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Vertex-Mean Graphs

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Abstract: Let $k \geq 0$ be an integer. A *Smarandachely vertex-mean k-labeling* of a (p, q) graph $G = (V, E)$ is such an injection $f : E \longrightarrow \{0, 1, 2, ..., q_* + k\}$, $q_* = \max(p, q)$ such that the function $f^V : V \longrightarrow \mathbb{N}$ defined by the rule $f^V(v) = Round\left(\frac{\sum_{e \in E_v} f(e)}{d(v)}\right) - k$ satisfies the property that $f^V(V) = \left\{f^V(u) : u \in V\right\} = \{1, 2, ..., p\}$, where E_v denotes the set of edges in G that are incident at v, $\mathbb{N}$ denotes the set of all natural numbers and *Round* is the *nearest integer function*. A graph that has a Smarandachely vertex-mean k-labeling is called *Smarandachely k vertex-mean graph* or *Smarandachely k V-mean graph*. Particularly, if $k = 0$, such a Smarandachely vertex-mean 0-labeling and Smarandachely 0 vertex-mean graph or Smarandachely 0 V-mean graph is called a *vertex-mean labeling* and a *vertex-mean graph* or *V-mean graph*, respectively. In this paper, we obtain necessary conditions for a graph to be V-mean and study V-mean behaviour of certain classes of graphs.

Key Words: Smarandachely vertex-mean k-labeling, vertex-mean labeling, edge labeling, Smarandachely k vertex-mean graph, vertex-mean graph.

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§1. Introduction

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge xy depending on the vertex labels. An *edge labeling* of a graph G is an assignment f of labels to the edges of G that induces a label for each vertex v depending on the labels of the edges incident on it. Vertex labelings such as *graceful labeling, harmonious labeling* and *mean labeling* and edge labelings such as *edge-magic labeling, (a,d)-anti magic labeling and vertex-graceful labeling* are some of the interesting labelings found in the dynamic survey of graph labeling by Gallian [3]. In fact B. D. Acharya [2] has introduced *vertex-graceful graphs*, as an edge-analogue of *graceful graphs*. Observe that, in a variety of practical problems, the arithmetic mean, X, of a finite set of real numbers $\{x_1, x_2, ..., x_n\}$ serves as a

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better estimate for it, in the sense that $\sum(x_i - X)$ is zero and $\sum(x_i - X)^2$ is the minimum. If it is required to use a single integer in the place of X then $Round(X)$ does this best, in the sense that $\sum(x_i - Round(X))$ and $\sum(x_i - Round(X))^2$ are minimum, where $Round(Y)$, *nearest integer function* of a real number, gives the integer closest to Y; to avoid ambiguity, it is defined to be the nearest even integer in the case of half integers. This motivates us to define the edge-analogue of the *mean labeling* introduced by R. Ponraj [1]. A *mean labeling* f is an injection from V to the set $\{0, 1, 2, ..., q\}$ such that the set of edge labels defined by the rule $Round(\dfrac{f(u)+f(v)}{2})$ for each edge uv is $\{1, 2, ..., q\}$. For all terminology and notations in graph theory, we refer the reader to the text book by D. B. West [4]. All graphs considered in the paper are finite and simple.

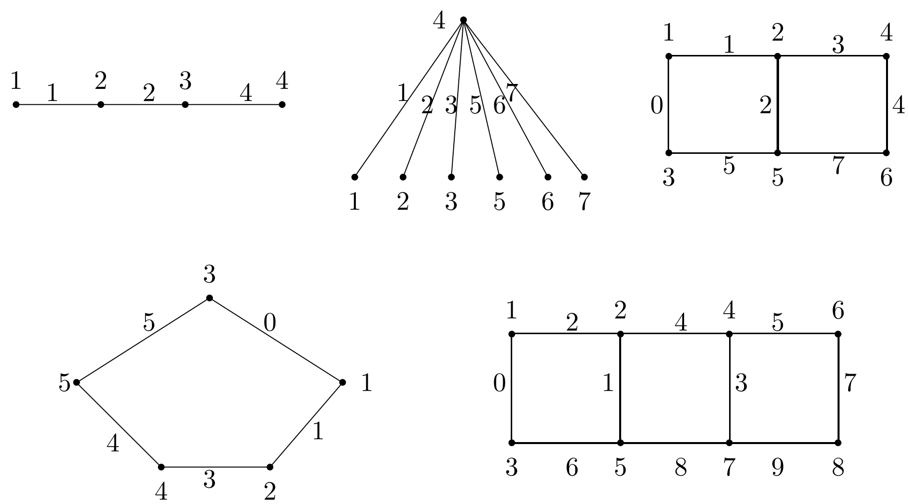

Fig.1 **Some V -mean graphs**

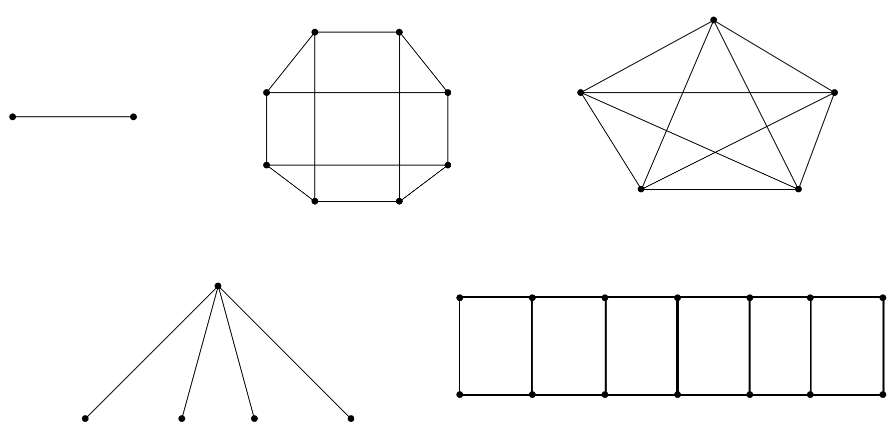

Fig.2

Definition 1.1 *Let $k \geq 0$ be an integer. A Smarandachely vertex-mean k-labeling of a (p,q) graph $G=(V,E)$ is such an injection $f: E \longrightarrow \{0,1,2,...,q_*+k\}$, $q_* = \max(p,q)$ such that the function $f^V: V \longrightarrow \mathbb{N}$ defined by the rule $f^V(v) = Round\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right) - k$ satisfies the property that $f^V(V) = \left\{f^V(u): u \in V\right\} = \{1,2,...,p\}$, where E_v denotes the set of edges in G that are incident at v, $\mathbb{N}$ denotes the set of all natural numbers and Round is the nearest integer function. A graph that has a Smarandachely vertex-mean k-labeling is called Smarandachely k vertex-mean graph or Smarandachely k V-mean graph. Particularly, if $k=0$, such a Smarandachely vertex-mean 0-labeling and Smarandachely 0 vertex-mean graph or Smarandachely 0 V-mean graph is called a vertex-mean labeling and a vertex-mean graph or V-mean graph, respectively.*

Henceforth we call vertex-mean as V-mean. To initiate the investigation, we obtain necessary conditions for a graph to be a V-mean graph and we present some results on this new notion in this paper. In Fig.1 we give some V-mean graphs and in Fig.2, we give some non V-mean graphs.

§2. Necessary Conditions

Following observations are obvious from Definition 1.1.

Observation 2.1 If G is a V-mean graph then no V-mean labeling assigns 0 to a pendant edge.

Observation 2.2 The graph K_2 and disjoint union of K_2 are not V-mean graphs, as any number assigned to an edge uv leads to assignment of same number to each of u and v. Thus every component of a V-mean graph has at least two edges.

Observation 2.3 The minimum degree of any V-mean graph is less than or equal to three ie, $\delta \leqslant 3$ as $Round(0+1+2+3)$ is 2. Thus graphs that contain a r-regular graph, where $r \geq 4$ as spanning sub graph are not V-mean graphs and any 3-edge-connected V-mean graph has a vertex of degree three.

Observation 2.4 If f is a V-mean labeling of a graph G then either (1) or (2) of the following is satisfied according as the induced vertex label $f^V(v)$ is obtained by rounding up or rounding down.

$$f^V(v)d(v) \leq \sum_{e\in E_v} f(e) + \frac{1}{2}d(v), \tag{1}$$

$$f^V(v)d(v) \geq \sum_{e\in E_v} f(e) - \frac{1}{2}d(v). \tag{2}$$

Theorem 2.5 *If G is a V-mean graph then the vertices of G can be arranged as $v_1, v_2, ..., v_p$ such that $q^2 - 2q \leq \sum_1^p kd(v_k) \leq 2qq_* - q^2 + 2q$.*

Proof Let f be a V-mean labeling of a graph G. Let us denote the vertex that has the induced vertex label k, $1 \leq k \leq p$ as v_k. Observe that, $\sum_{v\in V} f^V(v)d(v)$ attains it maximum/minimum when each induced vertex label is obtained by rounding up/down and the first

q largest/smallest values of the set $\{0, 1, 2, \cdots, q_*\}$ are assigned as edge labels by f. This with Observation 2.4 completes the proof. $\square$

Corollary 2.6 *Any 3-regular graph of order $2m$, $m \geq 4$ is not a V-mean graph.*

Corollary 2.7 *The ladder $L_n = P_n \times P_2$, $n \geq 7$ is not a V-mean graph.*

A V-mean labeling of ladders L_3 and L_4 are shown in Figure 1.

§3. Classes of V-Mean Graphs

Theorem 3.1 *If $n \geq 3$ then the path P_n is V-mean graph.*

Proof Let $\{e_1, e_2, ..., e_{n-1}\}$ be the edge set of P_n such that $e_i = v_i v_{i+1}$. We define $f : E \longrightarrow \{0, 1, 2, ..., q_* = p\}$ as follows:

$$f(e_i) = \begin{cases} i, & if\ 1 \leq i \leq p-2, \\ i+1, & if\ i = p-1. \end{cases}$$

It can be easily verified that f is a V-mean labeling. $\square$

A V-mean labeling of P_{10} is shown in Fig.3.

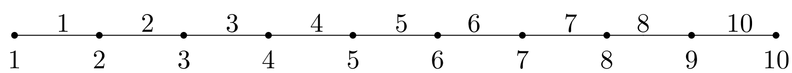

Fig.3

Theorem 3.2 *If $n \geq 3$ then the cycle C_n is V-mean graph.*

Proof Let $\{e_1, e_2, ..., e_n\}$ be the edge set of C_n such that $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$, $e_n = v_n v_1$. Let $\zeta = \lceil \frac{n}{2} \rceil - 1$. The edges of C_n are labeled as follows: The numbers $0, 1, 2, \cdots, n$ except ζ are arranged in an increasing sequence $\alpha_1, \alpha_2, \cdots, \alpha_n$ and α_k is assigned to e_k. Clearly the edges of C_n receive distinct labels and the vertex labels induced are $1, 2, \cdots, n$. Thus C_n is V-mean graph. $\square$

The *corona* $G_1 \odot G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2. The graph $C_n \odot K_1$ is called a *crown*.

Theorem 3.3 *The corona $P_n \odot K_m^C$, where $n \geq 2$ and $m \geq 1$ is V-mean graph.*

Proof Let the vertex set and the edge set of $G = P_n \odot K_m^C$ be as follows:

$V(G) = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n\ and\ 1 \leq j \leq m\}$,

$E(G) = A \bigcup B$,

where $A = \{e_i = u_iu_{i+1} : 1 \leq i \leq n-1\}$ and $B = \{e_{ij} = u_iu_{ij} : 1 \leq i \leq n \; and \; 1 \leq j \leq m\}$. We observe that G has order $(m+1)n$ and size $(m+1)n-1$. The edges of G are labeled in three steps as follows :

Step 1. The edges e_1 and $e_{1j}, 1 \leq j \leq m$ are assigned distinct integers from 1 to $(m+1)$ in such a way that e_1 receives the number $Round(\frac{\sum_{j=1}^{m+1} j}{m+1})$.

Step 2. For each i, $2 \leq i \leq n-1$, the edges e_i and $e_{ij}, 1 \leq j \leq m$ are assigned distinct integers from $(m+1)(i-1)+1$ to $(m+1)i$ in such a way that e_i receives the number

$$Round(\frac{f(e_{i-1}) + \sum_{j=1}^{m+1}(m+1)(i-1)+j}{m+2}).$$

Step 3. The edges $e_{nj}, 1 \leq j \leq m$ are assigned distinct integers from $(m+1)(n-1)+1$ to $(m+1)n$ in such a way that non of these edges receive the number

$$Round(\frac{f(e_{n-1}) + \sum_{j=1}^{m+1}(m+1)(n-1)+j}{m+2}).$$

Then the edges of G receive distinct labels and the vertex labels induced are $1, 2, ..., (m+1)n$. Thus G is V-mean graph.

Fig.4 displays a V-mean labeling of $P_5 \odot K_4^C$.

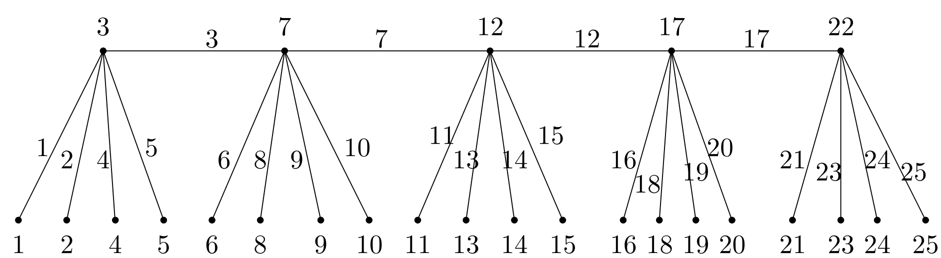

Fig.4 **A** V **-mean labeling of** $P_5 \odot K_4^C$

Theorem 3.4 *The star graph* $K_{1,n}$ *is* V*-mean graph if and only if* $n \cong 0(mod2)$.

Proof Necessity: Suppose $G = K_{1,n}$, $n = 2m+1$ for some $m \geq 1$ is V-mean and let f be a V-mean labeling of G. As no V-mean labeling assigns zero to a pendant edge, f assigns $2m+1$ distinct numbers from the set $\{1, 2, ..., 2m+2\}$ to the edges of G. Observe that, whatever be the labels assigned to the edges of G, label induced on the central vertex of G will be either $m+1$ or $m+2$. In both cases two vertex labels induced on G will be identical. This contradiction proves *necessity.*

Sufficiency: Let $G = K_{1,n}$, $n = 2m$ for some $m \geq 1$. Then assignment of $2m$ distinct numbers except $m+1$ from the set $\{1, 2, ..., 2m+1\}$ gives the desired V-mean labeling of G.□

Theorem 3.5 *The crown* $C_n \odot K_1$ *is* V*-mean graph.*

Proof Let the vertex set and the edge set of $G = C_n \odot K_1$ be as follows: V(G) = $\{u_i, v_i : 1 \le i \le n\}$, $E(G) = A \bigcup B$ where $A = \{e_i = u_i u_{i+1} : 1 \le i \le n-1\} \bigcup \{e_n = u_n u_1\}$ and $B = \left\{e_i^{'} = u_i v_i : \ 1 \le i \le n\right\}$. Observe that G has order and size both equal to $2n$. For $3 \le n \le 5$, V-mean labeling of G are shown in Fig.5. For $n \ge 6$, define $f : E(G) \longrightarrow \{0, 1, 2, ..., 2n\}$ as follows:

Case 1 $n \equiv 0(mod\ 3)$.

$$f(e_i) = \begin{cases} 2i-2 & if\ 1 \le i \le \dfrac{n}{3} - 1, \\ 2i & if\ i = \dfrac{n}{3}, \\ 2i-1 & if\ \dfrac{n}{3} + 1 \le i \le n, \end{cases}$$

$$f(e_i^{'}) = \begin{cases} 2i-1 & if\ 1 \le i \le \dfrac{n}{3}, \\ 2i & if\ \dfrac{n}{3} + 1 \le i \le n. \end{cases}$$

Case 2 $n \not\equiv 0(mod\ 3)$.

$$f(e_i) = \begin{cases} 2i-2 & if\ 1 \le i \le \left\lfloor \dfrac{n}{3} \right\rfloor, \\ 2i-1 & if\ \left\lfloor \dfrac{n}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(e_i^{'}) = \begin{cases} 2i-1 & if\ 1 \le i \le \left\lfloor \dfrac{n}{3} \right\rfloor, \\ 2i & if\ \left\lfloor \dfrac{n}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

It can be easily verified that f is a V-mean labeling of G. □

A V-mean labeling of some crowns are shown in Fig.5.

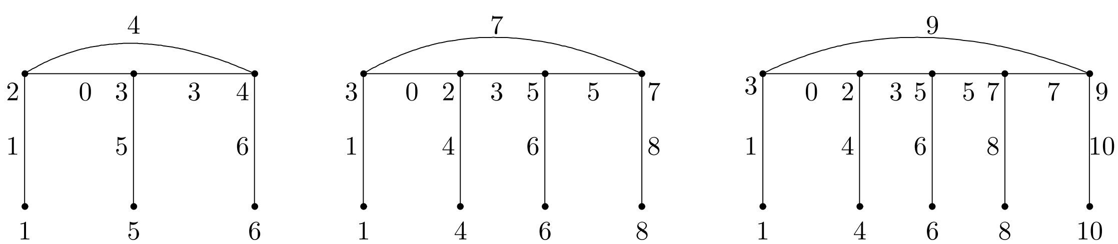

Fig.5 V **-mean labeling of crowns for** $n = 3, 4, 5$

Problem 3.6 *Determine new classes of trees and unicyclic graphs which are V-mean graphs.*

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