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Complementary Signed Domination Number of Certain Graphs

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Abstract: Let $G = (V, E)$ be a simple graph, $k \geq 1$ an integer and let $f : V(G) \rightarrow \{-k, k-1, \cdots, -1, 1, \cdots, k-1, k\}$ be $2k$ valued function. If $\sum\limits_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v, then f is a Smarandachely complementary k-signed dominating function on G. The weight of f is defined as $w(f) = \sum_{v \in V} f(v)$ and the Smarandachely complementary k-signed domination number of G is defined as $\gamma_{cs}^{S}(G) = \min\{w(f) : f$ is a minimal complementary signed dominating function of $G\}$. Particularly, a Smarandachely complementary 1-signed dominating function or family is called a complementary singed dominating function or family on G with abbreviated notation $\gamma_{cs}(G)$, the Smarandachely complementary 1-signed domination number of G. In this paper, we determine the value of complementary signed domination number for some special class of graphs. We also determine bounds for this parameter and exhibit the sharpness of the bounds. We also characterize graphs attaining the bounds in some special classes.

Key Words: Smarandachely complementary k-signed dominating function, Smarandachely complementary k-signed dominating number, dominating function, signed dominating function, complementary signed dominating function.

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§1. Introduction

By a graph we mean a finite, undirected connected graph without loops or multiple edges. Terms not defined here are used in the sense of Haynes et. al. [3] and Harary [2].

Let $G = (V, E)$ be a graph with n vertices and m edges. A subset $S \subseteq V$ is called a dominating set of G if every vertex in V-S is adjacent to at least one vertex in S.

A function $f : V \rightarrow \{0, 1\}$ is called a dominating function of G if $\sum_{u \in N[v]} f(u) \geq 1$ for every $v \in V$. Dominating function is a natural generalization of dominating set. If S is a dominating set, then the characteristic function is a dominating function.

Generally, let $f : V(G) \rightarrow \{-k, k-1, \cdots, -1, 1, \cdots, k-1, k\}$ be $2k$ valued function. If $\sum\limits_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v, then f is a *Smarandachely complementary k-signed dominating function* on G. The weight of f is defined

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as $w(f) = \sum_{v \in V} f(v)$ and the *Smarandachely complementary k-signed domination number* of G is defined as $\gamma_{cs}^{S}(G) = \min\{w(f) : f$ is a minimal complementary signed dominating function of $G\}$. Particularly, if $k = 1$, a Smarandachely complementary 1-signed dominating function is a function $f : V \to \{+1, -1\}$ such that $\sum_{u \in N[v]} f(u) \geq 1$ for every $v \in V$ on G with abbreviated notation $\gamma_{cs}^{S}(G) = \gamma_{cs}(G) = \min\{w(f) : f$ is a minimal complementary signed dominating function of $G\}$, the Smarandachely complementary 1-signed domination number of G. Signed dominating function is defined in [1].

Definition 1.1 *A caterpillar is a tree T for which removal of all pendent vertices leaves a path.*

Definition 1.2 *The wheel W_n is defined to be the graph $K_1 + C_{n-1}$ for $n \geq 4$.*

§2. Main Results

Definition 2.1 *A function $f : V \to \{+1, -1\}$ is called a complementary signed dominating function of G if $\sum_{u \notin N[v]} f(u) \geq 1$ for every $v \in V$ with $deg(v) \neq n - 1$. The weight of a complementary signed dominating function f is defined as $w(f) = \sum_{v \in V} f(v)$.*

The complementary signed domination number of G is defined as

$$\gamma_{cs}(G) = \min\{w(f) : f \text{ is a minimal complementary signed dominating function of } G\}.$$

Example 2.2 Consider the graph G given in Fig 2.1

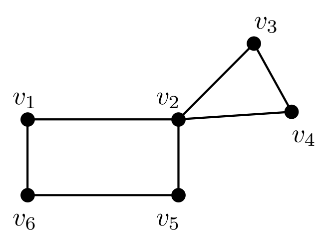

Fig.2.1

Define $f : V(G) \to \{+1, -1\}$ by $f(v_1) = f(v_3) = f(v_4) = f(v_6) = 1$ and $f(v_2) = f(v_5) = -1$. It is easy to observe that f is a minimal complementary signed dominating function with minimum weight and so $\gamma_{cs}(G) = 2$.

Theorem 2.3 *Let T_n be a caterpillar on $2n$ vertices obtained from a path $v_1, v_2, \ldots, v_n$ on n vertices by adding n new vertices $u_1, u_2, \ldots, u_n$ and joining u_i to v_i with an edge for each i. Then $\gamma_{cs}(T_n) = 4$.*

Proof The proof is divided into cases following.

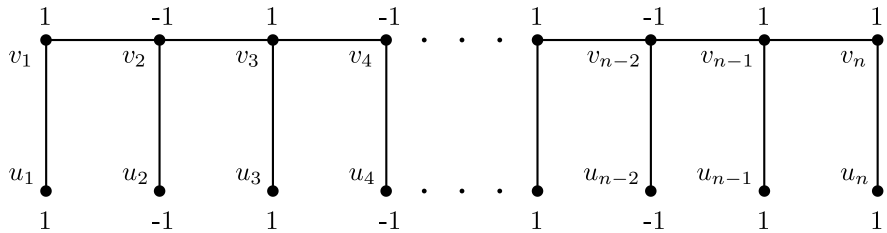

Fig.2.2

Case i n is even.

Define $f : V(T_n) \to \{+1, -1\}$ as follows :

$$f(v_i) = f(u_i) = \begin{cases} +1 & \text{if } 1 \leq \text{i} < \text{n and i is odd,} \\ -1 & \text{if } 2 \leq \text{i} < \text{n and i is even.} \end{cases}$$

$f(v_n) = f(u_n) = +1$. We claim that f is a complementary signed dominating function.

For odd i with $1 \leq i < n$,

$$\sum_{w \notin N[u_i]} f(w) = -(n-2) + [(n-2) - 2] + 4 = 2.$$

Also,

$$\sum_{w \notin N[u_n]} f(w) = -(n-2) + [(n-2) - 2] + 4 = 2.$$

For even i with $2 \leq i < n$,

$$\sum_{w \notin N[u_i]} f(w) = -[(n-2) - 2] + (n-2) + 4 = 6.$$

For $2 \leq i < n-2$,

$$\sum_{w \notin N[v_i]} f(w) = -[(n-2) - 2] + (n-2) - 2 + 4 = 4,$$

$$\sum_{w \notin N[v_1]} f(w) = -[(n-2) - 1] + [(n-2) - 2] + 4 = 3,$$

$$\sum_{w \notin N[v_{n-1}]} f(w) = -[(n-2) - 1] + (n-2) + 4 - 3 = 2,$$

$$\sum_{w \notin N[v_{n-2}]} f(w) = -[(n-2) - 2] + (n-2) - 1 + 4 - 1 = 4,$$

$$\sum_{w \notin N[v_n]} f(w) = -(n-2) + (n-2) + 4 - 3 = 1.$$

Therefore f is a complementary signed dominating function. Since $\sum_{w \notin N[v_n]} f(w) = 1$, the labeling is minimum with respect to the vertices $v_1, v_2, \ldots, v_{n-2}$ and $u_1, u_2, \ldots, u_{n-1}$.

If u_{n-1} is given value -1, then $\sum_{u\notin N[u_n]} f(u) = 0$. It is easy to observe that $\sum_{v\in V[T_n]} f(v) = 4$ is minimum for this particular complementary signed dominating function. Hence $\gamma_{cs}(T_n) = 4$ if n is even.

Case ii n is odd.

Define $f : V(T_n) \to \{+1, -1\}$ as follows :

$$f(v_i) = \begin{cases} +1 & \text{for } 1 \le \text{i} \le \text{n} - 2 \text{ and i is odd,} \\ -1 & \text{for } 2 \le \text{i} \le \text{n} - 3 \text{ and i is even.} \end{cases}$$

and $f(v_{n-1}) = f(v_n) = +1$.

$$f(u_i) = \begin{cases} +1 & \text{if } 1 \le \text{i} \le \text{n} \text{ and i is odd,} \\ -1 & \text{if } 2 \le \text{i} \le \text{n} - 1 \text{ and i is even.} \end{cases}$$

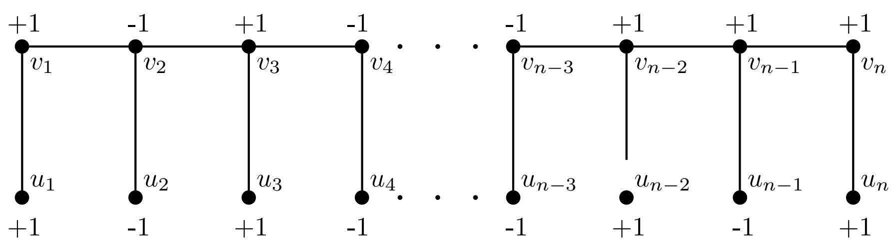

Fig.2.3

We claim that f is a complementary signed dominating function.

For odd i with $1 \le i \le n-4$,

$$\sum_{w\notin N[u_i]} f(w) = -[(n-3)+1] + [(n-3)+1-2] + 4 = 2.$$

For even i with $2 \le i \le n-3$,

$$\sum_{w\notin N[u_i]} f(w) = -[(n-3)+1-2] + (n-3)+1+4 = 6.$$

Also

$$\sum_{w\notin N[u_{n-2}]} f(w) = -[(n-3)+1] + (n-3)+1+4-2 = 2,$$

$$\sum_{w\notin N[u_{n-1}]} f(w) = -[(n-3)+1-1] + [(n-3)+1-1] + 4 = 4$$

and

$$\sum_{w\notin N[u_n]} f(w) = -[(n-3)+1] + [(n-3)+1] + 4 - 2 = 2.$$

For $2 \le i \le n-4$,

$$\sum_{w \notin N[v_i]} f(w) = -[(n-3)+1-2] + [(n-3)+1-2] + 4 = 4,$$
$$\sum_{w \notin N[v_1]} f(w) = -[(n-3)+1-1] + [(n-3)+1-2] + 4 = 3,$$
$$\sum_{w \notin N[v_{n-3}]} f(w) = -[(n-3)+1-2] + [(n-3)+1-1] + 4 - 1 = 4,$$
$$\sum_{w \notin N[v_{n-2}]} f(w) = -[(n-3)+1-1] + [(n-3)+1-1] + 4 - 2 = 2,$$
$$\sum_{w \notin N[v_{n-1}]} f(w) = -[(n-3)+1-1] + [(n-3)+1-1] + 4 - 2 = 2,$$
$$\sum_{w \notin N[v_n]} f(w) = -[(n-3)+1] + [(n-3)+1-1] + 4 - 2 = 1.$$

Therefore f is a complementary signed dominating function. Since $\sum_{w \notin N[v_n]} f(w) = 1$, the labeling is minimum with respect to the vertices $v_1, v_2, \ldots, v_{n-2}$ and $u_1, u_2, \ldots, u_{n-1}$.

If u_{n-2} is given value -1, then $\sum_{w \notin N[v_{n-1}]} f(w) = 0$. It is easy to observe that $\sum_{v \in V[T_n]} f(v) = 4$ is minimum for this particular complementary signed dominating function. Hence $\gamma_{cs}(T_n) = 4$ if n is odd. Therefore $\gamma_{cs}(T_n) = 4$ for all n. □

Theorem 2.4 *Let P_n be a path on n vertices and each vertex of P_n is a support which is adjacent to exactly two pendent vertices. Such a graph is called a caterpillar and denoted by T. Then*

$$\gamma_{cs}(T) = \begin{cases} 3 & \text{if } n \text{ is odd, } n \ge 3 \\ 4 & \text{if } n \text{ is even, } n \ge 4. \end{cases}$$

Proof Let the vertices of the path P_n be $v_1, v_2, \ldots, v_n$ and let each vertex v_i be adjacent to exactly two pendent vertices namely u_i and w_i.

Case i n is odd.

Define $f : V(T) \to \{+1, -1\}$ as follows :

$$f(v_i) = \begin{cases} +1 & \text{if } i \text{ is odd,} \\ -1 & \text{if } i \text{ is even.} \end{cases}$$

$f(u_i) = +1$ for all i and

$$f(w_i) = \begin{cases} +1 & \text{if } i = 1, \\ -1 & \text{if } 2 \le i \le n. \end{cases}$$

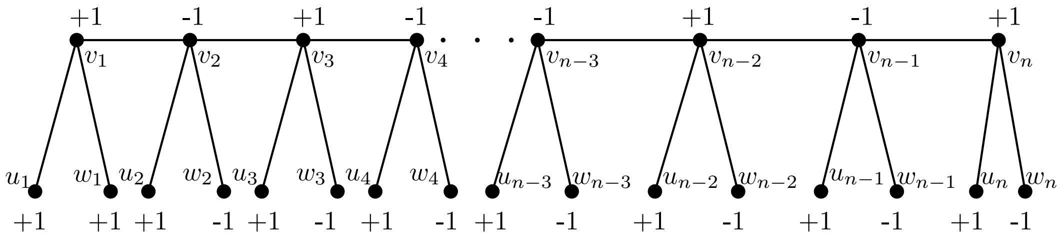

Fig.2.4

We claim that f is a complementary signed dominating function. We have,

$$\sum_{w\notin N[u_1]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left(\frac{n-1}{2}\right) + 2 - 1 + (n-1) - (n-1) = 1.$$

For even i with $2 \le i \le n-1$,

$$\sum_{w\notin N[u_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - \left[\left(\frac{n-1}{2}\right) - 1\right] + 2 + (n-1) - 1 - (n-1) = 3.$$

For odd i with $3 \le i \le n$,

$$\sum_{w\notin N[u_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left(\frac{n-1}{2}\right) + 2 + (n-1) - 1 - (n-1) = 1,$$

$$\sum_{w\notin N[w_1]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left(\frac{n-1}{2}\right) + 2 - 1 + (n-1) - (n-1) = 1,$$

For even i with $2 \le i \le n-1$,

$$\sum_{w\notin N[w_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - \left[\left(\frac{n-1}{2}\right) - 1\right] + 2 + (n-1) - [(n-1) - 1] = 5.$$

For odd i with $3 \le i \le n$,

$$\sum_{w\notin N[w_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left(\frac{n-1}{2}\right) + 2 + (n-1) - [(n-1) - 1] = 3,$$

$$\sum_{w\notin N[v_1]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left[-1 + \left(\frac{n-1}{2}\right)\right] + 2 - 2 + (n-1) - (n-1) = 1.$$

For even i with $2 \le i \le n-1$,

$$\sum_{w\notin N[v_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 2 - \left[\left(\frac{n-1}{2}\right) - 1\right] + 2 + (n-1) - 1 - [(n-1) - 1] = 2.$$

For odd i with $3 \le i < n$,

$$\sum_{w\notin N[v_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left[\frac{n-1}{2} - 2\right] + 2 + (n-1) - 1 - [(n-1) - 1] = 4,$$

$$\sum_{w\notin N[v_n]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left[\left(\frac{n-1}{2}\right) - 1\right] + 2 + (n-1) - 1 - [(n-1) - 1] = 3.$$

Therefore f is a complementary signed dominating function. Since $\sum_{w\notin N[w_1]} f(w) = 1$, the labeling is minimum with respect to the vertices $v_2, v_3, \ldots, v_n$ and $w_2, w_3, \ldots, w_n$, $u_1, u_2, \ldots, u_n$.

If u_1 is given value -1, then $\sum_{w\notin N[v_i]} f(w) = 0$ for even i with $2 \le i \le n-1$. It is easy to observe that $\sum_{v\in V(T)} f(v) = 3$ is minimum for this particular complementary signed dominating function. Therefore $\gamma_{cs}(T) = 3$ if n is odd and $n \ge 3$.

Case ii n is even.

Define $f : V(T) \to \{+1, -1\}$ as follows :

$$f(v_i) = \begin{cases} +1 & \text{for } 1 \le i \le 4 \text{ and } 5 \le i \le n, i \text{ is odd,} \\ -1 & \text{for } 6 \le i \le n \text{and } i \text{ is even.} \end{cases}$$

$f(u_i) = +1$ for $1 \le i \le n$ and $f(w_i) = -1$ for $1 \le i \le n$

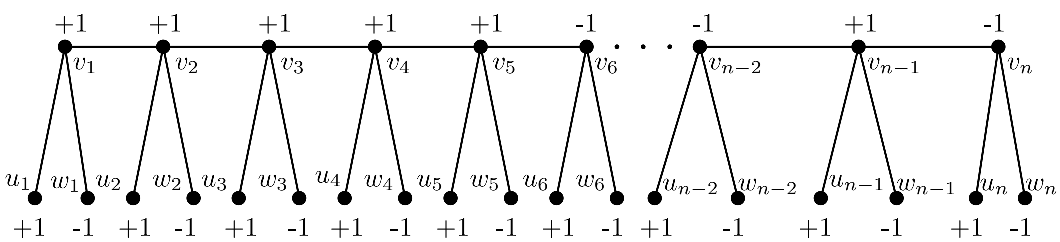

Fig.2.5

We claim that f is a complementary signed dominating function.

$$\sum_{w\notin N[v_1]} f(w) = 4 - 2 + \left(\frac{n-4}{2}\right) - \left(\frac{n-4}{2}\right) + (n-1) - (n-1) = 2.$$

For $i = 2, 3$,

$$\sum_{w\notin N[v_i]} f(w) = 4 - 3 + \left(\frac{n-4}{2}\right) - \left(\frac{n-4}{2}\right) + (n-1) - (n-1) = 1,$$

$$\sum_{w\notin N[v_4]} f(w) = 4 - 2 + \left(\frac{n-4}{2}\right) - 1 - \left(\frac{n-4}{2}\right) + (n-1) - (n-1) = 1,$$

$$\sum_{w\notin N[v_5]} f(w) = 4 - 1 + \left(\frac{n-4}{2}\right) - 1 - \left[\left(\frac{n-4}{2}\right) - 1\right] + (n-1) - (n-1) = 3.$$

For odd i with $7 \le i \le n-1$,

$$\sum_{w\notin N[v_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 1 - \left[\frac{n-4}{2} - 2\right] + (n-1) - (n-1) = 5.$$

For even i with $6 \le i < n$,

$$\sum_{w\notin N[v_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 2 - \left[\left(\frac{n-4}{2}\right) - 1\right] + (n-1) - (n-1) = 3,$$

$$\sum_{w\notin N[v_n]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 1 - \left[\left(\frac{n-4}{2}\right) - 1\right] + (n-1) - (n-1) = 4.$$

For $1 \le i \le 4$,

$$\sum_{w \notin N[u_i]} f(w) = 4 - 1 + \left(\frac{n-4}{2}\right) - \left(\frac{n-4}{2}\right) + (n-1) - n = 2.$$

For odd i with $5 \le i \le n-1$,

$$\sum_{w \notin N[u_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 1 - \left(\frac{n-4}{2}\right) + (n-1) - n = 2.$$

For even i with $6 \le i \le n$,

$$\sum_{w \notin N[u_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - \left[\frac{n-4}{2} - 1\right] + (n-1) - n = 4.$$

For $1 \le i \le 4$,

$$\sum_{w \notin N[w_i]} f(w) = (4-1) + \frac{n-4}{2} - \left(\frac{n-4}{2}\right) + n - (n-1) = 4.$$

For odd i with $5 \le i \le n-1$,

$$\sum_{w \notin N[w_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 1 - \left(\frac{n-4}{2}\right) + n - [n-1] = 4.$$

For even i with $6 \le i \le n$,

$$\sum_{w \notin N[w_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - \left[\left(\frac{n-4}{2}\right) - 1\right] + n - (n-1) = 6.$$

Therefore f is a complementary signed dominating function. Since $\sum_{w \notin N[v_2]} f(w) = 1$, the labeling is minimum with respect to the vertices $v_4, v_5, \ldots, v_n$, $u_1, u_3, \ldots, u_n$ and $w_1, w_3, \ldots, w_n$.

If v_4 is given value -1, then $\sum_{w \notin N[v_1]} f(w) = 0$. It is easy to observe that $\sum_{v \in V(T)} f(v) = 4$ is minimum for this particular complementary signed dominating function. Therefore $\gamma_{cs}(T) = 4$ if n is even and $n \ge 4$. □

Theorem 2.5 *For a bipartite graph $K_{m,n}$,*

$$\gamma_{cs}(K_{m,n}) = \begin{cases} 5 & \text{if exactly one of } m, n \text{ is odd,} \\ 6 & \text{if both } m \text{ and } n \text{ are odd,} \\ 4 & \text{if both } m \text{ and } n \text{ are even,} \end{cases}$$

where $2 \le m \le n$.

Proof Let (V_1, V_2) be the partition of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Let the vertices of V_1 be $v_1, v_2, \ldots, v_m$ and let the vertices of V_2 be $u_1, u_2, \ldots, u_n$. Define $f : V(K_{m,n}) \to \{+1, -1\}$ as follows :

$$f(v_i) = \begin{cases} -1 & \text{if } 1 \le i \le \dfrac{m-2}{2}, \\ +1 & \text{if } \dfrac{m-2}{2} < i \le m, \end{cases}$$

when m is even and

$$f(v_i) = \begin{cases} -1 & \text{if } 1 \le i \le \dfrac{m-3}{2}, \\ +1 & \text{if } \dfrac{m-3}{2} < i \le m, \end{cases}$$

when m is odd

$$f(u_i) = \begin{cases} -1 & \text{if } 1 \le i \le \dfrac{n-2}{2}, \\ +1 & \text{if } \dfrac{n-2}{2} < i \le n, \end{cases}$$

when n is even and

$$f(u_i) = \begin{cases} -1 & \text{if } 1 \le i \le \dfrac{n-3}{2}, \\ +1 & \text{if } \dfrac{n-3}{2} < i \le n, \end{cases}$$

when n is odd.

Case i m is even.

Let v_i be a vertex with $f(v_i) = -1$. Then

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left[\frac{m-2}{2} - 1\right] + m - \left(\frac{m-2}{2}\right)$$
$$= -(m-2) + 1 + m = 3$$

Let v_i be a vertex with $f(v_i) = +1$. Then

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(\frac{m-2}{2}\right) + m - \left(\frac{m-2}{2}\right) - 1$$
$$= -(m-2) + m - 1 = 1$$

Case ii m is odd.

Let v_i be a vertex with $f(v_i) = -1$. Then

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left[\left(\frac{m-3}{2}\right) - 1\right] + m - \left(\frac{m-3}{2}\right)$$
$$= -(m-3) + 1 + m = 4$$

Let v_i be a vertex with $f(v_i) = +1$. Then

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(\frac{m-3}{2}\right) + m - \left(\frac{m-3}{2}\right) - 1 = 2$$

Case iii n is even.

The proof is similar to case (i) replacing m and v_i by n and u_i.

Case iv n is odd.

The proof is similar to case (ii) replacing m and v_i by n and u_i.

If the number of vertices with function -1 is increased by 1, a vertex with function value $+1$ will not satisfy the condition necessary for a complementary signed dominating function. Therefore f is a complementary signed dominating function.

Case I Exactly one of m or n is odd.

When m is even and n is odd, then

$$\begin{aligned}\gamma_{cs}(K_{m,n}) &= \sum_{v\in V(K_{m,n})} f(v)\\ &= (-1)\left(\frac{m-2}{2}\right) + m - \left(\frac{m-2}{2}\right) + (-1)\left(\frac{n-3}{2}\right) + n - \left(\frac{n-3}{2}\right)\\ &= -(m-2) + m - (n-3) + n = 5\end{aligned}$$

When m is odd and n is even

$$\begin{aligned}\gamma_{cs}(K_{m,n}) &= \sum_{v\in V(K_{m,n})} f(v)\\ &= -\left(\frac{m-3}{2}\right) + m - \left(\frac{m-3}{2}\right) - \left(\frac{n-2}{2}\right) + n - \left(\frac{n-2}{2}\right)\\ &= -(m-3) + m - (n-2) + n = 5\end{aligned}$$

Case II Both m and n are even.

$$\begin{aligned}\gamma_{cs}(K_{m,n}) &= -\left(\frac{m-2}{2}\right) + m - \left(\frac{m-2}{2}\right) - \left(\frac{n-2}{2}\right) + n - \left(\frac{n-2}{2}\right)\\ &= -(m-2) + m - (n-2) + n = 4\end{aligned}$$

Case III Both m and n are odd.

$$\begin{aligned}\gamma_{cs}(K_{m,n}) &= -\left(\frac{m-3}{2}\right) + m - \left(\frac{m-3}{2}\right) - \left(\frac{n-3}{2}\right) + n - \left(\frac{n-3}{2}\right)\\ &= -(m-3) + m - (n-3) + n = 6\end{aligned}$$

□

Remark 2.6 $\gamma_{cs}(K_{m,n}) = \gamma_s(K_{m,n})$ for $m, n > 3$.

We observe that $\gamma_{cs}(W_5) = 3$, $\gamma_{cs}(W_6) = 4$, $\gamma_{cs}(W_7) = 1$, $\gamma_{cs}(W_8) = 4$, $\gamma_{cs}(W_9) = 3$ and $\gamma_{cs}(W_{10}) = 2$. We determine $\gamma_{cs}(W_n)$ for $n \geq 11$.

Theorem 2.7 *For the Wheel $W_n = K_1 + C_{n-1}$,*

$$\gamma_{cs}(W_n) = \begin{cases} 4 & \textit{if } n \textit{ is even}\\ 3 & \textit{if } n \textit{ is odd}\end{cases}$$

Proof Let $v_1, v_2, \ldots, v_{n-1}, v$ be the vertices of W_n, where v is the center of the Wheel.

Case i n is even.

Define $f : V(W_n) \longrightarrow \{+1, \ -1\}$ by $f(v_1) = f(v_2) = f(v_3) = f(v_4) = f(v_5) = +1$ and for $6 \le i \le n-1$,

$$f(v_i) = \begin{cases} -1 & \text{if } i \text{ is even,} \\ +1 & \text{if } i \text{ is odd} \end{cases}$$

and $f(v) = -1$. We claim that f is a complementary signed dominating function.

$$\sum_{u \notin N[v_1]} f(u) = 5 - 2 + \left[\left(\frac{n-6}{2}\right) - 1\right] - \left(\frac{n-6}{2}\right) = 2.$$

For $i = 2, \ 3, \ 4$

$$\sum_{u \notin N[v_i]} f(u) = 5 - 3 + \left(\frac{n-6}{2}\right) - \left(\frac{n-6}{2}\right) = 2,$$

$$\sum_{u \notin N[v_5]} f(u) = 5 - 2 + \left(\frac{n-6}{2}\right) - \left[\left(\frac{n-6}{2}\right) - 1\right] = 4,$$

$$\sum_{u \notin N[v_6]} f(u) = 5 - 1 + \left(\frac{n-6}{2}\right) - 1 - \left[\left(\frac{n-6}{2}\right) - 1\right] = 4.$$

If i is odd and $7 \le i \le n-3$, then

$$\sum_{u \notin N[v_i]} f(u) = 5 + \left(\frac{n-6}{2}\right) - 1 - \left[\left(\frac{n-6}{2}\right) - 2\right] = 5 - 1 + 2 = 6.$$

If i is even and $8 \le i < n-1$, then

$$\sum_{u \notin N[v_i]} f(u) = 5 + \left(\frac{n-6}{2}\right) - 2 - \left[\left(\frac{n-6}{2}\right) - 1\right] = 4.$$

Also

$$\sum_{u \notin N[v_{n-1}]} f(u) = 5 - 1 + \left(\frac{n-6}{2}\right) - 1 - \left[\left(\frac{n-6}{2}\right) - 1\right] = 4.$$

Therefore f is a complementary signed dominating function. Since $\sum\limits_{u \notin N[v_4]} f(u) = 2$, the labeling is minimum with respect to the vertices $v_1, \ v_2, \ v_6, \ldots, \ v_{n-1}$. If $f(v_1) = -1$, then $\sum\limits_{u \notin N[v_3]} f(u) = 0$. It is easy to observe that

$$\sum_{u \in V(W_n)} f(u) = 5 + \left(\frac{n-6}{2}\right) - \left(\frac{n-6}{2}\right) - 1 = 4$$

is minimum. Hence $\gamma_{cs}(W_n) = 4$ if n is even.

Case ii n is odd.

Define $f : V(W_n) \longrightarrow \{+1, \ -1\}$ by $f(v_1) = f(v_2) = f(v_3) = f(v_4) = +1$ and for $5 \le i \le n-1$,

$$f(v_i) = \begin{cases} -1 & \text{if } i \text{ is even,} \\ +1 & \text{if } i \text{ is odd} \end{cases}$$

and $f(v) = -1$. We claim that f is a complementary signed dominating function.

$$\sum_{u \notin N[v_1]} f(u) = 4 - 2 + \left(\frac{n-5}{2}\right) - \left[\left(\frac{n-5}{2}\right) - 1\right] = 3.$$

For $i = 2, \ 3$

$$\sum_{u \notin N[v_i]} f(u) = 4 - 3 + \left(\frac{n-5}{2}\right) - \left(\frac{n-5}{2}\right) = 1,$$

$$\sum_{u \notin N[v_4]} f(u) = 4 - 2 + \left(\frac{n-5}{2}\right) - 1 - \left(\frac{n-5}{2}\right) = 1,$$

$$\sum_{u \notin N[v_5]} f(u) = 4 - 1 + \left(\frac{n-5}{2}\right) - 1 - \left[\left(\frac{n-5}{2}\right) - 1\right] = 3.$$

If i is even and $6 \le i \le n-3$, then

$$\sum_{u \notin N[v_i]} f(u) = 4 + \left(\frac{n-5}{2}\right) - 2 - \left[\left(\frac{n-5}{2}\right) - 1\right] = 3.$$

If i is odd and $5 < i < n-1$, then

$$\sum_{u \notin N[v_i]} f(u) = 4 + \left(\frac{n-5}{2}\right) - 1 - \left[\left(\frac{n-5}{2}\right) - 2\right] = 5,$$

$$\sum_{u \notin N[v_{n-1}]} f(u) = 4 - 1 + \left(\frac{n-5}{2}\right) - 1 - \left[\left(\frac{n-5}{2}\right) - 1\right] = 3.$$

Therefore f is a complementary signed dominating function. Since $\sum\limits_{u \notin N[v_2]} f(u) = 1$, the labeling is minimum with respect to the vertices $v_4, \ v_5, \ldots, \ v_{n-1}$. If $f(v_5) = -1$, then $\sum\limits_{u \notin N[v_3]} f(u) < 0$. It is easy to observe that $\sum\limits_{u \in V(W_n)} f(u) = 3$ is minimum. Hence $\gamma_{cs}(W_n) = 3$ if n is odd. □

Theorem 2.8 *For the wheel* $W_n = K_1 + C_{n-1}$, $n \ge 4$, $\gamma_{cs}(W_n) = \gamma_{cs}(C_{n-1}) - 1$.

Proof Let $v_1, v_2, \ldots, v_n, v$ be the vertices of W_n. Now,

$$\begin{aligned} \gamma_{cs}(W_n) &= \sum_{i=1}^{n-1} f(v_i) + f(v) \\ &= \gamma_{cs}(C_{n-1}) - 1 \end{aligned}$$

Hence $\gamma_{cs}(W_n) = \gamma_{cs}(C_{n-1}) - 1$. □

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