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Incidence Algebras and Labelings of Graph Structures

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Abstract: Ancykutty Joseph, *On Incidence Algebras and Directed Graphs*, IJMMS, 31:5(2002), 301-305, studied the incidence algebras of directed graphs. We have extended it to undirected graphs also in our earlier paper. We established a relation between incidence algebras and the labelings and index vectors introduced by R.H. Jeurissen in *Incidence Matrix and Labelings of a Graph*, Journal of Combinatorial Theory, Series B, Vol 30, Issue 3, June 1981, 290-301, in that paper. In this paper, we extend the concept to graph structures introduced by E. Sampathkumar in On Generalized Graph Structures, Bull. Kerala Math. Assoc., Vol 3, No.2, Dec 2006, 65-123.

Key Words: Graph structure, R_i-labeling, R_i-index vector, labelling matrix, index matrix, incidence algebra.

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§1. Introduction

Ancykutty Joseph introduced the concept of incidence algebras of directed graphs in [1]. She used the number of directed paths from one vertex to another for introducing the incidence algebras of directed graphs. Stefan Foldes and Gerasimos Meletiou [10] has discussed the incidence algebras of pre-orders also. This motivated us in our study on the incidence algebras of undirected graphs in [8]. We used the number of paths for introducing the concept of incidence algebras of undirected graphs. We also established a relation between incidence algebras and the labelings and index vectors of a graph as given by Jeurissen [12](based on the works of Brouwer [2], Doob [9] and Stewart [15]) in that paper.

E. Sampathkumar introduced the concept of a graph structure in [13] as a generalization of signed graphs. In this paper, we extend the results of our paper on graphs to graph structures and prove that the collection of all R_i-labelings for the collection of all admissible R_i- index vectors, the collection of all R_i-labelings for the index vector 0 and the collection of all R_i-labelings for the index vector $\lambda_i j_i$, ($\lambda_i \in F, F$, a commutative ring j_i an all 1-vector) of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$ are subalgebras of the incidence algebra $I(V, F)$. We also

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prove that the set of labeling matrices for all admissible index matrices of a graph structure is a subalgebra of $I(V^k, F^k)$.

§2. Preliminaries

Throughout this paper, by a ring we mean an associative ring with identity. First We go through the definitions of commutative ring, partially ordered set, pre-ordered set etc. The following definitions are adapted from [16].

Definition 2.1 *A (left) A-module is an additive abelian group M with the operation of (left) multiplication by elements of the ring A that satisfies the following properties.*

(i) $a(x+y) = ax + ay$ *for any* $a \in A, x, y \in M$;

(ii) $(a+b)x = ax + bx$ *for any* $a, b \in A, x \in M$;

(iii) $(ab)x = a(bx)$ *for any* $a, b \in A, x \in M$;

(iv) $1x = x$ *for an* $x \in M$.

By an A-module, we mean a left A-module.

Definition 2.2 *A set* $\{x_1, x_2, ..., x_n\}$ *of elements of M is a basis for M if*

(i) $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$ *for* $a_i \in A$ *only if* $a_1 = a_2 = ... = a_n = 0$ *and*

(ii) M is generated by $\{x_1, x_2, ..., x_n\}$, *i.e., M is the collection of all linear combinations of* $\{x_1, x_2, ..., x_n\}$ *with scalars from A.*

A finitely generated module that has a basis is called free.

Definition 2.3 *An algebra A is a set over a field K with operations of addition, multiplication and multiplication by elements of K that have the following properties.*

(i) A is a vector space with respect to addition and multiplication by elements of the field.

(ii) A is a ring with respect to addition and multiplication.

iii. $(\lambda a)b = a(\lambda b) = \lambda(ab)$ *for any* $\lambda \in K, a, b \in A$.

A subset S of an algebra A is called a subalgebra if it is simultaneously a subring and a subspace of A.

Definition 2.4([14]) *A set X with a binary relation $\leq$ is a pre-ordered set if $\leq$ is reflexive and transitive. If $\leq$ is reflexive, transitive and antisymmetric, then X is a partially ordered set (poset).*

E. Spiegel and C.J. O'Donnell [14] defined incidence algebra as follows.

Definition 2.5([14]) *The incidence algebra $I(X, R)$ of the locally finite partially ordered set X over the commutative ring R with identity is* $I(X, R) = \{f : X \times X \to R | f(x, y) =$

0 *if* x *is not less than or equal to* $y\}$ *with operations given by*

$$\begin{aligned}(f+g)(x,y) &= f(x,y)+g(x,y)\\ (f.g)(x,y) &= \sum_{x\le z\le y} f(x,z).g(z,y)\\ (r.f)(x,y) &= r.f(x,y)\end{aligned}$$

for $f,g\in I(X,R)$ *with* $r\in R$ *and* $x,y,z\in X$.

Ancykutty Joseph [1] established a relation between incidence algebras and directed graphs. The incidence algebra $I(G,Z)$ for digraph without cycles and multiple edges $(G,\le)$ representing the finite poset $(V,\le)$ is defined in [1] as follows.

Definition 2.6([1]) *For* $u,v\in V$, *let* $p_k(u,v)$ *denote the number of directed paths of length* k *from* u *to* v *and* $p_k(v,u)=-p_k(u,v)$. *For* $i=0,1,\cdots,n-1$, *define* $f_i,f_i^*:V\times V\to Z$ *by* $f_i(u,v)=p_i(u,v), f_i^*(u,v)=-p_i(u,v)$. *The incidence algebra* $I(G,Z)$ *of* $(G,\le)$ *over the commutative ring* Z *with identity is defined by* $I(G,Z)=\{f_i,f_i^*:V\times V\to Z, i=0,1,...,n-1\}$ *with operations defined as*

(*i*) *For* $f\ne g$,$(f+g)(u,v)=f(u,v)+g(u,v)$;
(*ii*) $(f.g)(u,v)=\sum_w f(u,w)g(w,v)$;
(*iii*) $(zf)(u,v)=z.f(u,v)\forall z\in Z; f,g\in I(G,Z)$.

In [10], Stefan Foldes and Gerasimos Meletiou says about incidence algebra of pre-order as follows.

Definition 2.7([10]) *Given a field* F, *the incidence algebra* $A(\rho)$, *of a pre-ordered set* $(S,\rho), S=\{1,2,...,n\}$ *over* F *is the set of maps* $\alpha:S^2\to F$ *such that* $\alpha(x,y)=0$ *unless* $x\rho y$. *The addition and multiplication in* $A(\rho)$ *are defined as matrix sum and product.*

Replacing field F by a commutative ring R with identity and following the definition of Foldes and Meletiou[10], we obtained in graphs [8] an analogue of the incidence algebra of a directed graph given by Ancykutty Joseph[1].

Theorem 2.1([8]) *Let* $G=(V,E)$ *be a graph without cycles and multiple edges with* V *and* E *finite. For* $u,v\in V$, *let* $f_i(u,v)$ *be the number of paths of length* i *between* u *and* v. *Then* $\{f_i\}$ *is an incidence algebra of* (G,ρ) *denoted by* $I(G,Z)$ *over the commutative ring* Z *with identity.*

§3. Graph Structure and Incidence Algebra

We recall some basic definitions on graph structure given by E. Sampathkumar[13].

Definition 3.1([13]) $G=(V,R_1,R_2,\cdots,R_k)$ *is a graph structure if* V *is a non empty set and* $R_1,R_2,\cdots,R_k$ *are relations on* V *which are mutually disjoint such that each* $R_i, i=1,2,\cdots,k$, *is symmetric and irreflexive.*

If $(u,v) \in R_i$ for some $i, 1 \leq i \leq k$, (u,v) is an R_i-edge. R_i-path between two vertices u and v consists only of R_i-edges. G is $R_1R_2\cdots R_k$ connected if G is R_i-connected for each i.

We define $R_{i_1i_2\cdots i_r}$-path, $1 \leq r \leq k$, in a similar way as follows.

Definition 3.2 *A sequence of vertices $x_0.x_1, \cdots, x_n$ of V of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$ is an $R_{i_1i_2\cdots i_r}$-path,$1 \leq r \leq k$, if $R_{i_1}, R_{i_2}, \cdots, R_{i_r}$ are some among $R_1, R_2, \cdots, R_k$ which are represented in it.*

Note that the above definition matches with the concepts introduced in [4] by the authors.

Theorem 3.1 *Let $f_i^j(u,v)$ be the number of R_i-paths of length j between u and $f_i^{j*}(u,v) = -f_i^j(u,v)$. $I_{R_i}(G,Z) = \{f_i^j, f_i^{j*} : V \times V \to Z, i = 0,1,...,n-1\}$ is an incidence algebra over Z.*

Proof Let f_i^r and f_i^s be R_i-paths of length r and s respectively. For $f_i^r \neq f_i^s \in I_{R_i}(G,Z)$, define $((f_i^r + f_i^s)(u,v))$ = number of R_i-paths of length either r or s between u and v= $f_i^r(u,v) + f_i^s(u,v)$. Then

$$(f_i^r.f_i^s)(u,v) = \text{number of } R_i\text{-paths of length } r+s \text{ between } u \text{ and } v$$
$$= \sum_{w:(u,w)\in R_i,(w,v)\in R_i} f_i^r(u,w) f_i^s(w,v).$$

$(zf_i^r)(u,v) = z.f_i^r(u,v)\forall z \in Z; f_i^r, f_i^s \in I_{R_i}(G,Z)$ (The operations are extended in the usual way if either or both are elements of the form f_i^{r*}).

So $I_{R_i}(G,Z)$ is an incidence algebra over Z. □

Note 1. We may also consider another type of incidence algebras. Let $f_{i_1i_2\cdots i_r}^l(u,v)$ be the number of $R_{i_1i_2\cdots i_r}$ paths of length l between u and v and $f_{i_1i_2\cdots i_r}^{l*}(u,v) = -f_{i_1i_2\cdots i_r}^l(u,v)$. Then $I_{i_1i_2\cdots i_r}(V,Z) = \{f_{i_1i_2\cdots i_r}^l, f_{i_1i_2\cdots i_r}^{l*} : V \times V \to Z, i = 0,1,\cdots,n-1\}$ with operations defined as follows is another subalgebra over Z.

(i) $(f_{i_1i_2\cdots i_r}^l + f_{i_1i_2\cdots i_r}^m)(u,v) = f_{i_1i_2\cdots i_r}^l(u,v) + f_{i_1i_2\cdots i_r}^m(u,v)$.

(ii) $(f_{i_1i_2\cdots i_r}^l.f_{i_1i_2\cdots i_r}^m)(u,v) = \displaystyle\sum_{w:(u,w),(w,v)\in \bigcup_{i=i_1}^{i_r} R_i} f_{i_1i_2\cdots i_r}^l(u,w) f_{i_1i_2\cdots i_r}^m(w,v)$.

(iii) $(zf_{i_1i_2\cdots i_r}^l)(u,v) = z.f_{i_1i_2\cdots i_r}^l(u,v)\forall z \in Z; f_{i_1i_2\cdots i_r}^l, f_{i_1i_2\cdots i_r}^m \in I_{i_1i_2\cdots i_r}(G,Z)$. (The operations are extended in the usual way if either or both are elements of the form f_i^{r*}).

Thus $I_{i_1i_2\ldots i_r}(V,Z)$ is an incidence algebra over Z.

Note 2. Another possibility is to consider a subalgebra consisting of various paths of the type $R_{i_1i_2\cdots i_r}$ with all of $i_1i_2\cdots i_r$ being different from $j_1j_2\cdots j_s$ for any two $u-v$ paths $f_{i_1i_2\cdots i_r}$ and $f_{j_1j_2\cdots j_s}$. Let $f_{l_1l_2\cdots l_r}^l, f_{m_1m_2\cdots m_s}^m$ be $R_{i_1i_2\cdots i_r}$ and $R_{j_1j_2\cdots,j_s}$-paths of length l and m respectively. Define

$$(f_{i_1i_2\cdots i_r}^l + f_{j_1j_2\cdots j_s}^m)(u,v) = f_{i_1i_2\cdots i_rj_1j_2\cdots j_s}^l(u,v) + f_{i_1i_2\cdots i_rj_1j_2\cdots j_s}^m(u,v),$$

$$(f^j_{i_1i_2\cdots i_r}.f^j_{j_1j_2\cdots j_s})(u,v) = \sum_{w:(u,w),(w,v)\in \bigcup_{i=i_1}^{i_r} R_i} f^l_{i_1i_2\ldots i_r}(u,w)f^m_{j_1j_2\ldots j_s}(w,v),$$

$$(zf^l_{l_1l_2\cdots l_r})(u,v) = z.f^l_{l_1l_2\cdots l_r}(u,v),$$

$$I_{path}(V,Z) = \{f, f^* : V\times V \to Z\},$$

where f is an $R_{i_1i_2\cdots i_r}$-path, $i_1, i_2, \cdots, i_r \in \{1,2,\cdots,k\}, 1 \le r \le k$ and $f^* = -f$. (The operations are extended in the usual way if either or both are elements of the form f^*).

Thus $I_{path}(V,Z)$ is an incidence algebra over Z.

§4. R_i-labelings and Incidence Algebra

Now consider R_i-labelings and R_i-index vectors of G. We recall the concepts of R_i-labelings and R_i-index vectors introduced in [5].

Definition 4.1([5]) *Let F be an abelian group or a ring and $G = (V, R_1, R_2, \cdots, R_k)$ be a graph structure with vertices $v_0, v_1, \cdots, v_{p-1}$ and q_i number of R_i-edges. A mapping $r_i : V \to F$ is an R_i-index vector with components $r_i(v_0), r_i(v_1), \cdots, r_i(v_{p-1}), i = 1, 2, \cdots, k$ and a mapping $x_i : R_i \to F$ is an R_i-labeling with components $x_i(e^1_i), x_i(e^2_i), \cdots, x_i(e^{q_i}_i), i = 1, 2, \cdots, k$.*

An R_i-labeling x_i is an R_i-labeling for the R_i-index vector r_i iff $r_i(v_j) = \sum_{e_r\in E^j_i} x_i(e_r)$, where E^j_i is the set of all R_i-edges incident with v_j. R_i-index vectors for which an R_i-labeling exists are called admissible R_i-index vectors.

Now we prove some results on R_i-labellings and incidence algebras. For that, first we recall the operations of addition and scalar multiplication mentioned in [5].

$$(r^1_i + r^2_i)(v_j) = r^1_i(v_j) + r^2_i(v_j),$$

$$(fr^1_i)(v_j) = fr^1_i(v_j),$$

$$(x^1_i + x^2_i)(e_j) = x^1_i(e_j) + x^2_i(e_j),$$

$$(fx^1_i)(e_j) = fx^1_i(e_j).$$

Now we define multiplication as follows.

Definition 4.2 *Let r^1_i, r^2_i be R_i-index vectors and x^1_i, x^2_i be R_i-labelings of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$.*

$$(r^1_i.r^2_i)(v_l) = \sum_{s:(v_l,v_s)\in R_i} r^1_i(v_l)r^2_i(v_s)$$

$$(x^1_i.x^2_i)(v_l,v_m) = 2.\sum_{s:(v_l,v_s)\in R_i,(v_s,v_m)\in R_i} x^1_i(v_l,v_s)x^2_i(v_s,v_m)$$ *(Multiplication by 2 is to adjust the duplication due to symmetric property of R_i-edges).*

Now we prove that with respect to these operations, the set of all R_i-labelings for all admissible R_i-index vectors is a subalgebra of the incidence algebra $I(V,F)$.

Theorem 4.1 *The set of R_i-labelings for all admissible R_i-index vectors of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$ is a subalgebra of $I_{L(A_i)}(V, F)$ where A_i is the collection of all admissible R_i-index vectors.*

Proof Let $I_{L(A_i)}(V, F)$ be the collection of R_i-labelings for elements of A_i. Let $x_i^1, x_i^2 \in I_{L(A_i)}(V, F)$. Then there exist $r_i^1, r_i^2 \in F$ such that

$$r_i^1(v_j) = \sum_{p:(v_j,v_p)\in R_i} x_i^1(v_j, v_p) \ \text{ and } \ r_i^2(v_j) = \sum_{p:(v_j,v_p)\in R_i} x_i^2(v_j, v_p).$$

$$\begin{aligned}(r_i^1 + r_i^2)(v_j) &= r_i^1(v_j) + r_i^2)(v_j) = \sum_{p:(v_j,v_p)\in R_i} x_i^1(v_j, v_p) + \sum_{p:(v_j,v_p)\in R_i} x_i^2(v_j, v_p) \\ &= \sum_{p:(v_j,v_p)\in R_i} (x_i^1 + x_i^2)(v_j, v_p).\end{aligned}$$

Therefore $x_i^1 + x_i^2$ is an R_i-labeling for $(r_i^1 + r_i^2$, i.e., $x_i^1 + x_i^2 \in I_{L(A_i)}(V, F)$.

$$\begin{aligned}(ri^1.r_i^2)(v_j) &= \sum_{s:(v_jv_s)\in R_i} r_i^1(v_j) r_i^2(v_s) \\ &= \sum_{s:(v_jv_s)\in R_i} [\sum_{l:(v_jv_l)\in R_i} x_i^1(v_j, v_l) \sum_{m:(v_sv_m)\in R_i} x_i^2(v_s, v_m)] \\ &= 2. \sum_{s:(v_jv_s)\in R_i} \sum_{m:(v_sv_m)\in R_i} x_i^1(v_j, v_s) x_i^2(v_s, v_m) \\ &= \sum_{n:(v_jv_n)\in R_i} (x_i^1 x_i^2)(v_j, v_n)\end{aligned}$$

Therefore $x_i^1.x_i^2$ is an R_i-labeling for $r_i^1.r_i^2$. i.e., $x_i^1.x_i^2 \in I_{L(A_i)}(V, F)$.

$$\begin{aligned}(fr_i^1)(v_j) &= f.r_i^1(v_j) \\ &= f. \sum_{n:(v_jv_n)\in R_i} x_i^1(v_j, v_n) \\ &= \sum_{n:(v_jv_n)\in R_i} fx_i^1(v_j, v_n) \\ &= \sum_{n:(v_jv_n)\in R_i} (fx_i^1)(v_j, v_n)\end{aligned}$$

i.e., $fx_i^1 \in I_{L(A_i)}(V, F)$. Hence $I_{L(A_i)}(V, F)$ is a subalgebra of $I(V, F)$. □

For the next few results, we require results from our previous papers [5] and [7].

Theorem 4.2([5]) *If F is an integral domain, the R_i-labelling of G for the R_i-index vector* 0 *form a free F-module.*

Theorem 4.3([7]) *Let F be an integral domain. Then $S_i(G)$, the collection of R_i-labelings for $\lambda_i j_i, \lambda_i \in F, j_i$ an all 1-vector, is a free F-module.*

Theorem 4.4 *The set of R_i-labellings for $\lambda_i j_i, \lambda_i \in F, j_i$ an all 1 vector of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$ forms a subalgebra of the incidence algebra $I(V, F)$.*

Proof Let $I_{L(\lambda_i)}(V,F)$ be the collection of R_i-labelings for $\lambda_i j_i$. Let $x_i^1, x_i^2 \in I_{L(\lambda_i)}(V,F)$. Then there exist $\lambda_i^1, \lambda_i^2 \in F$ such that

$$\lambda_i^1(v_j) = \sum_{p:(v_j v_p)\in R_i} x_i^1(v_j,v_p) \text{ and } \lambda_i^2(v_j) = \sum_{p:(v_j v_p)\in R_i} x_i^2(v_j,v_p).$$

By Theorem 4.3, $\lambda_i j_i$ is an F-module. Hence it is enough if we prove that $x_i^1.x_i^2$ is an R_i-labeling for $(\lambda_i^1.\lambda_i^2)j$

$$\begin{aligned}
(\lambda_i^1.\lambda_i^2)(v_j) &= \sum_{s:(v_j v_s)\in R_i} \lambda_i^1(v_j)\lambda_i^2(v_s)\\
&= \sum_{s:(v_j v_s)\in R_i} [\sum_{l:(v_j v_l)\in R_i} x_i^1(v_j,v_l) \sum_{m:(v_s v_m)\in R_i} x_i^2(v_s,v_m)]\\
&= 2. \sum_{s:(v_j v_s)\in R_i,(v_s v_n)\in R_i} x_i^1(v_j,v_s)x_i^2(v_s,v_n)\\
&= \sum_{n:(v_j v_n)\in R_i} (x_i^1 x_i^2)(v_j,v_n)
\end{aligned}$$

Therefore $x_i^1.x_i^2$ is an R_i-labeling for $\lambda_i^1.\lambda_i^2 = \lambda_i^3$. i.e., $x_i^1.x_i^2 \in I_{L(\lambda_i)}(V,F)$. Hence $I_{L(\lambda_i)}(V,F)$ is a subalgebra of $I(V,F)$. □

Theorem 4.5 *The set of R_i-labelings for 0 of a graph structure $G=(V,R_1,R_2,...,R_k)$ forms a subalgebra of the incidence algebra $I(V,F)$.*

Let $I_{L(0_i)}$ be the collection of all R_i-labelings for 0. By Theorem 4.2, the collection is an F-module. So it is enough if we prove that $x_i^1.x_i^2 \in I_{L(0_i)}(V,F) \forall x_i^1, x_i^2 \in I_{L(0_i)}(V,F)$.

$$\begin{aligned}
\sum_{n:(v_j v_n)\in R_i} (x_i^1.x_i^2)(v_j,v_n) &= 2. \sum_{n:(v_j v_n)\in R_i} [\sum_{s:(v_j v_s)\in R_i,(v_s v_n)\in R_i} x_i^1(v_j,v_s)x_i^2(v_s,v_n)]\\
&= \sum_{s:(v_j v_s)\in R_i} x_i^1(v_j,v_s)[\sum_{n:(v_s v_n)\in R_i} x_i^2(v_s,v_n)]\\
&= \sum_{s:(v_j v_s)\in R_i} x_i^1(v_j,v_s).0(v_s)\\
&= 0
\end{aligned}$$

Therefore $x_i^1.x_i^2$ is an R_i-labeling for 0. ie., $x_i^1.x_i^2 \in I_{L(0_i)}(V,F)$. So $I_{L(0_i)}(V,F)$ is a subalgebra of $I(V,F)$. □

§5. Labeling Matrices and Incidence Algebras

We now establish the relation between labeling matrices and incidence algebras. For that first we recall the concepts of labeling matrices and index matrices of a graph structure introduced by the authors in [6].

Definition 5.1([6]) *Let F be an abelian group or a ring. Let R_i be an R_i-index vector and x_i be an R_i-labeling for $i=1,2,\cdots,k$. Then*

$$x = \begin{bmatrix} x_1 & 0 & . & . & . & 0 \\ 0 & x_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & . & x_k \end{bmatrix}$$

is a labeling matrix and

$$r = \begin{bmatrix} r_1 & 0 & . & . & . & 0 \\ 0 & r_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & . & r_k \end{bmatrix}$$

is an index matrix for the graph structure $G = (V, R_1, R_2, \cdots, R_k)$.

$$x : \begin{bmatrix} R_1 \\ R_2 \\ . \\ . \\ . \\ R_k \end{bmatrix} \to F^k$$

is a labeling for $r : V^k \to F^k$ *if* $\sum_{m \in E_s} x_i(m) = r_i(x_s)$ *for* $s = 0, 1, \cdots, p-1; i = 1, 2, \cdots, k$. *If* r_i *is an admissible* R_i*-index vector* $i = 1, 2, \cdots, k$, *then* r *is called an admissible index matrix for* G.

Now we establish some relations between these and incidence algebras.

Theorem 5.1 *The set of labeling matrices for all admissible index matrices of a graph structure* $G = (V, R_1, R_2, \cdots, R_k)$ *is a subalgebra of* $I(V^k, F^k)$.

Proof Let $I_{L(A)}(V^k, F^k)$ be the set of all labeling matrices for the elements of A, the set of all admissible index matrices. Let $x_1, x_2 \in I_{L(A)}(V^k, F^k)$. Then $x_i^1, x_i^2 \in I_{L(A_i)}(V, F)$, the set of all R_i-labelings for the elements of the set A_i of all admissible R_i-index vectors for $i = 1, 2, \cdots, k$. Then as proved in Theorem 4.1, $x_i^1 + x_i^2, x_i^1.x_i^2, fx_i^1 \in I_{L(A_i)}(V, F)$ where $f \in F$. Hence $x^1 + x^2, x^1.x^2, fx^1$ are labelings for some $r^1 + r^2, r^1.r^2, fr^1$ respectively. i.e., $x^1 + x^2, x^1.x^2, fx^1 \in I_{L(A)}(V^k, F^k)$. So $I_{L(A)}(V^k, F^k)$ is a subalgebra of $I(V^k, F^k)$. □

Theorem 5.2 *The set of labeling matrices for* ΛJ *with*

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ 0 & 0 & . & . & 0 & \lambda_k \end{bmatrix}, \quad J = \begin{bmatrix} j_1 & 0 & . & . & . & 0 \\ 0 & j_2 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ 0 & 0 & . & . & 0 & j_k \end{bmatrix},$$

j_i, *an all 1-vector for* $i = 1, 2, ..., k$ *of a graph structure* $G = (V, R_1, R_2, \cdots, R_k)$ *is a subalgebra of* $I(V^k, F^k)$.

Proof Let $I_{L(\Lambda)}(V^k, F^k)$ be the set of all labeling matrices for the index matrix Λ. Let $x_1, x_2 \in I_{L(\Lambda)}(V^k, F^k)$. Then $x_i^1, x_i^2 \in I_{L(\lambda_i)}(V, F)$, the set of all R_i-labellings for λ_i for $i = 1, 2, ..., k$. Then as proved in Theorem 4.4, $x_i^1 + x_i^2, x_i^1.x_i^2, fx_i^1 \in I_{L(\lambda_i)}(V, F)$ where $f \in F$. Hence $x^1 + x^2, x^1.x^2, fx^1$ are labelings for $\Lambda^1 + \Lambda^2, \Lambda^1.\Lambda^2, f\Lambda^1$ respectively, i.e.,$x^1 + x^2, x^1.x^2, fx^1 \in I_{L(\Lambda)}(V^k, F^k)$. So $I_{L(\Lambda)}(V^k, F^k)$ is a subalgebra of $I(V^k, F^k)$. □

Theorem 5.3 *The set of labeling matrices for* 0 *of a graph structure* $G = (V, R_1, R_2, ..., R_k)$ *is a subalgebra of* $I(V^k, F^k)$.

Proof Let $I_{L(0)}(V^k, F^k)$ be the set of all labeling matrices for the index matrix 0. Let $x_1, x_2 \in I_{L(0)}(V^k, F^k)$. Then $x_i^1, x_i^2 \in I_{L(0_i)}(V, F)$, the set of all R_i-labelings for 0 for $i = 1, 2, ..., k$. Then as proved in Theorem 4.5, $x_i^1 + x_i^2, x_i^1.x_i^2, fx_i^1 \in I_{L(0_i)}(V, F)$ where $f \in F$. Hence $x^1 + x^2, x^1.x^2, fx^1$ are labelings for $0 + 0 = 0, 0.0 = 0, f0 = 0$ respectively, i.e., $x^1 + x^2, x^1.x^2, fx^1 \in I_{L(0}(V^k, F^k)$. So $I_{L(0)}(V^k, F^k)$ is a subalgebra of $I(V^k, F^k)$. □

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