International J.Math. Combin. Vol.3 (2011), 82-87

Forcing (G,D)-number of a Graph

K.Palani

(Department of Mathematics A.P.C. Mahalaxmi College for Women, Thoothukudi, India)

A.Nagarajan

(Department of Mathematics V.O.C. College, Thoothukudi, India)

E-mail: kp5.6.67apcm@gmail.com, nagarajan.voc@gmail.com

Abstract: In [7], we introduced the new concept (G,D)-set of graphs. Let $G = (V, E)$ be any graph. A (G,D)-set of a graph G is a subset S of vertices of G which is both a dominating and geodominating(or geodetic) set of G. The minimum cardinality of all (G,D)-sets of G is called the (G,D)-number of G and is denoted by $\gamma_G(G)$. In this paper, we introduce a new parameter called forcing (G,D)-number of a graph G. Let S be a γ_G-set of G. A subset T of S is said to be a forcing subset for S if S is the unique γ_G-set of G containing T. A forcing subset T of S of minimum cardinality is called a minimum forcing subset of S. The forcing (G,D)-number of S denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of S. The forcing (G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all γ_G-sets S of G and it is denoted by $f_{G,D}(S)$.

Key Words: (G,D)-number, Forcing (G,D)-number, Smarandachely k-dominating set.

AMS(2010): 05C69

§1. Introduction

By a graph G=(V,E), we mean a finite, undirected connected graph without loops and multiple edges. For graph theoretic terminology, we refer [5]. A set of vertices S in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a dominating set of G, i.e., every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number $\gamma(G)$ of G[6]. A u-v geodesic is a u-v path of length d(u,v). A set S of vertices of G is a geodominating (or geodetic) set of G if every vertex of G lies on an x-y geodesic for some x,y in S. The minimum cardinality of a geodominating set is the geodomination (or geodetic) number of G and it is denoted by g(G)[1[-[4]. A (G,D)-set of G is a subset S of V(G) which is both a dominating and geodetic set of G. The minimum cardinality of all (G,D)-sets of G is called the (G,D)-number of G and is denoted by γ_G(G).

[1]Received January 21, 2011. Accepted August 30, 2011.

Any (G,D)-set of G of cardinality γ_G is called a γ_G-set of G[7].In this paper, we introduce a new parameter called forcing (G,D)-number of a graph G. Let S be a γ_G-set of G. A subset T of S is said to be a forcing subset for S if S is the unique γ_G-set of G containing T. A forcing subset T of S of minimum cardinality is called a minimum forcing subset of S. The forcing (G,D)-number of S denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of S. The forcing (G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all γ_G-sets S of G and it is denoted by $f_{G,D}$(S).

§2. Forcing (G,D)-number

Definition 2.1 *Let G be a connected graph and S be a γ_G-set of G. A subset T of S is called a forcing subset for S if S is the unique γ_G-set of G containing T. A forcing subset T of S of minimum cardinality is called a minimum forcing subset for S. The forcing (G,D)-number of S denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of S. The forcing (G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all γ_G-sets S of G and it is denoted by $f_{G,D}(G)$. That is, $f_{G,D}(G) = \min\{f_{G,D}(S)$: S is any γ_G-set of G$\}$.*

Example 2.2 In the following figure,

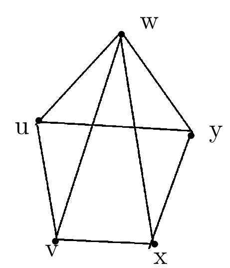

Fig.2.1

$S_1 = \{u, x\}$ and $S_2 = \{v, y\}$ are the only two γ_G-sets of G. $\{u\}, \{x\}$ and $\{u, x\}$ are forcing subsets of S_1. Therefore, $f_{G,D}(S_1) = 1$. Similarly, $\{v\}$, $\{y\}$ and $\{v, y\}$ are the forcing subsets of $f_{G,D}(S_2)$. Therefore, $f_{G,D}(S_2) = 1$. Hence $f_{G,D}(G) = min\{1, 1\} = 1$. For G, we have, $0 < f_{G,D}(G) = 1 < \gamma_G(G) = 2$.

Remark 2.3 1. For every connected graph G, $0 \leqslant f_{G,D}(G) \leqslant \gamma_G(G)$.

2. Here the lower bound is sharp, since for any complete graph $S = V(G)$ is a unique γ_G-set. So, $T = \Phi$ is a forcing subset for S and $f_{G,D}(K_p) = 0$.

3. Example 2.2 proves the bounds are strict.

Theorem 2.4 *Let G be a connected graph. Then,*

(*i*) $f_{G,D}(G) = 0$ *if and only if G has a unique γ_G-set;*

(*ii*) $f_{G,D}(G) = 1$ *if and only if G has at least two γ_G-sets, one of which, say, S has forcing (G,D)-number equal to* 1;

(iii) $f_{G,D}(G) = \gamma_G(G)$ *if and only if every* γ_G*-set* S *of* G *has the property,* $f_{G,D}(S) = |S| = \gamma_G(G)$.

Proof (i) Suppose $f_{G,D}(G) = 0$. Then, by Definition 2.1, $f_{G,D}(S) = 0$ for some γ_G-set S of G. So, empty set is a minimum forcing subset for S. But, empty set is a subset of every set. Therefore, by Definition 2.1, S is the unique γ_G-set of G. Conversely, let S be the unique γ_G-set of G. Then, empty set is a minimum forcing subset of S. So, $f_{G,D}(G) = 0$.

(ii) Assume $f_{G,D}(G) = 1$. Then, by (i), G has at least two γ_G-sets. $f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is any } \gamma_{\rm G} - \text{setof } G\}$. So, $f_{G,D}(S) = 1$ for at least one γ_G-set S. Conversely, suppose G has at least two γ_G-sets satisfying the given condition. By (i), $f_{G,D}(G) \neq 0$. Further, $f_{G,D}(G) \geqslant 1$. Therefore, by assumption, $f_{G,D}(G) = 1$.

(iii) Let $f_{G,D}(G) = \gamma_G(G)$. Suppose S is a γ_G-set of G such that $f_{G,D}(S) < |S| = \gamma_G(\mathrm{G})$. So, S has a forcing subset T such that $|T| < |S|$. Therefore, $f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is a } \gamma_{\rm G} - \text{set of } G\} \leqslant |T| < |S| = \gamma_G(G)$. This is a contradiction. So, every γ_G-set S of G satisfies the given condition. The converse is obvious. Hence the result. □

Corollary 2.5 $f_{G,D}(P_n) = 0$ *if* $n \equiv 1(mod 3)$.

Proof Let $P_n = (v_1, v_2, \ldots, v_{3k+1})$, $k \geqslant 0$. Now, $S = \{v_1, v_4, v_7, \ldots, v_{3k+1}\}$ is the unique γ_G-set of P_n. So, by Theorem 2.4, $f_{G,D}(P_n) = 0$. □

Observation 2.6 *Let* G *be any graph with at least two* γ_G*-sets. Suppose* G *has a* γ_G*-set* S *satisfying the following property:*

$$S \text{ has a vertex } u \text{ such that } u \in S' \text{ for every } \gamma_G\text{-set } S' \text{ different from } S \qquad \text{(I)},$$

Then, $f_{G,D}(G) = 1$.

Proof As G has at least two γ_G-sets, by Theorem 2.4, $f_{G,D}(G) \neq 0$. If G satisfies (I), then we observe that $f_{G,D}(S) = 1$. So, by Definition 2.1, $f_{G,D}(G) = 1$. □

Corollary 2.7 *Let* G *be any graph with at least two* γ_G*-sets. Suppose* G *has a* γ_G*-set* S *such that* $S \bigcap S' = \phi$ *for every* γ_G*-set* S' *different from* S*. Then* $f_{G,D}(G) = 1$.

Proof Given that G has a γ_G-set S such that $S \bigcap S' = \phi$ for every γ_G-set S' different from S. Then, we observe that S satisfies property (I) in Observation 2.6. Hence, we have, $f_{G,D}(G) = 1$. □

Corollary 2.8 *Let* G *be any graph with at least two* γ_G*-sets. If pair wise intersection of distinct* γ_G*-sets of* G *is empty, then* $f_{G,D}(G) = 1$.

Proof The proof proceeds along the same lines as in Corollary 2.7. □

Corollary 2.9 $f_{G,D}(C_n) = 1$ *if* $n = 3k,\ k > 1$.

Proof Let $n = 3k,\ k > 1$. Let $V(C_n) = \{v_1, v_2, \ldots, v_{3k}\}$. Note that the only γ_G-sets of C_n are $S_1 = \{v_1, v_4, \ldots, v_{3(k-1)+1}\}$, $S_2 = \{v_2, v_5, \ldots, v_{3(k-1)+2}\}$ and $S_3 = \{v_3, v_6, \ldots, v_{3k}\}$.

Further, we have, $S_1 \bigcap S_2 = S_1 \bigcap S_3 = S_2 \bigcap S_3 = \emptyset$. That is, pair wise intersection of distinct γ_G-sets of C_n is empty. Hence, from Corollary 2.8, we have $f_{G,D}(C_n) = 1$ if $n = 3k$. □

Definition 2.10 *A vertex v of G is said to be a (G,D)-vertex of G if v belongs to every γ_G-set of G.*

Remark 2.11 1. All the extreme vertices of a graph G are (G,D)-vertices of G.

2. If G has a unique γ_G-set S, then every vertex of S is a (G,D)-vertex of G.

Lemma 2.12 *Let $G = (V, E)$ be any graph and $u \in V(G)$ be a (G,D)-vertex of G. Suppose S is a γ_G-set of G and T is a minimum forcing subset of S, then $u \notin T$.*

Proof Since u is a (G,D)-vertex of G, u is in every γ_G-set of G. Given that S is a γ_G-set of G and T is a minimum forcing subset of S. Suppose $u \in T$. Then, there exists a γ_G-set S' of G different from S such that $T - \{u\} \subseteq S'$. Otherwise, $T - \{u\}$ is a forcing subset of S. Since $u \in S'$, $T \subseteq S'$. This contradicts the fact that T is a minimum forcing subset of S. Hence, from the above arguments, we have $u \notin T$. □

Corollary 2.13 *Let W be the set of all (G,D)-vertices of G. Suppose S is a γ_G-set of G and T is a forcing subset of S. If W is non-empty, then $T \neq S$.*

Definition 2.14 *Let G be a connected graph and S be a γ_G-set of G. Suppose T is a minimum forcing subset of S. Let $E = S - T$ be the relative complement of T in its relative γ_G-set S. Then, $\mathscr{L}$ is defined by*

$$\begin{aligned}\mathscr{L} \;=\; & \{E | E \; is \; a \; relative \; complement \; of \; a \; minimum \\ & forcing \; subset \; T \; in \; its \; relative \; \gamma_G - set \; S \; of \; G\}.\end{aligned}$$

Theorem 2.15 *Let G be a connected graph and ζ = The intersection of all $E \in \mathscr{L}$. Then, ζ is the set of all (G,D)-vertices of G.*

Proof Let W be the set of all (G,D)-vertices of G.

Claim $W = \zeta$, the intersection of all $E \in \mathscr{L}$. Let $v \in W$. By Definition 2.10, v is in every γ_G-set of G. Let S be a γ_G-set of G and T be a minimum forcing subset of S. Then, $v \in S$. From Lemma 2.12, we have, $v \notin T$. So, $v \in E = S - T$. Hence, $v \in E$ for every $E \in \mathscr{L}$. That is, $v \in \zeta$. Conversely, let $v \in \zeta$. Then, $v \in E = S - T$, where T is a minimum forcing subset of the γ_G-set S. So, $v \in S$ for every γ_G-set S of G. That is, $v \in W$. □

Corollary 2.16 *Let S be a γ_G-set of a graph G and T is a minimum forcing subset of S. Then, $W \bigcap T = \emptyset$.*

Remark 2.17 *The above result holds even if G has a unique γ_G-set.*

Corollary 2.18 *Let W be the set of all (G,D)-vertices of a graph G. Then, $f_{G,D}(G) \leqslant \gamma_G(G) - |W|$.*

Remark 2.19 In the above corollary, the inequality is strict. For example, consider the following graph G.

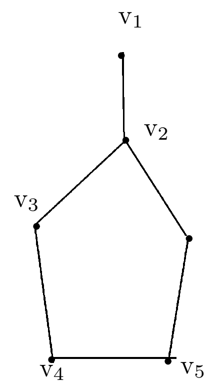

Fig.2.2

For G, $S_1 = \{v_1, v_4, v_5\}, S_2 = \{v_1, v_3, v_5\}, S_3 = \{v_1, v_4, v_6\}$ are the only distinct γ_G-sets. Therefore, $\gamma_G(G) = 3$. But, $f_{G,D}(S_1) = 2$ and $f_{G,D}(S_2) = f_{G,D}(S_3) = 1$. So, $f_{G,D}(G) = \min\{f_{G,D}(S)$: S is a γ_G-set of $G\} = 1$. Also, $W = \{1\}$. Now, $\gamma_G(G) - |W| = 3 - 1 = 2$. Hence $f_{G,D}(G) \leqslant \gamma_G(G) - |W|$.

Also the upper bound is sharp. For example, consider the following graph G.

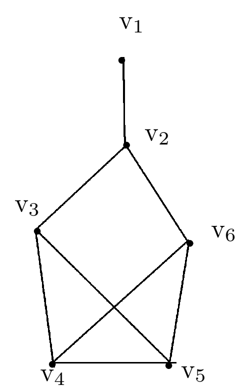

Fig.2.3

For G, $S_1 = \{v_1, v_4, v_5\}, S_2 = \{v_1, v_3, v_6\}$ are different γ_G-sets. Therefore, $\gamma_G(G) = 3$. But, $f_{G,D}(S_1) = f_{G,D}(S_2) = 2$. So, $f_{G,D}(G) = \min\{f_{G,D}(S)$: S is a γ_G-set of $G\} = 2$. Also, $W = \{1\}$. Now, $\gamma_G(G) - |W| = 3 - 1 = 2$. Hence, $f_{G,D}(G) = \gamma_G(G) - |W|$.

Corollary 2.20 *$f_{G,D}(G) \leqslant \gamma_G(G) - k$ where k is the number of extreme vertices of G.*

Proof The result follows from $|W| \geqslant k$. □

Theorem 2.21 *For a complete graph $G = K_p$, $f_{G,D}(G) = 0$ and $|W| = p$.*

Proof $V(K_p)$ is the unique γ_G-set of K_p. Hence by Theorem 2.4, $f_{G,D}(K_p) = 0$. By Remark 2.11, $W = V(G)$ with $|W| = p$. □

References

[1] G.Chartrand , F.Harary and P.Zhang, Geodetic sets in graphs, *Discussiones Mathematicae Graph Theory*, 20(2000),129-138e

[2] G.Chartrand, F.Harary and H.C.Swart P.Zhang, Geodomination in graphs, *Bulletin of the ISA*, 31(2001), 51-59.

[3] G.Chartrand , M.Palmer and P.Zhang, The Geodetic number of a graph-A Survey, *Congressus Numerantium*, 156 (2002), 37-58.

[4] G.Chartrand, F.Harary and P.Zhang, On the Geodetic number of a graph, *Networks*, Vol 39(1)(2002),1-6.

[5] F. Harary, *Graph Theory*, Addison Wesley Reading Mass, 1972.

[6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Decker, Inc., New York 1998.

[7] K.Palani and A.Nagarajan, (G,D)-Number of a Graph, *International Journal of Mathematics Research*, Vol.3, No.3 (2011), 285 - 299.