International J.Math. Combin. Vol.3 (2011), 76-81

Graph Theoretic Parameters Applicable to Social Networks

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Abstract: Let $G = (V, E)$ be a graph. When G is used to model the network of a group of individuals, the vertex set V stands for individuals and the edge set E is used to represent the relations between them. If we want a set of representatives having relations with other members of the group, choose a dominating set of the graph. For a smallest set of representatives, choose a minimal dominating set of the graph. In this paper we generalize this concept by allowing the division of the group into a number of subgroups. We introduce the concept of class domination (greed domination) and study it's properties. A dominating set S of G is a class dominating set or a greed dominating set, if $S \bigcap V_i \neq \phi$ for all i. Here V_i such that $i = 1, 2, \ldots, n$ is a partition of V. We also discuss different versions of domination in the context of social networks.

Key Words: Minimal dominating set, greed dominating set, minimal greed dominating set, proportionate greed dominating set.

AMS(2010): 05C69

§1. Introduction

A graph $G = (V, E)$ is a discrete mathematical structure which contains the nonempty set V of vertices and the set E of unordered pairs of elements of V called edges. In this paper we restrict our attention to finite simple graphs. For basic terminology and definitions which are not explained in this paper, reader may refer Harary [4].

Graph is an efficient tool for modeling group of individuals (represented by vertices) and various relationships among them (represented by edges). Consider the problem of selecting representatives from the group, who have good relationship with the remaining members of the group. A dominating set of the graph which model the problem is the solution. The *dominating set* (DS) of a graph $G = (V, E)$ is a subset S of V such that all vertices in $V - S$ is adjacent to at least one vertex in S. A *minimal dominating set* (MDS) is a dominating set S such that $S - \{v\}$ is not a dominating set for all vertex $v \in S$. The *domination number* $\gamma(G)$ and the

[1]Support by sanctioning a minor project No : MRP (S) - 653 / 2007 (X Plan) / KLKE 018 / UGC - SWRO.
[2]Received November 1, 2010. Accepted August 30, 2011.

upper domination number $\Gamma(G)$ of the graph G are defined as follows.

$$\gamma(G) = \min \{|S| : \text{ S is a minimal dominating set of } G\}$$

and

$$\Gamma(G) = \max \{|S| : \text{ S is a minimal dominating set of } G\}.$$

Although the mathematical study of dominating sets in graphs began around 1960, the subject has historical root dating back to 1862 when de Jaenisch [3] studied the problem of determining the minimum number of queens which are necessary to cover an $n \times n$ chessboard. In 1958 Claud Berg [1] wrote a book on graph theory, in which he defined for the first time the concept of domination number of a graph (he called the number, the *coefficient of external stability*). In 1977 Cockayne and Hedetniemi [2] published a survey of the few results known at that time about the dominating sets in graphs. Later the subject has developed as an important area of research with many related areas such as independence, irredundance, packing, covering etc. A comprehensive text on domination is available, which is edited by T. W. Haynes et al. [5]. For advanced research topics, reader may refer another text edited by T. W. Haynes et al. [6].

§2. Greed Domination

A group of people contains Hindus, Christians and Muslims. It is possible that a member of a particular religion has good relation with members of other religion. As a consequence, if we select a minimal set of representatives having good relationship with all other members of the group, the representatives may not contain members from some religion. This situation results into imbalance of social relations. A possible solution is to give due consideration to all subgroups while selecting the representatives. This motivates us to generalize the concept of dominating sets in graphs.

Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots V_n\}$ be a mutually disjoint partition of V. Total number of subsets in the partition P is denoted by $|P|$. A subset S of V is called a *greed dominating set (class dominating set)* of G w.r.t to the partition P, if S dominate all vertices of $V - S$ and $V_i \cap S \neq \phi$ for all $i = 1, 2, \ldots, n$. A greed dominating set S is a *minimal greed dominating set* if no proper subset of S is a greed dominating set. The *greed domination number* $\gamma_{gP}(G)$ and the *upper greed domination number* $\Gamma_{gP}(G)$ of the graph G are defined as follows.

$$\gamma_{gP}(G) = \min \{|S| : \text{ S is a minimal greed dominating set of } G\}$$

and

$$\Gamma_{gP}(G) = \max \{|S| : \text{ S is a minimal greed dominating set of } G\}.$$

When $P = \{V\}$, greed domination coincides with ordinary domination. For any partition P of V, at least one minimal greed dominating set exists. Hence the definitions of $\gamma_{gP}(G)$ and $\Gamma_{gP}(G)$ are meaningful. Let P_1 and P_2 are two partitions of V. We say that P_2 is bigger than P_1 or P_1 is smaller than P_2 if P_2 is obtained by further partitioning one or more subsets of P_1. Two partitions P_1 and P_2 are incomparable, if P_2 is not bigger than P_1 or vice versa.

Theorem 2.1 *If P is a partition of G such that $|P| = n$, then $\gamma_{gP}(G) \geq n$.*

Proof Any minimal dominating set S of G w.r.t the partition $P = \{V_1, V_2, \ldots, V_n\}$ satisfies $S \cap V_i \neq \phi$ for all $i = 1, 2, \ldots, n$. Hence $|S| \geq n$ and $\gamma_{gP}(G) \geq n$. □

Is it possible that for partition P, $\gamma_{gP}(G) > |P|$? The answer is YES. It is illustrated below.

Example 2.2 Consider the graph G, which is the union of the cycles (v_1, v_2, v_3), (v_6, v_7, v_8) and the path (v_3, v_4, v_5, v_6). Clearly $\gamma = 2$. Consider the partition $P = \{V_1, V_2\}$ of V such that, $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V_2 = \{v_7, v_8\}$. For this partition, $\gamma_{gP}(G) = 3 > |P|$.

Theorem 2.3 *If P is a partition such that, $\gamma_{gP}(G) = |P|$, and for any partition P' where P' is bigger than P, obtained by partitioning exactly one subset of P and $|P'| = |P| + 1$, then $\gamma_{gP'}(G) = |P| + 1$.*

Proof Let $S = \{v_1, v_2, \ldots, v_{|P|}\}$ be a minimal greed dominating set of G w.r.t $P = \{V_1, V_2, \ldots, V_{|P|}\}$ such that $v_i \in V_i$ for $i = 1, 2, \ldots, |P|$. Let P' be obtained by further partitioning exactly one of the subsets, say V_1 into to subsets V_{11} and V_{12}. If $v_1 \in V_{11}$ then $v_1 \notin V_{12}$ and vice versa. For the time being let $v_1 \in V_{11}$. Now consider $S' = \{v, v_1, v_2 \ldots, v_{|P|}\}$, where $v \in V_{12}$. Clearly S' is a minimal greed dominating set of G w.r.t the new partition P'. Hence the result. □

Corollary 2.4 *If $P_1, P_2, \ldots, P_n$ are partitions of $V(G)$ satisfying the conditions,*

(i) *P_{i+1} is bigger than P_i;*
(ii) *$|P_{i+1}| = |P_i| + 1$ for each i;*
(iii) *$\gamma_{gP_1}(G) = |P_1|$,*

then $\gamma_{gP_{i+1}}(G) = \gamma_{gP_i}(G) + 1$ for each i.

Next we shall characterize the graphs such that $\gamma_{gP}(G) = |P|$ for each partition P of $V(G)$.

Theorem 2.5 *For the graph G, $\gamma_{gP}(G) = |P|$ for all partition P of $V(G)$ if and only if there exists a vertex $v \in V$ such that $N[v] = V(G)$.*

Proof Suppose the graph G has the property, $\gamma_{gP}(G) = |P|$ for for each partition P of $V(G)$. Consider the partition $P = \{V\}$. Then $\gamma_{gP}(G) = 1$. Hence there exists a vertex $v \in V$ such that $N[v] = V(G)$.

Conversely, Let there exists a vertex $v \in V$ such that $N[v] = V(G)$. Take any partition $P = \{V_1, V_2, \ldots, V_n\}$ of $V(G)$. With no loss of generality we can assume that, $v \in V_1$. Now consider the set $S = \{v, v_2, v_3, \ldots, v_n\}$ made by selecting v from V_1 and an arbitrary vertex v_i from V_i for $i = 2, 3, \ldots, n$. This set is a minimal greed dominating set of G w.r.t the partition P. Hence $\gamma_{gP}(G) = |P|$, by Theorem 2.1. □

Theorem 2.6 *Let P_1 and P_2 are two partitions of V such that P_2 is bigger than P_1, then $\gamma_{gP_1}(G) \leq \gamma_{gP_2}(G)$.*

Proof Suppose that S is a minimal greed dominating set of the graph G w.r.t the partition P_2 such that $\gamma_{gP_2}(G) = |S|$. Then S is a greed dominating set of G w.r.t the partition P_1. Hence $\gamma_{gP_1}(G) \leq \gamma_{gP_2}(G) = |S|$. □

Theorem 2.7 *If γ is the domination number of the graph G, then $V(G)$ has a partition P such that $\gamma_{gP}(G) = \gamma$.*

Proof Let $S = \{v_1, v_2, \ldots, v_\gamma\}$ be a minimal dominating set of G. Consider the partition $P = \{V_1, V_2, \ldots, V_\gamma\}$ of V such that $v_i \in V_i$ for all $i = 1, 2, \ldots, \gamma$. Now $\gamma_{gP}(G) = \gamma$. □

Theorem 2.8 *If P is a partition such that $\gamma_{gP}(G) = \gamma$, then $\gamma_{gP'}(G) = \gamma$ for all partition P' smaller than P.*

Proof Let P' be smaller than P. Then P' is obtained by combining two or more subsets of P. Suppose S' is the smallest minimal greed dominating set of G w.r.t the partition P' and $|S| > \gamma$. Since $\gamma_{gP}(G) = \gamma$, there exists a minimal greed dominating set S w.r.t P such that $|S| = \gamma$. But intersection of S with any subset of P' is nonempty. This gives another minimal greed dominating set of G w.r.t P'. Also $|S| < |S'|$. This is a contradiction. □

§3. Proportionate Greed Domination

A greed dominating set S of the graph G is called a *proportionate greed dominating set* (PGDS) w.r.t. the partition $P = \{V_1, V_2, \ldots, V_n\}$, if $\dfrac{|S \cap Vi|}{|V_i|} = \dfrac{|S \cap Vj|}{|V_j|}$ for all $i, j = 1, 2, \ldots, n$. This idea is a special case of the concept of greed dominating set. A proportionate greed dominating set S is called a minimal proportionate greed dominating set (MPGDS) if no proper subset of S is a proportionate greed dominating set. MPGDS is used to model the problem of selecting representatives from a group of individuals, so that the number of representatives is proportionate to the strength of the subgroups.

Theorem 3.1 *The graph $G = (V, E)$ has a PGDS w.r.t the partition P where $|P| \neq |V|$ if and only if $|V|$ is not a prime number.*

Proof Let S be a PGDS w.r.t the partition $P = \{V_1, V_2, \ldots, V_n\}$ of the graph G. Then by definition of PGDS, $\dfrac{|S \cap Vi|}{|V_i|} = \dfrac{|S \cap Vj|}{|V_j|} = \dfrac{p}{q}$ for all $i, j = 1, 2, \ldots, n$, where p and q are relatively prime positive integers and $q \neq 0$. Clearly, q divides $|S \cap Vi|$ and p divides $|Vi|$ for all i. Then $|V| = \sum_i |V_i|$ is divisible by p. If $p = 1$, then $|V_i| = q \times |S \cap Vi|$ for all i. Now $|V|$ is divisible by q. Hence always $|V|$ is not a prime number.

Conversely, let $|V| = qr$, where $q, r > 1$ and $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of V such that $|V_i| = qr_i$ for all i and $\sum_i r_i = r$. Then the set $S = V$ itself is a PGDS of G w.r.t the given partition. □

If a graph has a PGDS w.r.t. a partition P, then it has an MPGDS. This fact leads to the following result.

Corollary 2.2 *The graph $G = (V, E)$ has an MPGDS w.r.t the partition P where $|P| \neq |V|$ if and only if $|V|$ is not a prime number.*

Theorem 3.3 *If S is a PGDS w.r.t the partition $P = \{V_1, V_2, \ldots, V_n\}$ of the graph G, then $\frac{|S \cap Vi|}{|V_i|} = \frac{|S|}{|V|} = \frac{p}{q}$ for all $i = 1, 2, \ldots, n$.*

Proof Since $\frac{|S \cap Vi|}{|V_i|} = \frac{|S|}{|V|} = \frac{p}{q}$ for all i and $(p, q) = 1$, $|S \cap Vi| = n_i p$ and $|V_i| = n_i q$ where n_i is some positive integer. Then $|S| = \sum_i |S \cap Vi| = \sum_i n_i p$ and $|V| = \sum_i |V_i| = \sum_i n_i q$. Hence the result. □

But in the graphs modeling real situations we cannot ensure the equality of the fractions $\frac{|S \cap Vi|}{|V_i|}$. To deal with these cases we allow variations of the values $\frac{|S \cap Vi|}{|V_i|}$, subject to the condition $|\frac{p}{q} - \frac{|S \cap Vi|}{|V_i|}| \leq \epsilon$, where ϵ has a prescribed value. Using Theorem 3.3 we get an approximate value of $\frac{|S \cap Vi|}{|V_i|}$ for graphs having no PGDS w.r.t the partition P.

§4. Cost Factor of a Partition

If the graph G models a set of people, then $\gamma(G)$ is the minimum number of representatives selected from the group. But in many situations, where considerations of group within group is strong, this is not practical. Consequently selection of more representatives than the minimum required increases the total cost. Another interesting situation arise while establishing communication networks. If radio stations are to be situated at different places in a country, naturally we select those places such that every part of the country receive signals from at least one station. To minimize the total cost, we try to minimize the number of places selected. Then some states may not get a radio station. To solve this problem, every state is given minimum one radio station, which undermines our objective. Keeping this fact in mind we introduce the *cost factor* of the partition P. The cost factor of the partition P is defined as $C_P(G) = \gamma_{gP}(G) - \gamma(G)$. A partition P of $V(G)$ is called a *cost effective partition* if $C_P(G) = 0$. Every graph has at least one cost effective partition.

Theorem 4.1 *Let $G = (V, E)$ be a graph, then*

(i) *G has at least one cost effective partition;*

(ii) *G has exactly one cost effective partition if and only if $\gamma(G) = |V|$.*

Proof The conclusion (i) follows from Theorem 2.7. For (ii), if $\gamma(G) = |V|$ and $P = \{V_1, V_2, \ldots, V_{|V|}\}$ is a partition of V, then $|V_i| = 1$ for each i. If there exists another partition P' such that $|P'| = |V|$, then $P = P'$.

To prove converse part, Let the graph G has exactly one cost effective partition, say $P = \{V_1, V_2, \ldots, V_\gamma\}$. Suppose $\gamma(G) < |V|$. Since P is cost effective, $\gamma_{gP}(G) = \gamma(G)$ and let S be the corresponding greed dominating set. Take the vertex $v \in (V - S)$. If necessary rename the subset of the partition such that, $v \in V_1$. Next consider the new partition $P' = \{V_1 - \{v\}, V_2 \cup \{v\}, V_3, \ldots, V_\gamma\}$. Clearly $|P| = |P'|$ and $\gamma_{gP'}(G) = \gamma(G)$. This contradicts the uniqueness of P. □

§5. Problems for Further Research

Here we present a set of questions which are intended for future research.

(i) We have proved in Theorem 2.6 that, for the partitions P_1 and P_2 of V such that P_2 bigger than P_1, $\gamma_{gP_1}(G) \leq \gamma_{gP_2}(G)$. Is there any relation between $\Gamma_{gP_1}(G)$ and $\Gamma_{gP_2}(G)$?

(ii) Is it possible to characterize the partitions of a graph, so that $\gamma_{gP}(G) = |P|$?

(iii) Find the total number of different partitions of the graph G having domination number γ, such that $\gamma_{gP}(G) = \gamma$.

(iv) The subset S of $V(G)$ is a total dominating set, if every vertex in V is adjacent to at least one vertex in S. Extend the idea of greed domination to total dominating sets of G.

(v) Design an algorithm for computing the values of $\gamma_{gP}(G)$ and $\Gamma_{gP}(G)$ for a given partition P of the graph G.

(vi) Find the total number of cost effective partitions of a given graph with n vertices and having domination number γ.

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