

Lucas Gracefulness of Almost and Nearly for Some Graphs

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Abstract: Let G be a (p, q) - graph. An injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_a\}, (a \in N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \dots, l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits Lucas graceful labeling, then G is said to be Lucas graceful graph. An injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_{a-1}, l_{a+1}\}, (a \in N)$, is said to be almost Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \dots, l_q\}$ or $\{l_1, l_2, \dots, l_{q-1}, l_{q+1}\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. Then G is called almost Lucas graceful graph if it admits almost Lucas graceful labeling. Also, an injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_a\}, (a \in N)$, is said to be nearly Lucas graceful labeling if the induced edge labeling $f_1(u, v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \dots, l_{i-1}, l_{i+1}, l_{i+2}, \dots, l_{j-1}, l_{j+1}, l_{j+2}, \dots, l_{k-1}, l_{k+1}, l_{k+2}, \dots, l_b\} (b \in N \text{ and } b \leq a)$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be nearly Lucas graceful graph. In this paper, we show that the graphs $S_{m,n}, S_{m,n} @ P_t$ and $F_m @ P_n$ are almost Lucas graceful graphs. Also we show that the graphs $S_{m,n} @ P_t$ and C_n are nearly Lucas graceful graphs.

Key Words: Smarandache-Fibonacci triple, super Smarandache-Fibonacci graceful graph, graceful labeling, Lucas graceful labeling, almost Lucas graceful labeling and nearly Lucas graceful labeling.

AMS(2010): 05C78

§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A cycle of length n is denoted by C_n . G^+ is a graph obtained from the graph G by attaching pendant vertex to each vertex of G . The concept of graceful labeling was introduced by Rosa [3] in 1967. A function f is called a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{1, 2, 3, \dots, q\}$ such that when each edge uv is assigned the label

¹Received May 26, 2011. Accepted September 6, 2011.

$|f(u) - f(v)|$, the resulting edge labels are distinct. The notion of Fibonacci graceful labeling was introduced by K.M.Kathiresan and S.Amutha [4]. We call a function f , a Fibonacci graceful label labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \dots, F_q\}$, where F_q is the q^{th} Fibonacci number of the Fibonacci series $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$ and each edge uv is assigned the label $|f(u) - f(v)|$. Based on the above concept we define the following.

A *Smarandache-Fibonacci triple* is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Clearly, it is a generalization of *Fibonacci sequence* and *Lucas sequence*. Let G be a (p, q) -graph and $\{S(n) | n \geq 0\}$ a Smarandache-Fibonacci triple. An bijection $f: V(G) \rightarrow \{S(0), S(1), S(2), \dots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \dots, S(q)\}$. Particularly, if $S(n), n \geq 0$ is just the Lucas sequence, such a labeling $f: V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_a\}$ ($a \in N$) is said to be *Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection on to the set $\{l_1, l_2, \dots, l_q\}$. If G admits Lucas graceful labeling, then G is said to be *Lucas graceful graph*. An injective function $f: V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_{a-1}, l_{a+1}\}$, ($a \in N$), is said to be *almost Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \dots, l_q\}$ or $\{l_1, l_2, \dots, l_{q-1}, l_{q+1}\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. Then G is called *almost Lucas graceful graph* if it admits almost Lucas graceful labeling. Also, an injective function $f: V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_a\}$, ($a \in N$), is said to be *nearly Lucas graceful labeling* if the induced edge labeling $f_1(u, v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \dots, l_{i-1}, l_{i+1}, l_{i+2}, \dots, l_{j-1}, l_{j+1}, l_{j+2}, \dots, l_{k-1}, l_{k+1}, l_{k+2}, \dots, l_b\}$ ($b \in N$ and $b \leq a$) with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be *nearly Lucas graceful graph*. In this paper, we show that the graphs $S_{m,n}, S_{m,n} @ P_t$ and $F_m @ P_n$ are almost Lucas graceful graphs. Also we show that the graphs $S_{m,n} @ P_t$ and C_n are nearly Lucas graceful graphs.

§2. Almost Lucas Graceful Graphs

In this section, we show that some graphs namely $S_{m,n}, S_{m,n} @ P_t$ and $F_m @ P_n$ are almost Lucas graceful graphs.

Definition 2.1 Let G be a (p, q) - graph. An injective function $f: V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_{a-1}, l_{a+1}\}$, $a \in N$, is said to be *almost Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \dots, l_q\}$ or $\{l_1, l_2, \dots, l_{q-1}, l_{q+1}\}$. Then G is called *almost Lucas graceful graph* if it admits almost Lucas graceful labeling.

Definition 2.2 ([2]) $S_{m,n}$ denotes a star with n spokes in which each spoke is a path of length m .

Theorem 2.3 $S_{m,n}$ is an almost Lucas graceful graph when $m \equiv 1(mod 2)$ and $n \equiv 0(mod 3)$

Proof Let $G = S_{m,n}$. Let $V(G) = \{u_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ be the vertex set of

G . Let $E(G) = \{u_0u_{i,1} : 1 \leq i \leq m\} \cup \{u_{i,j}u_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}$ be the edge set of G . So, $|V(G)| = mn + 1$ and $|E(G)| = mn$. Define $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_a\}, a \in N$ by $f(u_0) = l_0$. For $i = 1, 2, \dots, m-2$ and $i \equiv 1 \pmod{2}$, $f(u_{i,j}) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$. For $i = 1, 2, \dots, m-1$ and $i \equiv 0 \pmod{2}$, $f(u_{i,j}) = l_{ni+2-2j}, 1 \leq j \leq n$. For $s = 1, 2, \dots, \frac{n-3}{3}$ $f(u_{m,j}) = l_{(m-1)n+2(j+1)-3s}, 3s-2 \leq j \leq 3s$. and for $s = \frac{n}{3}, f(u_{m,j}) = l_{(m-1)n+2(j+1)-3s}, 3s-2 \leq j \leq 3s-1$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^m \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^m \{|f(u_0) - f(u_{i,1})|\} \\ &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^m \{|l_0 - l_{n(i-1)+1}|\} = \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^m \{l_{n(i-1)+1}\} \\ &= \{l_1, l_{2n+1}, l_{4n+1}, \dots, l_{(m-1)n+1}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^m \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^m \{|f(u_0) - f(u_{i,1})|\} \\ &= \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^m \{|l_0 - l_{ni}|\} = \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^m \{l_{ni}\} \\ &= \{l_{2n}, l_{4n}, \dots, l_{(m-1)n}\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\ &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \bigcup_{j=1}^{n-1} \{|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}|\} = \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \bigcup_{j=1}^{n-1} \{l_{n(i-1)+2j}\} \\ &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \{l_{n(i-1)+2}, l_{n(i-1)+4}, \dots, l_{n(i-1)+2n-2}\} \\ &= \{l_2, l_4, \dots, l_{2n-2}\} \cup \{l_{2n+2}, l_{2n+4}, \dots, l_{4n-2}\} \cup \\ &\quad \dots \cup \{l_{(m-3)n+2}, l_{(m-3)n+4}, \dots, l_{(m-3)n+2n-2}\} \\ &= \{l_2, l_4, \dots, l_{2n-2}, l_{2n+4}, \dots, l_{4n-2}, \dots, l_{(m-3)n+2}, l_{(m-3)n+4}, \dots, l_{mn-n-2}\}, \end{aligned}$$

$$\begin{aligned}
E_4 &= \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
&= \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} = \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} \\
&= \bigcup_{i \equiv 0 \pmod{2}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-1}, l_{ni-3}, \dots, l_{ni-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \dots, l_3\} \cup \{l_{4n-1}, l_{4n-3}, \dots, l_{2n+3}\} \cup \\
&\quad \dots \cup \{l_{(m-1)n-1}, l_{(m-1)n-3}, \dots, l_{mn-3n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \dots, l_3, l_{4n-1}, l_{4n-3}, \dots, l_{2n+3}, \dots, l_{(m-1)n-1}, l_{(m-1)n-3}, \dots, l_{mn-3n+3}\},
\end{aligned}$$

$$\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|l_{n(m-1)+2j-3s+2} - l_{n(m-1)+2j-3s+4}| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{l_{n(m-1)+2j-3s+3} : 3s-2 \leq j \leq 3s-1\} \\
&= \{l_{n(m-1)+2}, l_{n(m-1)+4}\} \cup \{l_{n(m-1)+5}, l_{n(m-1)+7}\} \cup \\
&\quad \dots \cup \{l_{n(m-1)+2n-10-n+3+3}, l_{n(m-1)+2n-8-n+3+3}\} \\
&= \{l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \dots, l_{n(m-1)+n-4}, l_{n(m-1)+n-2}\} \\
&= \{l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \dots, l_{mn-4}, l_{mn-2}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, \dots, \frac{n-3}{3}$. Let

$$\begin{aligned}
E_6 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_{m,j}u_{m,j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : j = 3s\} \\
&= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \dots, |f(u_{m,n-3}) - f(u_{m,n-2})|\} \\
&= \{|l_{(m-1)n+8-3} - l_{(m-1)n+10-6}|, |l_{(m-1)n+14-6} - l_{(m-1)n+16-9}|, \\
&\quad \dots, |l_{(m-1)n+2n-4-n+3} - l_{(m-1)n+2n-2-n}|\} \\
&= \{|l_{(m-1)n+5} - l_{(m-1)n+4}|, |l_{(m-1)n+8} - l_{(m-1)n+7}|, \dots, |l_{(m-1)n+n-1} - l_{(m-1)n+n-2}|\} \\
&= \{|l_{(m-1)n+3}, l_{(m-1)n+6}, \dots, l_{(m-1)n+n-3}|\} \\
&= \{l_{(m-1)n+3}, l_{(m-1)n+6}, \dots, l_{mn-3}\}.
\end{aligned}$$

be the vertex set and edge set of G , respectively. Thus $|V(G)| = mn + t + 1$ and $|E(G)| = mn + t$. Define $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_a\}, a \in N$ by $f(u_0) = l_0$. For $i = 1, 2, \dots, m$ and for $i \equiv 1(\text{mod } 2)$, $f(u_{i,j}) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$. For $i = 1, 2, \dots, m$ and for $i \equiv 1(\text{mod } 2)$, $f(u_{i,j}) = l_{ni-2j+2}, 1 \leq j \leq n$. For $s = 1, 2, \dots, \frac{t-3}{3}$, $f(v_k) = l_{mn+2k-3s+2}, 3s-2 \leq k \leq 3s$ and for $s = \frac{t}{3}$, $f(v_k) = l_{mn+2k-3s+2}, 3s-2 \leq k \leq 3s-1$. We claim that the edge labels are distinct. Let

$$\begin{aligned}
 E_1 &= \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{f_1(u_0 u_{i,1})\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{|f(u_0) - f(u_{i,1})|\} = \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{|l_0 - l_{n(i-1)+1}|\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{l_{n(i-1)+1}\} = \{l_1, l_{2n+1}, l_{4n+1}, \dots, l_{n(m-1)+1}\},
 \end{aligned}$$

$$\begin{aligned}
 E_2 &= \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{f_1(u_0 u_{i,1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{|f(u_0) - f(u_{i,1})|\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{|l_0 - l_{ni}|\} = \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{l_{ni}\} = \{l_{2n}, l_{4n}, \dots, l_{mn}\},
 \end{aligned}$$

$$\begin{aligned}
 E_3 &= \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \bigcup_{j=1}^{n-1} \{f_1(u_{i,j} u_{i,j+1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \bigcup_{j=1}^{n-1} \{|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}|\} = \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \bigcup_{j=1}^{n-1} \{l_{n(i-1)+2j}\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 1(\text{mod } 2)}}^m \{l_{n(i-1)+2}, l_{n(i-1)+4}, \dots, l_{n(i-1)+2n-2}\} \\
 &= \{l_2, l_4, \dots, l_{2n-2}\} \cup \{l_{2n+2}, l_{2n+4}, \dots, l_{4n-2}\} \cup \\
 &\quad \dots \cup \{l_{n(m-2)+2}, l_{n(m-2)+4}, \dots, l_{mn-2}\} \\
 &= \{l_2, l_4, \dots, l_{2n-2}, l_{2n+2}, l_{2n+4}, \dots, l_{4n-2}, \dots, l_{n(m-2)+2}, l_{n(m-2)+4}, \dots, l_{mn-2}\},
 \end{aligned}$$

$$\begin{aligned}
E_4 &= \bigcup_{i \equiv 1 \pmod{2}}^m \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{i \equiv 1 \pmod{2}}^m \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
&= \bigcup_{i \equiv 1 \pmod{2}}^m \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} \\
&= \bigcup_{i \equiv 1 \pmod{2}}^m \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} = \bigcup_{i \equiv 0 \pmod{2}}^m \{l_{ni-1}, l_{ni-3}, \dots, l_{ni-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \dots, l_3\} \cup \{l_{4n-1}, l_{4n-3}, \dots, l_{2n+3}\} \cup \dots \cup \{l_{mn-1}, l_{mn-3}, \dots, l_{mn-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \dots, l_3, l_{4n-1}, l_{4n-3}, \dots, l_{2n+3}, \dots, l_{mn-1}, l_{mn-3}, \dots, l_{mn-2n+3}\},
\end{aligned}$$

$$E'_1 = \{f_1(u_0v_1)\} = \{|f(u_0) - f(v_1)|\} = \{|l_0 - l_{mn+1}|\} = \{l_{mn+1}\},$$

$$\begin{aligned}
E'_2 &= \bigcup_{s=1}^{\frac{t-3}{3}} \{f_1(v_kv_{k+1}) : 3s-2 \leq k \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{|f(v_k) - f(v_{k+1})| : 3s-2 \leq k \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{|l_{mn+2k+2-3s} - l_{mn+2k+4-3s}| : 3s-2 \leq k \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{l_{mn+2k+3-3s} : 3s-2 \leq k \leq 3s-1\} \\
&= \{l_{mn+2}, l_{mn+4}\} \cup \{l_{mn+5}, l_{mn+7}\} \cup \dots \cup \{l_{mn+t-4}, l_{mn+t-2}\} \\
&= \{l_{mn+2}, l_{mn+4}, l_{mn+5}, l_{mn+7}, \dots, l_{mn+t-4}, l_{mn+t-2}\}
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for integers $s = 1, 2, \dots, \frac{t-3}{3}$. Let

$$\begin{aligned}
E'_3 &= \bigcup_{s=1}^{\frac{t-3}{3}} \{f_1(u_{3s}u_{3s+1})\} \\
&= \{|f(u_{3s}) - f(u_{3s+1})|\} \\
&= \{|f(u_3) - f(u_4)|, |f(u_6) - f(u_7)|, \dots, |f(u_{t-3}) - f(u_{t-2})|\} \\
&= \{|l_{mn+8-3} - l_{mn+10-6}|, |l_{mn+14-6} - l_{mn+16-9}|, \dots, |l_{mn+2t-4-t+3} - l_{mn+2t-2-t}|\} \\
&= \{|l_{mn+5} - l_{mn+4}|, |l_{mn+8} - l_{mn+7}|, \dots, |l_{mn+t-1} - l_{mn+t-2}|\} \\
&= \{l_{mn+3}, l_{mn+6}, \dots, l_{mn+t-3}\}.
\end{aligned}$$

For $s = \frac{t}{3}$, let

$$\begin{aligned}
 E'_4 &= \{f_1(v_k v_{k+1}) : 3s - 2 \leq k \leq 3s - 1\} \\
 &= \{|f(v_k) - f(v_{k+1})| : 3s - 2 \leq k \leq 3s - 1\} \\
 &= \{|l_{mn+2t-4+2-t} - l_{mn+2t-2+2-t}|, |l_{mn+2t-2+2-t} - l_{mn+2t+2-t}|\} \\
 &= \{|l_{mn+t-2} - l_{mn+t+2}|, |l_{mn+t} - l_{mn+t+1}|\} = \{l_{mn+t-1}, l_{mn+t+1}\}.
 \end{aligned}$$

Now, $E = \bigcup_{i=1}^4 (E_i \cup E'_i) = \{l_1, l_2, \dots, l_{mn}, \dots, l_{mn+t-1}, l_{mn+t+1}\}$. So, the edge labels of G are distinct. Therefore, f is an almost Lucas graceful graph. Thus $G = S_{m,n} @ P_t$ is an almost Lucas graceful graph when $m \equiv 0(\text{mod } 2)$ and $t \equiv 0(\text{mod } 3)$.

Example 2.7 An almost Lucas graceful labeling on $S_{4,7} @ P_6$ is shown in Fig.2.2.

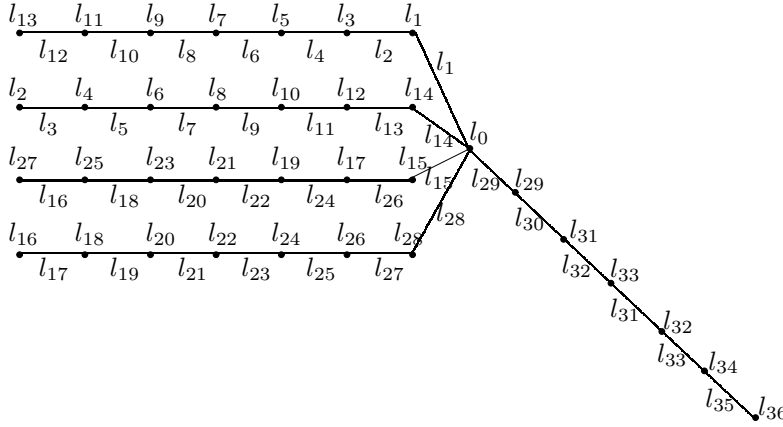


Fig.2.2 $S_{4,7} @ P_6$

Definition 2.8([2]) The graph $G = F_m @ P_n$ consists of a fan F_m and a path P_n of length n which is attached with the maximum degree of the vertex of F_m .

Theorem 2.9 $F_m @ P_n$ is almost Lucas graceful graph when $n \equiv 0(\text{mod } 3)$.

Proof Let $v_1, v_2, \dots, v_{m+1}$ and u_0 be the vertices of a Fan F_m . Let $u_1, u_2, \dots, u_n$ be the vertices of a path P_n . Let $G = F_m @ P_n$, $|V(G)| = m + n + 2$ and $|E(G)| = 2m + n + 1$. Define $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_{q+2}\}$ by $f(u_0) = l_0$; $f(v_i) = l_{2i-1}$; $f(u_j) = l_{2m+2j-3s+3}$, $3s - 2 \leq j \leq 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned}
 E_1 &= \bigcup_{i=1}^m \{f_1(v_i v_{i+1})\} = \bigcup_{i=1}^m \{|f(v_i) - f(v_{i+1})|\} \\
 &= \bigcup_{i=1}^m \{|l_{2i-1} - l_{2i+1}|\} \\
 &= \bigcup_{i=1}^m \{l_{2i}\} = \{l_2, l_4, \dots, l_{2m}\},
 \end{aligned}$$

$$\begin{aligned}
E_2 &= \bigcup_{i=1}^{m+1} \{f_1(u_0v_i)\} = \bigcup_{i=1}^{m+1} \{|f(u_0) - f(v_i)|\} \\
&= \bigcup_{i=1}^{m+1} \{|l_0 - l_{2i-1}|\} = \bigcup_{i=1}^{m+1} \{l_{2i-1}\} = \{l_1, l_3, \dots, l_{2m+1}\},
\end{aligned}$$

$$E_3 = \{f_1(u_0u_1)\} = \{|f(u_0) - f(u_1)|\} = \{|l_0 - l_{2m+2}|\} = \{l_{2m+2}\},$$

$$\begin{aligned}
E_4 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_ju_{j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_j) - f(u_{j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|\} \cup \{|f(u_4) - f(u_5)|, |f(u_5) - f(u_6)|\} \cup \\
&\quad \dots \cup \{|f(u_{n-5}) - f(u_{n-4})|, |f(u_{n-4}) - f(u_{n-3})|\} \\
&= \{|l_{2m+2} - l_{2m+4}|, |l_{2m+4} - l_{2m+6}|\} \cup \{|l_{2m+5} - l_{2m+7}|, |l_{2m+7} - l_{2m+9}|\} \cup \\
&\quad \dots \cup \{|l_{2m+2n-10+3-n+3} - l_{2m+2n-8+3-n+3}|, |l_{2m+2n-8+3-n+3} - l_{2m+2n-6+3-n+3}|\} \\
&= \{l_{2m+3}, l_{2m+5}\} \cup \{l_{2m+6}, l_{2m+8}\} \cup \dots \cup \{l_{2m+n-3}, l_{2m+n-1}\} \\
&= \{l_{2m+3}, l_{2m+5}, l_{2m+6}, l_{2m+8}, \dots, l_{2m+n-3}, l_{2m+n-1}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for $s = 1, 2, \dots, \frac{n}{3} - 1$. Let

$$\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n}{3}-1} \{f_1(u_ju_{j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n}{3}-1} \{|f(u_j) - f(u_{j+1})| : j = 3s\} \\
&= \{|l_{2m+6+3-3} - l_{2m+8+3-6}|, |l_{2m+12+3-6} - l_{2m+14+3-9}|\}, \\
&\quad \dots, |l_{2m+2n-6+3-n+3} - l_{2m+2n-4+3-n}|\} \\
&= \{|l_{2m+6} - l_{2m+5}|, |l_{2m+9} - l_{2m+8}|, |l_{2m+n} - l_{2m+n-1}|\} \\
&= \{l_{2m+4}, l_{2m+7}, \dots, l_{2m+n-2}\}.
\end{aligned}$$

For $s = \frac{n}{3}$, let

$$\begin{aligned}
E_6 &= \{f_1(u_ju_{j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \{|f(u_j) - f(u_{j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\
&= \{|l_{2m+2n-4+3-n} - l_{2m+2n-2+3-n}|, |l_{2m+2n-2+3-n} - l_{2m+2n+3-n}|\} \\
&= \{|l_{2m+n-1} - l_{2m+n+1}|, |l_{2m+n+1} - l_{2m+n+3}|\} \\
&= \{l_{2m+n}, l_{2m+n+2}\}.
\end{aligned}$$

Now, $E = \bigcup_{i=1}^6 E_i = \{l_1, l_2, \dots, l_{2m}, l_{2m+1}, l_{2m+2}, \dots, l_{2m+n-2}, l_{2m+n-1}, l_{2m+n}, l_{2m+n+2}\}$. So, the edge labels of G are distinct. Therefore, f is an almost Lucas graceful labeling.

Thus $G = F_m @ P_n$ is an almost Lucas graceful graph when $n \equiv 0 \pmod{3}$. $\square$

Example 2.10 An almost Lucas graceful labeling on $F_5 @ P_6$ is shown in Fig.2.3.

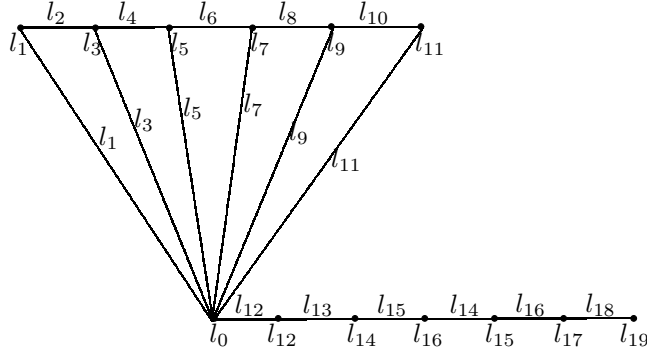


Fig.2.3 $F_5 @ P_6$

§3. Nearly Lucas Graceful Graphs

In this section, we show that the graphs $S_{m,n} @ P_t$ and C_n are nearly Lucas graceful graphs.

Definition 3.1 Let G be a (p, q) - graph. An injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_a\}$, ($a \in \mathbb{N}$), is said to be nearly Lucas graceful labeling if the induced edge labeling $f_1(u, v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \dots, l_{i-1}, l_{i+1}, l_{i+2}, \dots, l_{j-1}, l_{j+1}, l_{j+2}, \dots, l_{k-1}, l_{k+1}, l_{k+2}, \dots, l_b\}$ ($b \in \mathbb{N}$ and $b \leq a$) with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be nearly Lucas graceful graph.

Theorem 3.2 $S_{m,n} @ P_t$ is a nearly Lucas graceful graph when $n \equiv 1, 2 \pmod{3}$ $m \equiv 1 \pmod{2}$ and $t = 1, 2 \pmod{3}$

Proof Let $G = S_{m,n} @ P_t$ with $V(G) = \{u_0, u_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \cup \{v_k : 1 \leq k \leq t\}$. Let $E(G) = \{u_0 u_{i,j} : 1 \leq i \leq m\} \cup \{u_{i,j} u_{i,j+1} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \cup \{u_0 v_1\} \cup \{v_k v_{k+1} : 1 \leq k \leq t-1\}$ be the edge set of G . So, $|V(G)| = mn + t + 1$ and $|E(G)| = mn + t$. Define $f : V(G) \rightarrow \{l_0, l_1, \dots, l_a\}$, $a \in \mathbb{N}$ by $f(u_0) = l_0$. For $i = 1, 2, \dots, m$ and for $i \equiv 1 \pmod{2}$ $f(u_{i,j}) = l_{n(i-1)+2j-1}$, $1 \leq j \leq n$. For $i = 1, 2, \dots, m$ and for $i \equiv 0 \pmod{2}$, $f(u_{i,j}) = l_{in-2j+2}$, $1 \leq j \leq n$. For $s = 1, 2, \dots, \frac{n-2}{3} - 1$ or $s = 1, 2, \dots, \frac{n-1}{3} - 1$ or $s = 1, 2, 3, \dots, \frac{n}{3} - 1$, $f(u_{m,j}) = l_{mn+2(j+1)-3s}$, $3s-2 \leq j \leq 3s$. For $s = \frac{n-2}{3}$ or $\frac{n-1}{3}$ or $\frac{n}{3}$, $f(u_{m,j}) = l_{mn+2(j+1)-3s}$, $3s-2 \leq j \leq 3s-1$. For $r = 1, 2, \dots, \frac{t-2}{3}$ or $r = 1, 2, \dots, \frac{t-1}{3}$ or $r = 1, 2, 3, \dots, \frac{t}{3}$, $f(v_k) = l_{mn+2k+3-3r}$, $3r-2 \leq j \leq 3r-1$. We claim that the edge labels

are distinct. Let

$$\begin{aligned}
 E_1 &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^m \{f_1(u_0 u_{i,1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^m \{|f(u_0) - f(u_{i,j})|\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^m \{|l_0 - l_{(i-1)n+1}|\} = \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^m \{l_{(i-1)n+1}\} = \{l_1, l_{2n+1}, \dots, l_{(m-1)n+1}\},
 \end{aligned}$$

$$\begin{aligned}
 E_2 &= \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^m \{f_1(u_0 u_{i,1})\} = \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^m \{|f(u_0) - f(u_{i,1})|\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^m \{l_0 - l_{in}\} = \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^m \{l_{in}\} = \{l_{2n}, l_{4n}, \dots, l_{(m-1)n}\},
 \end{aligned}$$

$$\begin{aligned}
 E_3 &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j} u_{i,j+1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \bigcup_{j=1}^{n-1} \{|l_{(i-1)n+2j-1} - l_{(i-1)n+2j+1}|\} = \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \bigcup_{j=1}^{n-1} \{l_{(i-1)n+2j}\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 1 \pmod{2}}}^{m-2} \{l_{(i-1)n+2}, l_{(i-1)n+4}, \dots, l_{(i-1)n+2n-2}\} \\
 &= \{l_2, l_4, \dots, l_{2n-2}\} \cup \{l_{2n+2}, l_{2n+4}, \dots, l_{4n-2}\} \cup \\
 &\quad \dots \cup \{l_{(m-3)n+2}, l_{(m-3)n+4}, \dots, l_{mn-n-2}\} \\
 &= \{l_2, l_4, \dots, l_{2n-2}, l_{2n+2}, l_{2n+4}, \dots, l_{4n-2}, \dots, l_{(m-3)n+2}, l_{(m-3)n+4}, \dots, l_{mn-n-2}\},
 \end{aligned}$$

$$\begin{aligned}
 E_4 &= \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^{m-1} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j} u_{i,j+1})\} = \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^{m-1} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^{m-1} \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} = \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} \\
 &= \bigcup_{\substack{i=1 \\ i \equiv 0 \pmod{2}}}^{m-1} \{l_{in-1}, l_{in-3}, \dots, l_{in-2n+3}\} \\
 &= \{l_{2n-1}, l_{2n-3}, \dots, l_3\} \cup \{l_{4n-1}, l_{4n-3}, \dots, l_{2n+3}\} \cup \\
 &\quad \dots \cup \{l_{(m-1)n-1}, l_{(m-1)n-3}, \dots, l_{mn-3n+3}\} \\
 &= \{l_{2n-1}, l_{2n-3}, \dots, l_{4n-1}, l_{4n-3}, \dots, l_{2n+3}, \dots, l_{(m-1)n-1}, l_{(m-1)n-3}, \dots, l_{mn-3n+3}\}.
 \end{aligned}$$

For $n \equiv 1 \pmod{3}$ and $s = 1, 2, \dots, \frac{n-4}{3}$, let

$$\begin{aligned}
 E_5 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \leq j \leq 3s-1\} \\
 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \leq j \leq 3s-1\} \\
 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{|l_{(m-1)n+2j-3s+2} - l_{(m-1)n+2j-3s+4}| : 3s-2 \leq j \leq 3s-1\} \\
 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{l_{(m-1)n+2j-3s+3} : 3s-2 \leq j \leq 3s-1\} \\
 &= \{l_{(m-1)n+2}, l_{(m-1)n+4}\} \cup \{l_{(m-1)n+5}, l_{(m-1)n+7}\} \cup \dots \cup \{l_{(m-1)n+n-4}, l_{(m-1)n+n-2}\} \\
 &= \{l_{(m-1)n+2}, l_{(m-1)n+4}, l_{(m-1)n+5}, l_{(m-1)n+7}, \dots, l_{mn-4}, l_{mn-2}\}.
 \end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for integers $s = 1, 2, \dots, \frac{n-4}{3}$. Let

$$\begin{aligned}
 E_6 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_{m,j}u_{m,j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : j = 3s\} \\
 &= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \dots, |f(u_{m,n-1}) - f(u_{m,n})|\} \\
 &= \{|l_{(m-1)n+5} - l_{(m-1)n+4}|, |l_{(m-1)n+7}|, \dots, |l_{(m-1)n+2n-2-n+1} - l_{(m-1)n+2n+2-n-2}|\} \\
 &= \{l_{(m-1)n+3}, l_{(m-1)n+6}, \dots, l_{mn-1}\}.
 \end{aligned}$$

For $s = \frac{n-1}{3}$, Let

$$\begin{aligned}
 E_7 &= \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \leq j \leq 3s-1\} \\
 &= \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \leq j \leq 3s-1\} \\
 &= \{|l_{(m-1)n+2n-6+2-n+1} - l_{(m-1)n+2n-4+2-n+1}|, \\
 &\quad |l_{(m-1)n+2n-4+2-n+1} - l_{(m-1)n+2n-2+2-n+1}|\} \\
 &= \{|l_{mn-3} - l_{mn-1}|, |l_{mn-1} - l_{mn+1}|\} = \{l_{mn-2}, l_{mn}\}
 \end{aligned}$$

Now, $E = \bigcup_{i=1}^7 E_i = \{l_1, l_2, \dots, l_{mn}\}$. For $n \equiv 2(\text{mod } 3)$ and integers $s = 1, 2, \dots, \frac{n-2}{3}$,

$$\begin{aligned}
E'_1 &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|l_{(m-1)n+2j+2-3s} - l_{(m-1)n+2j+4-3s}| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{l_{(m-1)n+2j+3-3s} : 3s-2 \leq j \leq 3s-1\} \\
&= \{l_{(m-1)n+2}, l_{(m-1)n+4}\} \cup \{l_{(m-1)n+5}, l_{(m-1)n+7}\} \cup \dots \cup \{l_{(m-1)n+n-3}, l_{(m-1)n+n-1}\} \\
&= \{l_{(m-1)n+2}, l_{(m-1)n+4}, l_{(m-1)n+5}, l_{(m-1)n+7}, \dots, l_{mn-3}, l_{mn-1}\}
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $s+1^{th}$ loop for integers $s = 1, 2, \dots, \frac{n-2}{3}$. Let

$$\begin{aligned}
E'_2 &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_{m,j}u_{m,j+1})j = 3s\} = \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : j = 3s\} \\
&= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \dots, |f(u_{m,n-2}) - f(u_{m,n-1})|\} \\
&= \{|l_{(m-1)n+8-3} - l_{(m-1)n+10-6}|, |l_{(m-1)n+14-6} - l_{(m-1)n+16-9}|, \\
&\quad \dots, |l_{(m-1)n+2n-2-n+2} - l_{(m-1)n+2n-n-1}|\} \\
&= \{|l_{(m-1)n+5} - l_{(m-1)n+4}|, |l_{(m-1)n+8} - l_{(m-1)n+7}|, \\
&\quad \dots, |l_{(m-1)n+n} - l_{(m-1)n+n-1}|\} \\
&= \{l_{(m-1)n+3}, l_{(m-1)n+6}, \dots, l_{mn-2}\}.
\end{aligned}$$

For $s = \frac{n+1}{3}$, let

$$\begin{aligned}
E'_3 &= \{f_1(u_{m,j}u_{m,j+1}) : j = 3s-2\} = \{|f(u_{m,j}) - f(u_{m,j+1})| : j = n-1\} \\
&= \{|f(u_{m,n-1}) - f(u_{m,n})|\} = \{|l_{(m-1)n+2n-n-1} - l_{(m-1)n+2n+2-n-1}|\} \\
&= \{|l_{mn-1} - l_{mn+1}|\} = \{l_{mn}\}.
\end{aligned}$$

Therefore, $E' = \bigcup_{i=1}^3 E'_i$. Let

$$E_0 = \{f_1(u_0v_1)\} = \{|f(u_0) - f(v_1)|\} = \{|l_0 - l_{mn+2}|\} = \{l_{mn+2}\}.$$

For $t \equiv 2 \pmod{3}$ and $r = 1, 2, \dots, \frac{t-2}{3}$, let

$$\begin{aligned}
E_1'' &= \bigcup_{r=1}^{\frac{t-2}{3}} \{f_1(v_k v_{k+1}) : 3r-2 \leq k \leq 3r-1\} \\
&= \bigcup_{r=1}^{\frac{t-2}{3}} \{|f(v_k) - f(v_{k+1})| : 3r-2 \leq k \leq 3r-1\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \cup \{|f(v_4) - f(v_5)|, |f(v_5) - f(v_6)|\} \cup \\
&\quad \dots \cup \{|f(v_{t-4}) - f(v_{t-3})|, |f(v_{t-3}) - f(v_{t-2})|\} \\
&= \{|l_{mn+3+2-3} - l_{mn+3+4-3}|, |l_{mn+3+4-3} - l_{mn+3+6-3}|\} \cup \\
&\quad \{|l_{mn+8+3-6} - l_{mn+10+3-6}|, |l_{mn+10+3-6} - l_{mn+12+3-6}|\} \cup \\
&\quad \dots \cup \{|l_{mn+3+2t-8-t+2} - l_{mn+3+2t-6-t+2}|, |l_{mn+3+2t-6-t+2} - l_{mn+3+2t-4-t+2}|\} \\
&= \{|l_{mn+2} - l_{mn+4}|, |l_{mn+4} - l_{mn+6}|\} \cup \{|l_{mn+5} - l_{mn+7}|, |l_{mn+7} - l_{mn+9}|\} \cup \\
&\quad \dots \cup \{|l_{mn+t-3} - l_{mn+t-1}|, |l_{mn+t-1} - l_{mn+t+1}|\} \\
&= \{l_{mn+3}, l_{mn+5}\} \cup \{l_{mn+6}, l_{mn+8}\} \cup \dots \cup \{l_{mn+t-2}, l_{mn+t}\} \\
&= \{l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \dots, l_{mn+t-2}, l_{mn+t}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop for integers $r = 1, 2, \dots, \frac{t-2}{3}$. Let

$$\begin{aligned}
E_2'' &= \bigcup_{r=1}^{\frac{t-2}{3}} \{f_1(v_k v_{k+1}) : k = 3r\} = \bigcup_{r=1}^{\frac{t-2}{3}} \{|f(v_k) - f(v_{k+1})| : k = 3r\} \\
&= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, \dots, |f(v_{t-2}) - f(v_{t-1})|\} \\
&= \{|l_{mn+3+6-3} - l_{mn+3+8-6}|, |l_{mn+3+12-6} - l_{mn+3+14-9}|, \\
&\quad \dots, |l_{mn+3+2t-4-t+2} - l_{mn+3+2t-2-t-1}|\} \\
&= \{|l_{mn+6} - l_{mn+5}|, |l_{mn+9} - l_{mn+8}|, \dots, |l_{mn+t+1} - l_{mn+t}|\} \\
&= \{l_{mn+4}, l_{mn+7}, \dots, l_{mn+t-1}\}.
\end{aligned}$$

For $s = \frac{t+1}{3}$, let

$$\begin{aligned}
E_3'' &= \{f_1(v_k v_{k+1}) : k = 3r-2\} = \{|f(v_k) - f(v_{k+1})| : k = 3r-2\} \\
&= \{|l_{mn+3+2t-2-t-1} - l_{mn+3+2t-t-1}|\} = \{|l_{mn+t} - l_{mn+t+2}|\} = \{l_{mn+t+1}\}
\end{aligned}$$

Therefore, $E'' = E_0 \cup E_1'' \cup E_2'' \cup E_3'' = \{l_{mn+2}, l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \dots, l_{mn+t-2}, l_{mn+t}, l_{mn+t+1}, l_{mn+4}, l_{mn+7}, \dots, l_{mn+t-1}\}$. Now, $E \cup E'' = \bigcup_{i=1}^7 E_i \cup E_0 \cup E_1'' \cup E_2'' \cup E_3'' = \{l_1, l_2, \dots, l_{mn}, l_{mn+2}, l_{mn+3}, l_{mn+4}, \dots, l_{mn+t-2}, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. So, the edge labels of G are

distinct. For $t \equiv 1 \pmod{3}$ and integers $r = 1, 2, \dots, \frac{t-1}{3}$, let

$$\begin{aligned}
 E_1''' &= \bigcup_{r=1}^{\frac{t-1}{3}} \{f_1(v_k v_{k+1}) : 3r-2 \leq k \leq 3r-1\} \\
 &= \bigcup_{r=1}^{\frac{t-1}{3}} \{|f(v_k) - f(v_{k+1})| : 3r-2 \leq k \leq 3r-1\} \\
 &= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \cup \{|f(v_4) - f(v_5)|, |f(v_5) - f(v_6)|\} \cup \\
 &\quad \dots \cup \{|f(v_{t-3}) - f(v_{t-2})|, |f(v_{t-2}) - f(v_{t-1})|\} \\
 &= \{|l_{mn+3+2-3} - l_{mn+3+4-3}|, |l_{mn+3+4-3} - l_{mn+3+6-3}|\} \\
 &\quad \cup \{|l_{mn+3+8-6} - l_{mn+3+10-6}|, |l_{mn+3+10-6} - l_{mn+3+12-6}|\} \cup \\
 &\quad \dots \cup \{|l_{mn+3+2t-6-t+1} - l_{mn+3+2t-4-t+1}|, |l_{mn+3+2t-4-t+1} - l_{mn+3+2t-2-t+1}|\} \\
 &= \{|l_{mn+2} - l_{mn+4}|, |l_{mn+4} - l_{mn+6}|\} \cup \{|l_{mn+5} - l_{mn+7}|, |l_{mn+7} - l_{mn+9}|\} \cup \\
 &\quad \dots \cup \{|l_{mn+t-2} - l_{mn+t}|, |l_{mn+t} - l_{mn+t+2}|\} \\
 &= \{l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \dots, l_{mn+t-1}, l_{mn+t+1}\}.
 \end{aligned}$$

We find the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop for integers $r = 1, 2, \dots, \frac{t-1}{3}$. Let

$$\begin{aligned}
 E_2''' &= \bigcup_{r=1}^{\frac{t-1}{3}} \{f_1(v_k v_{k+1}) : k = 3r\} \\
 &= \bigcup_{r=1}^{\frac{t-1}{3}} \{|f(v_k) - f(v_{k+1})| : k = 3r\} \\
 &= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, \dots, |f(v_{t-1}) - f(v_t)|\} \\
 &= \{|l_{mn+3+6-3} - l_{mn+3+8-6}|, \dots, |l_{mn+3+2t-2-t+1} - l_{mn+3+2t-t-2}|\} \\
 &= \{|l_{mn+6} - l_{mn+5}|, |l_{mn+9} - l_{mn+8}|, \dots, |l_{mn+t+2} - l_{mn+t+1}|\} \\
 &= \{l_{mn+4}, l_{mn+7}, \dots, l_{mn+t}\}
 \end{aligned}$$

Therefore $E''' = E_0 \cup E_1''' \cup E_2''' = \{l_{mn+2}, l_{mn+3}, \dots, l_{mn+t-1}, l_{mn+t+1}, l_{mn+4}, l_{mn+7}, \dots, l_{mn+t}\} = \{l_{mn+2}, l_{mn+3}, l_{mn+4}, \dots, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. Now, $E \cup E' \cup E''' = \bigcup_{i=1}^4 E_i \cup \left\{ \bigcup_{i=1}^3 E_i' \right\} \cup \{E_0 \cup E_1''' \cup E_2'''\} = \{l_1, l_2, \dots, l_{mn}, l_{mn+2}, l_{mn+3}, \dots, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. So, the edge labels of G are distinct. In both cases, f is a nearly Lucas graceful labeling. Thus $G = S_{m,n} @ P_t$ is a nearly Lucas graceful graph when $m \equiv 1 \pmod{2}$, $n \equiv 1, 2 \pmod{3}$ and $t \equiv 1, 2 \pmod{3}$.

Example 3.3 A nearly Lucas graceful labeling of $S_{5,7} @ P_7$ is shown in Fig.3.1.

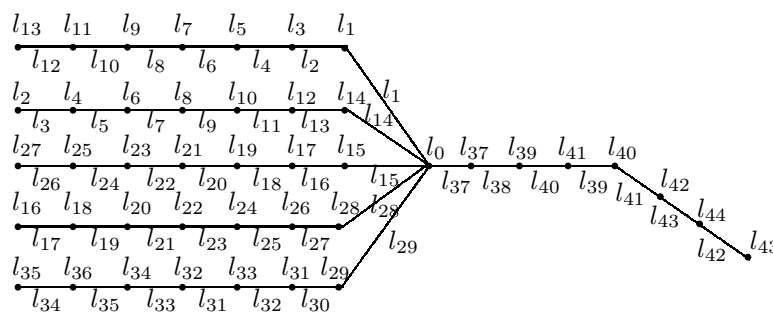


Fig.3.1 $S_{5,7}@P_7$

Proof Let $G = C_n$ with $V(G) = \{u_i : 1 \leq i \leq n\}$. Let $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\}$ be the edge set of G . So, $|V(G)| = n$ and $|E(G)| = n$.

Define $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_a\}$, $a \in N$ by $f(u_1) = l_0$. For $s = 1, 2, \dots, \frac{n-4}{3}$, $f(u_i) = l_{2i-3s}$, $3s-1 \leq i \leq 3s+1$ and for $s = \frac{n-1}{3}$, $f(u_i) = l_{2i-3s}$, $3s-1 \leq i \leq 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \{f_1(u_1u_2), f_1(u_nu_1)\} = \{|f(u_1) - f(u_2)|, |f(u_n) - f(u_1)|\} \\ &= \{|l_0 - l_1|, |l_{2n-n+1} - l_0|\} = \{l_1, l_{n+1}\}. \end{aligned}$$

For $s = 1, 2, \dots, \frac{n-1}{3}$, let

$$\begin{aligned}
E_2 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_i u_{i+1}) \mid 3s-1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_i) - f(u_{i+1})| \mid 3s-1 \leq i \leq 3s\} \\
&= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \bigcup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\
&= \{|l_1 - l_3|, |l_3 - l_5|\} \bigcup \{|l_4 - l_6|, |l_6 - l_8|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{2n-4-n+1} - l_{2n-2-n+1}|, |l_{2n-2-n+1} - l_{2n-n+1}|\} \\
&= \{l_2, l_4\} \bigcup \{l_5, l_7\} \bigcup \{l_{n-2}, l_n\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$

loop for integers $s = 1, 2, \dots, \frac{n-1}{3} - 1$. Let

$$\begin{aligned}
 E_3 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{f_1(u_i u_{i+1}) : i = 3s + 1\} \\
 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{|f(u_i) - f(u_{i+1})| : i = 3s + 1\} \\
 &= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \dots, |f(u_{n-3} - f(u_{n-2}))|\} \\
 &= \{|l_{8-3} - l_{10-6}|, |l_{14-6} - l_{16-9}|, \dots, |l_{2n-6-n+4} - l_{2n-4-n+1}|\} \\
 &= \{|l_5 - l_4|, |l_8 - l_7|, \dots, |l_{n-2} - l_{n-3}|\} = \{l_3, l_6, \dots, l_{n-4}\}
 \end{aligned}$$

Now, $E = \bigcup_{i=1}^3 E_i = \{l_1, l_2, l_3, l_4, \dots, l_{n-2}, l_n, l_{n+1}\}$.

Case 2 $n \equiv 2 \pmod{3}$.

Define $f : V(G) \rightarrow \{l_0, l_1, l_2, \dots, l_n\}$, $a \in N$ by $f(u_1) = l_0$, $f(u_n) = l_{n+2}$. For $s = 1, 2, \dots, \frac{n-2}{3} - 1$, $f(u_i) = l_{2i-3s}$, $3s-1 \leq i \leq 3s+1$ and for $s = \frac{n-2}{3}$, $f(u_i) = l_{2i-3s}$, $3s-1 \leq i \leq 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned}
 E_1 &= \{f_1(u_1 u_2), f_1(u_{n-1} u_n), f_1(u_n u_1)\} \\
 &= \{|f(u_1) - f(u_2)|, |f(u_{n-1}) - f(u_n)|, |f(u_n) - f(u_1)|\} \\
 &= \{|l_0 - l_1|, |l_{2n-2-n+2} - l_{n+2}|, |l_{n+2} - l_0|\} = \{l_1, l_{n+1}, l_{n+2}\},
 \end{aligned}$$

$$\begin{aligned}
 E_2 &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_i u_{i+1}) : 3s-1 \leq i \leq 3s\} \\
 &= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\} \\
 &= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \cup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \cup \\
 &\quad \dots \cup \{|f(u_{n-3}) - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})|\} \\
 &= \{|l_{4-3} - l_{6-3}|, |l_{6-3} - l_{8-3}|\} \cup \{|l_{10-6} - l_{12-6}|, |l_{12-6} - l_{14-6}|\} \cup \\
 &\quad \dots \cup \{|l_{2n-6-n+2} - l_{2n-4-n+2}|\} \\
 &= \{|l_1 - l_3|, |l_3 - l_5|\} \cup \{|l_4 - l_6|, |l_6 - l_8|\} \cup \\
 &\quad \dots \cup \{|l_{n-4} - l_{n-2}|, |l_{n-2} - l_n|\} \\
 &= \{l_2, l_4, l_5, l_7, \dots, l_{n-3}, l_{n-1}\}.
 \end{aligned}$$

We find the edge labeling between the end vertex of $(s-1)^{th}$ loop and the starting vertex of

s^{th} loop for integers $s = 1, 2, \dots, \frac{n-5}{3}$. Let

$$\begin{aligned}
 E_3 &= \bigcup_{s=1}^{\frac{n-5}{3}} \{f_1(u_i u_{i+1}) : i = 3s + 1\} \\
 &= \bigcup_{s=1}^{\frac{n-5}{3}} \{|f(u_i) - f(u_{i+1})| : i = 3s + 1\} \\
 &= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \dots, |f(u_{n-4}) - f(u_{n-3})|\} \\
 &= \{|l_5 - l_4|, |l_8 - l_7|, \dots, |l_{2n-8-n+5} - l_{2n-6-n+2}|\} = \{l_3, l_6, \dots, l_{n-2}\}
 \end{aligned}$$

Now, $E = \bigcup_{i=1}^3 E_i = \{l_1, l_2, l_3, l_4, \dots, l_{n-3}, l_{n-2}, l_{n-1}, l_{n+1}, l_{n+2}\}$ So, all these edge labels of G are distinct. In both the cases, f is a nearly Lucas graceful graph. Thus $G = C_n$ is a nearly Lucas graceful graph when $n \equiv 1, 2 \pmod{3}$. $\square$

Example 3.5 A nearly Lucas graceful labeling on C_{13} in Case 1 is shown in Fig.3.2.

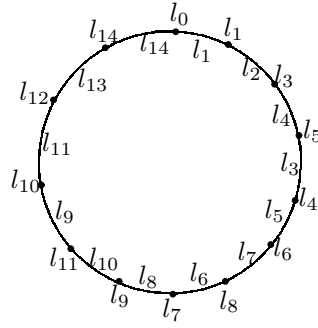


Fig.3.2 C_{13}

Example 3.6 A nearly Lucas graceful labeling on C_{14} in Case 2 is shown in Fig.3.3.

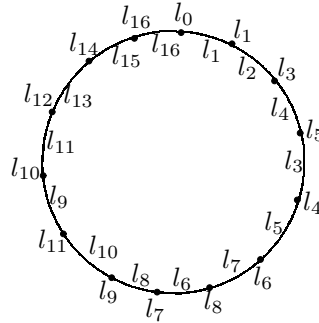


Fig.3.3 C_{14}

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