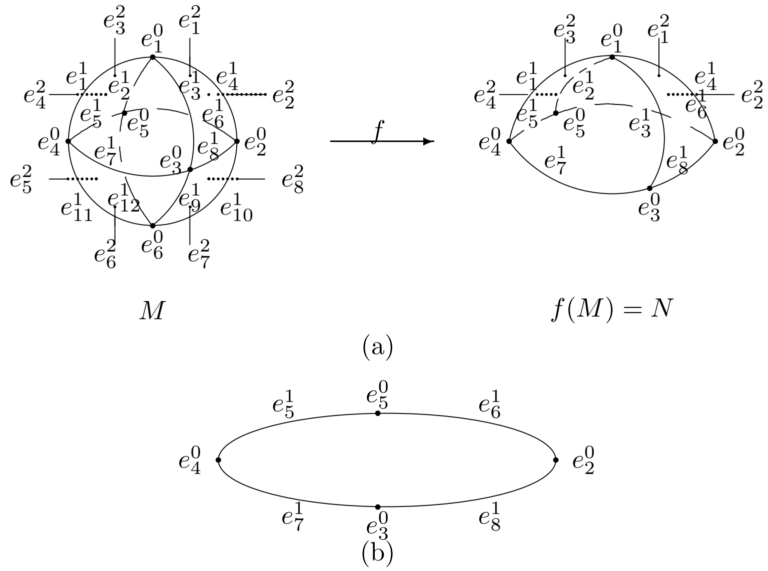


Fig.1

Now, let $g : N \to N$ be given by : $g(e_1^0, e_2^0, e_3^0, e_4^0, e_5^0) = (e_1^0, e_2^0, e_3^0, e_4^0, e_5^0)$, $g(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1) = (e_4^1, e_2^1, e_3^1, e_4^1, e_6^1, e_6^1, e_8^1, e_8^1)$, $g(e_1^2, e_2^2, e_3^2, e_4^2) = (e_1^2, e_2^2, e_1^2, e_2^2)$. See Fig.2(a). Again g is a cellular folding and the folding graphs Γ_g and $f^{-1}(\Gamma_g)$ are shown in Fig.2(b).

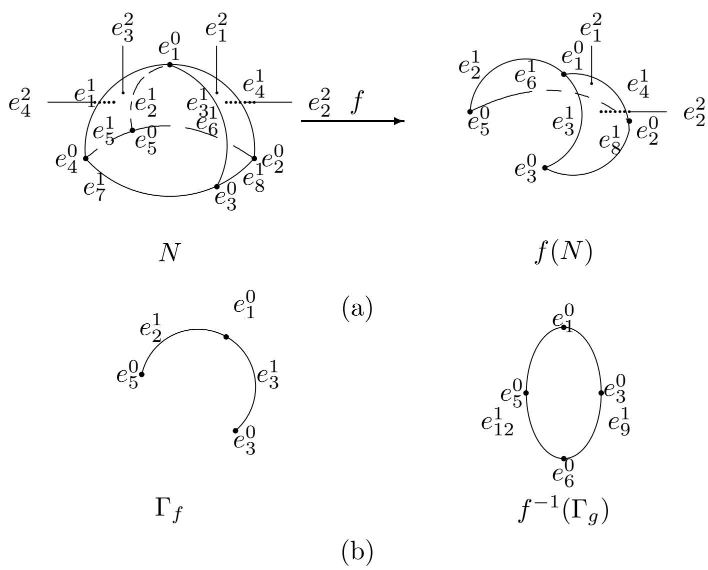
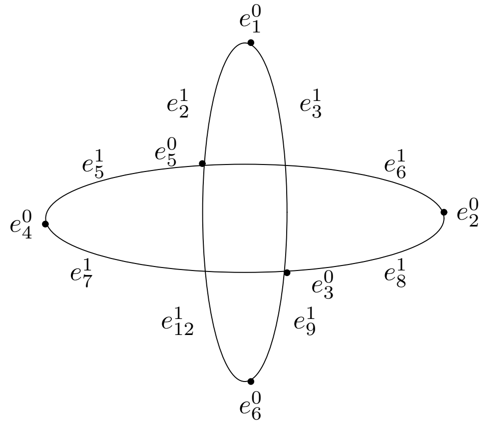


Fig.2

Then $g \circ f : M \to L$ is a cellular folding with folding graph $\Gamma_{g \circ f}$ shown in Fig.3.



$\Gamma_{g\circ f} = \Gamma_f \bigcup f^{-1}(\Gamma_g)$

Fig.3

Theorem 2.1 can be generalized for a series of cellular foldings as follows:

Theorem 2.3 *Let $M, M_1, M_2, \cdots, M_n$ be complexes of the same dimension 2 such that $M_n \subset M_{n-1} \subset M_1 \subset M$, and consider the cellular maps $M \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \cdots \xrightarrow{f_n} M_n$. Then the composition of these cellular maps $\phi :: M \to M_n$ is a cellular folding iff each $f_r,\ r = 1, 2, \cdots, n$ is a cellular folding. In this case the folding graphs satisfy the condition*

$$\begin{aligned}\Gamma_\phi &= \Gamma_{f_1} \bigcup f_1^{-1}(\Gamma_{f_2}) \bigcup (f_1 \circ f_1)^{-1}(\Gamma_{f_3} \bigcup (f_3 \circ f_2 \circ f_1)^{-1}(\Gamma_{f_4}))\\ &\quad \bigcup \cdots \bigcup (f_{n-1} \circ f_{n-2} \circ \cdots \circ f_1)^{-1}(\Gamma_{f_n}).\end{aligned}$$

Theorem 2.4 *Let M, N and L be complexes of the same dimension 2 such that $L \subset N \subset M$. Let $f : M \to N$ be a cellular folding such that $f(M) = N$. Then a cellular map $g : N \to L$ is a cellular folding iff $g \circ f : M \to L$ is a cellular folding. In this case $\Gamma_g = f[(\Gamma_{g\circ f} \setminus E(\Gamma_f)) \setminus \{V\}]$, where $E(\Gamma_f)$ is the set of edges of Γ_f and $\{V\}$ is the set of the isolated vertices remains in $\Gamma_{g\circ f}$.*

Proof Suppose $g \circ f$ is a cellular folding, $f \in C(M, N)$, $\sum f \neq \emptyset$. Let $\sigma \in M^{(i)}, i = 0, 1, 2$ be an arbitrary i-cell in M such that σ has S vertices. Since $g \circ f$ is a cellular folding, then $g \circ f(\sigma) = \sigma'$ is an i-cell in L such that σ' has S distinct vertices. But $g \circ f(\sigma) = g(f(\sigma))$ and $f(\sigma)$ is an i-cell in N such that $\overline{f(\sigma)}$ has S distinct vertices, then g maps i-cells to i-cells and satisfies the second condition of cellular folding, consequently, g is a cellular folding. In this case, $\Gamma_g = f[(\Gamma_{g\circ f} \setminus E(\Gamma_f)) \setminus \{V\}]$,where $E(\Gamma_f)$ is the set of edges of Γ_f and $\{V\}$ is the set of the isolated vertices remains in $\Gamma_{g\circ f}$.

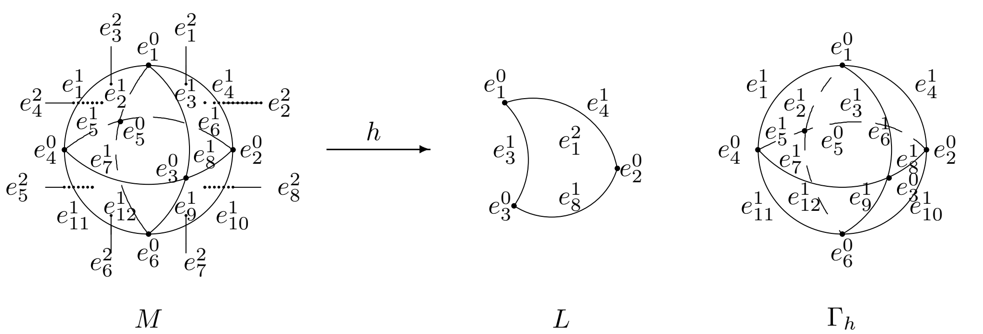
Conversely, suppose $g : N \to L$ is a cellular folding. Since $f : M \to N$ is a cellular folding, by Theorem 2.1, $g \circ f$ is a cellular folding. Notice that this conclusion is also true if we consider g and $g \circ f$ neat cellular foldings instead of cellular foldings. □

Example 2.5 Consider a complex on $|M| = S^2$ with cellular subdivision consisting of six vertices, twelve 1-cells and eight 2-cells. Let $f : M \to M, f(M) = N$ be a cellular folding given as shown in Fig.1(a) with folding graph Γ_f shown in Fig.1(b).

Now, let L be a 2-cell with boundary consists of three 0-cells and three 1-cells , see Fig.4(a) and let $h : M \to L$ be a cellular folding defined by:

$$h(e^0_1, e^0_2, e^0_3, e^0_4, e^0_5, e^0_6) = (e^0_1, e^0_2, e^0_3, e^0_2, e^0_3, e^0_1),$$
$$h(e^1_1, e^1_2, e^1_3, e^1_4, e^1_5, e^1_6, e^1_7, e^1_8, e^1_9, e^1_{10}, e^1_{11}, e^1_{12}) = (e^1_4, e^1_4, e^1_3, e^1_4, e^1_8, e^1_8, e^1_8, e^1_8, e^1_3, e^1_4, e^1_4, e^1_4),$$
$$h(e^2_1, e^2_2, e^2_3, e^2_4, e^2_5, e^2_6, e^2_7, e^2_8) = (e^2_1).$$

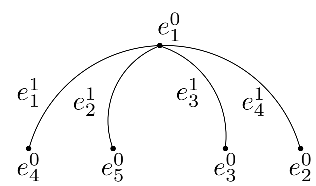
The folding graph Γ_h is shown in Fig.4(b).



(a) (b)

Fig.4

The cellular folding h is the composition of f with a cellular folding $g : N \to L$ which folds N onto L. The graph Γ_g is given is given in Fig.5.



$\Gamma_g = f[(\Gamma_{g\circ f} \setminus E(\Gamma_f)) \setminus \{V\}]$

Fig.5

where $E(\Gamma_f)$ is the edges of Γ_f and $\{V\}$ is the set of the isolated vertices remains in $\Gamma_{g\circ f} = \Gamma_h$.

Theorem 2.4 can be generalized for a finite series of cellular foldings as follows:

Theorem 2.6 *Let $M, M_1, M_2, \cdots, M_n$ be complexes of the same dimension* 2 *such that $M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M$, and consider the cellular maps $M \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \cdots \xrightarrow{f_{n-1}} M_{n-1}$. Then a cellular map $f_n : M_{n-1} \to M_n$ is a cellular folding iff the composition $f_n \circ f_{n-1} \circ \cdots \circ f_1$:*

$M \to M_n$ is a cellular folding. In this case the folding graph of f_n is given by:

$$\Gamma_{f_n} = (f_{n-1} \circ \cdots \circ f_1)[(\Gamma_{f_{n-1} \circ \cdots \circ f_1} \setminus E(\Gamma_{f_{n-1} \circ \cdots \circ f_1}) \setminus \{V\}],$$

where $E(\Gamma_{f_{n-1} \circ \cdots \circ f_1})$ is the set of edges of $\Gamma_{f_{n-1} \circ \cdots \circ f_1}$ and $\{V\}$ is the set of the isolated vertices remains in $\Gamma_{f_n \circ f_{n-1} \circ \cdots \circ f_1}$.

Theorem 2.7 *Suppose K, L, X and Y are complexes of the same dimension 2. Let $f : K \to X$ and $g : L \to Y$ be cellular maps. Then $f \times g : K \times L \to X \times Y$ is a cellular folding iff f and g are cellular foldings. In this case, $\Gamma_{f \times g} = (\Gamma_f \times L) \bigcup (\Gamma_g \times K)$.*

Proof Suppose f and g are cellular foldings. We claim that $f \times g$ is a cellular folding. Let e^i be an arbitrary i-cell in K, $e'j$ be an arbitrary j-cell in L. Then $(e^i, e^{'j})$ is an $(i+j)$-cell in $K \times L$. Since $(f \times g)[(e^i, e^{'j})] = (f(e^i), g(e'j))$, thus $(f \times g)(e^i, e^{'j})$ is an $(i+j)$-cell in $X \times Y$ (since $f(e^i)$ is an i-cell in X, $g(e^{'j})$ is a j - cell in Y, f and g are cellular foldings). Then $f \times g$ sends cells to cells of the same dimension. Also, if $\sigma = (e^i, e^{'j})$, $\overline{\sigma}$ and $\overline{(f \times g)(\sigma)}$ contains the same number of vertices because each of f and g is a cellular folding.

Suppose now $f \times g$ is a cellular folding, then $f \times g$ maps p-cells to p-cells, i.e., if (e, e') is a p-cell in $K \times L$, then $(f \times g)(e, e') = (f(e), g(e'))$ is a p-cell in $X \times Y$. Let e be an i-cell in K and e' be a $(p-i)$-cell in L. The all cellular maps must map i-cells to j-cells such that $j \leq i$. If $i = j$, there are nothing needed to prove. So let $i > j$. In this case g will map $(p-i)$-cells to $(p-j)$-cells and hence it is not a cellular map. This is a contradiction and hence $i = j$ is the only possibility. The second condition of cellular folding certainly satisfied in this case. □

It should be noted that this conclusion is also true for neat cellular foldings, but it is not true for simplecial complexes since the product of two positive-dimensional simplexes is not a simplex any more.

Example 2.8 Let K be complex such that $|K| = S^1$ with four vertices and four 1-cells, and let $f : K \to K$ be a cellular folding defined by $f(v^1, v^2, v^3, v^4) = (v^1, v^2, v^1, v^4)$ and L a complex such that $|L| = I$ with three vertices and two 1-cells and let $g : L \to L$ be a neat cellular folding $g(u^1, u^2, u^3) = (u^1, u62, u^1)$, see Fig.6.

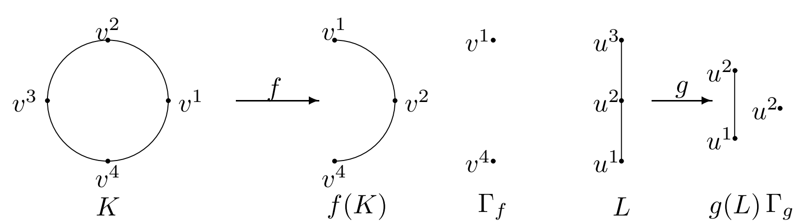
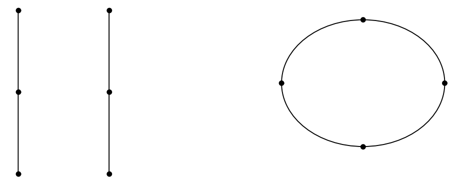


Fig.6

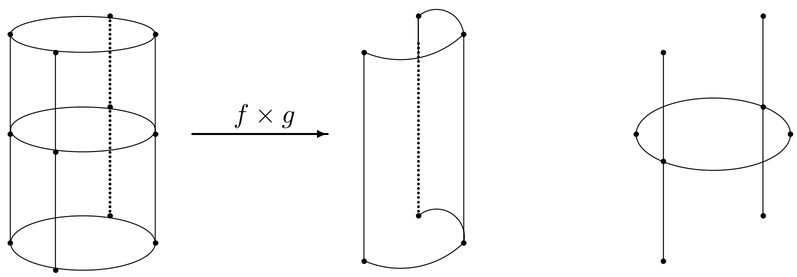
Then the folding graphs $\Gamma_f \times L$ and $\Gamma_g \times K$ have the form shown in Fig.7.



$\Gamma_f \times L$ $\Gamma_g \times K$

Fig.7

Now $f \times g : K \times L \to K \times L$ is a cellular folding but not neat. The cell decomposition of $K \times L$ and $(f \times g)(K \times L)$ are shown in Fig.8(a). In this case, $\Gamma_{f\times g}$ has the form shown in Fig.8(b).



$K \times L$ $(f \times g)(K \times L)$ $\Gamma_{f\times g} = (\Gamma_f \times L)\bigcup(\Gamma_g \times K)$

Fig.8

Theorem 2.7 can be generalized for the product of finite numbers of complexes as follows:

Theorem 2.9 *Suppose $K_1, K_2, \cdots, K_n$ and $X_1, X_2, \cdots, X_n$ are complexes of the same dimension 2 and $f_i : K_i \to X_i$ for $i = 1, 2, \cdots, n$ are cellular maps. Then the product map $f_1 \times f_2 \times \cdots f_n : K_1 \times K_2 \times \cdots \times K_n \to X_1 \times X_2 \times \cdots \times X_n$ is a cellular folding iff each of f_i is a cellular folding for $i = 1, 2, \cdots, n$. In this case,*

$$\begin{aligned}\Gamma_{f_1\times f_2\times\cdots\times f_n} &= \Gamma_{f_1} \times (K_2 \times K_3 \times \cdots \times K_n)\bigcup \Gamma_{f_2} \times (K_1 \times K_3 \times \cdots \times K_n)\\ &\bigcup \cdots \bigcup \Gamma_{f_n} \times (K_1 \times K_2 \times \cdots \times K_{n-1}).\end{aligned}$$

Theorem 2.10 *Let A, B, A_1, A_2, B_1, B_2 be complexes and let $f : A \to A_1$, $g : B \to B_1$, $h : A_1 \to A_2$, $k : B_1 \to B_2$ be cellular foldings. Then $(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g)$ is a cellular folding with folding graph*

$$\Gamma_{(h\times k)\circ(f\times g)} = \Gamma_{f\times g}\bigcup(f \times g)^{-1}(\Gamma_{h\times k}) = \Gamma_{(h\circ f)\times(k\circ g)} = (\Gamma_{h\circ f} \times B)\bigcup(\Gamma_{k\circ g} \times A).$$

Proof Since $h : A_1 \to A_2$, $k : B_1 \to B_2$ are cellular foldings, then $h \times k : A_1 \times B_1 \to A_2 \times B_2$ is a cellular folding. Also, since $f : A \to A_1$, $g : B \to B_1$ are cellular foldings, then so is

$f \times g : A \times B \to A_1 \times B_1$. Thus $(h \times k) \circ (f \times g) : A \times B \to A_2 \times B_2$ is a cellular folding with folding graph $\Gamma_{(h\times k)\circ(f\times g)} = \Gamma_{f\times g} \bigcup (f \times g)^{-1}(\Gamma_{h\times k})$.

On the other hand, because both of $(h\circ f)$ and $(k\circ g)$ are cellular foldings, then $(h\circ f)\times(k\circ g)$ is a cellular folding with folding graph

$$\Gamma_{(h\circ f)\times(k\circ g)} = (\Gamma_{h\circ f} \times B) \bigcup (\Gamma_{k\circ g} \times A). \qquad \square$$

The above theorem can be generalized for a finite number of cellular foldings.

Example 2.11 Suppose A, B, A_1, A_2, B_1, B_2 are complexes such that $A = S^1$, $B = |A_1| = |A_2| = |B_1| = |B_2| = I$ with cell decompositions shown in Fig.9.

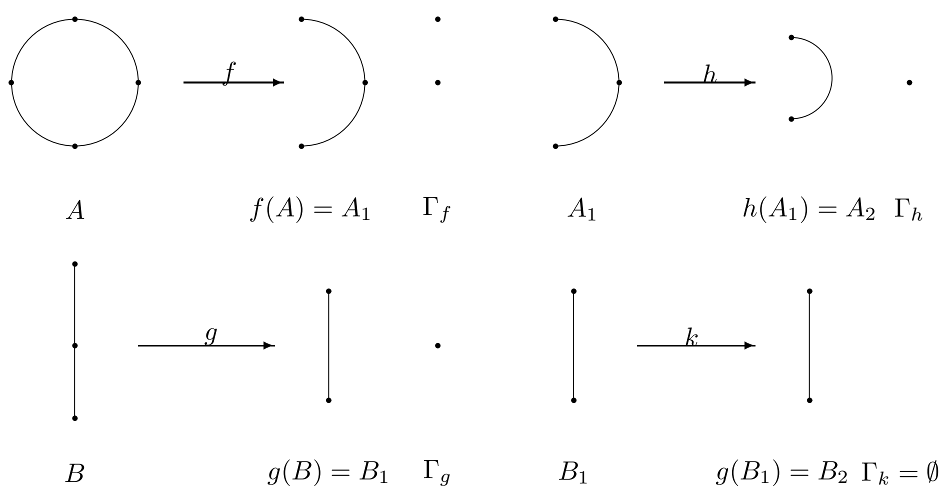


Fig.9

Suppose $f : A \to A_1$, $g : B \to B_1$, $h : A_1 \to A_2$ and $k : B_1 \to B_2$ are cellular foldings. The cellular foldings $f \times g$, $h \times k$ and the folding graphs $\Gamma_{f\times g}$, $\Gamma_{h\times k}$, $\Gamma_{(h\times k)\circ(f\times g)}$ are shown in Fig.10.

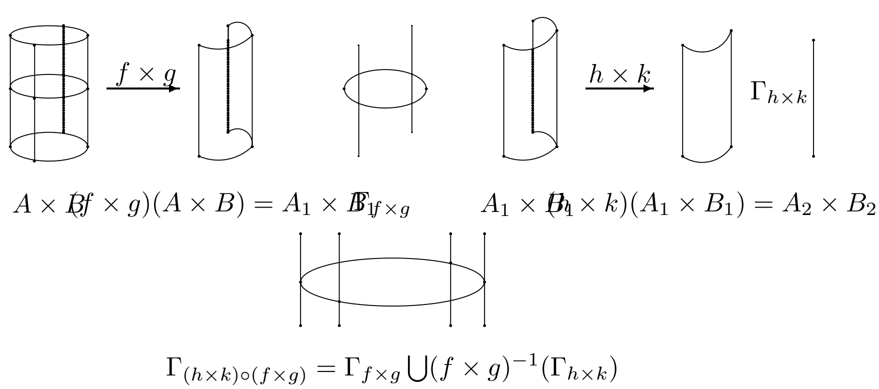
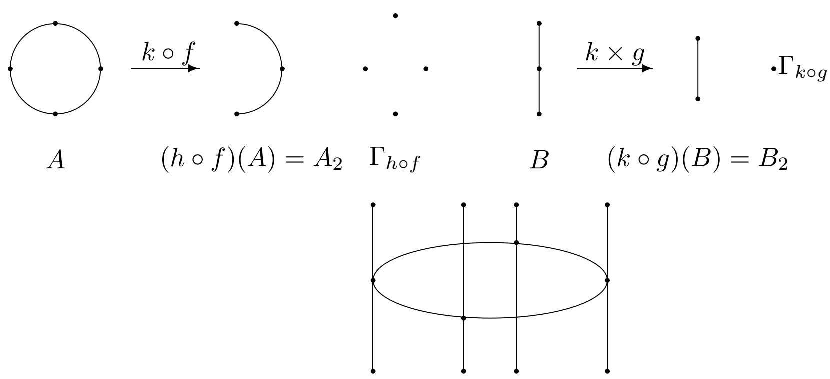


Fig.10

Also the cellular folding $h \circ f$, $k \circ g$ and the folding graphs $\Gamma_{h\circ f}$, $\Gamma_{k\circ g}$, $\Gamma_{(h\circ f)\times(k\circ g)}$ are shown in Fig.11.



$\Gamma_{(h\circ f)\times(k\circ g)} = (\Gamma_{h\circ f} \times B) \bigcup (\Gamma_{k\circ g} \times A)$

Fig.11

Proposition 2.11 *Let X be a complex and $f : X \to X$ any neat cellular folding. Then f restricted to any subcomplex A of X is again a neat cellular folding over the image $f(X) = Y$.*

This is due to the fact that f_{e^i} with e^i an i-cell of X, is a homeomorphism onto its image and in the case of neat cellular folding of surfaces the image, Y must has only one 2-cell, Int$Y$, and thus the restriction of f to any subcomplex of X will maps each 2-cells of A onto the 2-cell of Y and it does so for the 0 and 1-cells of A since f in fact is cellular. Consequently $f|_A$ is a neat cellular folding of A to Y.

Example 2.12 Consider a complex X such that $|X|$ is a torus with a cellular subdivision shown in Fig.12 and let $f : X \to X$ be given by

$$f(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_2^0, e_1^0, e_1^0),$$
$$f(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1) = (e_1^1, e_1^1, e_1^1, e_1^1, e_5^1, e_8^1, e_5^1, e_8^1),$$
$$f(e_n^2) = e_1^2 \quad \text{for } n = 1, 2, 3, 4.$$

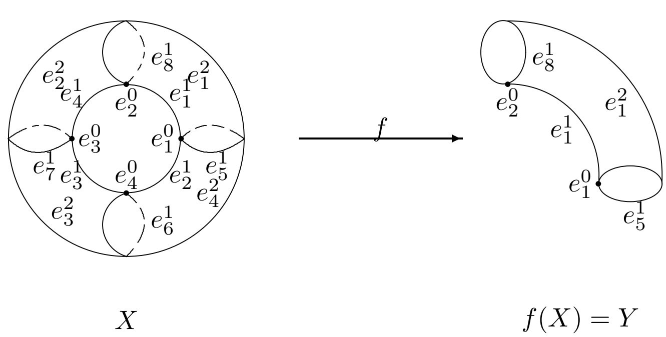


Fig.12

The map f is a neat cellular folding with image $f(X) = Y$ which is a subcomplex of X consists of two 0-cells, three 1-cells and a single 2-cell. Now let $A \subset X$ shown in Fig.13. Then $f|_A : A \to Y$ given by

$$f|_A(e_1^0, e_2^0, e_3^0) = (e_1^0, e_2^0, e_1^0),$$
$$f|_A(e_1^1, e_4^1, e_5^1, e_7^1, e_8^1) = (e_1^1, e_1^1, e_5^1, e_1^1, e_8^1),$$
$$f|_A(e_n^2) = e_1^2 \text{ for} n = 1, 2, 3, 4$$

is a neat cellular folding.

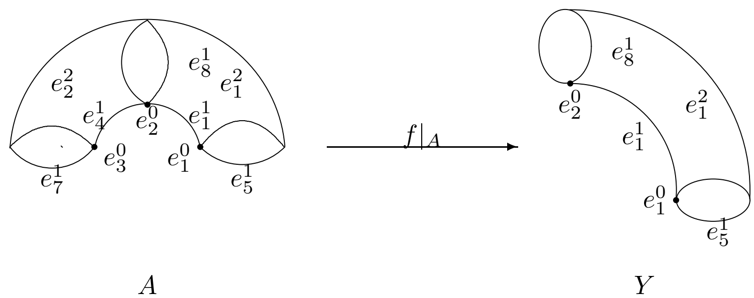


Fig.13

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