International J.Math. Combin. Vol.3 (2011), 55-60

Bounds for Distance$-g$ Domination Parameters in Circulant Graphs

T.Tamizh Chelvam

Department of Mathematics, Manonmaniam Sundaranar University,

Tirunelveli-627 012,Tamil Nadu, India

L.Barani Kumar

Department of Mathematics, Adhiparasakthi Engineering College,

Melmaruvathur 603319, Tamil Nadu, India

E-mail: tamche59@gmail.com, barani_apec@yahoo.com

Abstract: A circulant graph is a Cayley graph constructed out of a finite cyclic group Γ and a generating set A is a subset of Γ. In this paper, we attempt to find upper bounds for distance-g domination , distance-g paired domination and distance-g connected domination number for circulant graphs. Exact values are also determined in certain cases.

Key Words: Circulant graph, Smarandachely distance-g paired-(U,V) dominating $\mathscr{P}$-set, distance-g domination, distance-g paired, total and connected domination, distance-g efficient domination.

AMS(2010): 05C69

§1. Introduction

Let Γ be a finite group with e as the identity. A *generating set* of the group Γ is a subset A such that every element of Γ can be expressed as the product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. The *Cayley graph* $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, xa) | x \in V(G), a \in A\}$ and it is denoted by $Cay(\Gamma, A)$. The exclusion of e from A eliminates the possibility of loops in the graph. When $\Gamma = Z_n$, the Cayley graph $Cay(\Gamma, A)$ is called as *circulant graph* and denoted by $Cir(n, A)$.

Suppose $G = (V, E)$ is a graph, the open neighbourhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of vertices adjacent to v. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. For a set $D \subseteq V$, the open neighbourhood $N(D)$ is defined to be $\bigcup_{v \in D} N(v)$, and the closed neighbourhood of D is $N[D] = N(D) \cup D$. Let $u, v \in V(G)$, then $d(u, v)$ is the length of the shortest $uv-$path. For any $v \in V(G), N^g(v) = \{u \in V(G) : d(u, v) \leq g\}$ and $N^g[v] = N^g(v) \cup \{v\}$. A set $D \subseteq V$,

[1]Received February 12, 2011. Accepted August 20, 2011.

of vertices in G is called a *dominating set* if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. That is $N[D] = V(G)$. The domination number $\gamma(G)$ of G is the minimum cardinality among all the dominating sets in G and the corresponding dominating set is called a γ-set. A set $D \subseteq V$, of vertices in G is called a *distance-g dominating set* if $N^g[D] = V(G)$. The distance$-g$ domination number $\gamma^g(G)$ of G is the minimum cardinality among all the distance$-g$ dominating sets in G and the corresponding distance$-g$ dominating set is called a γ^g-set.

Let G be a graph, $D,\ U,\ V \subset V(G)$ with $U \bigcup V = V(G)$, $U \bigcap V = \emptyset$, $g \geq 1$ an integer and $\langle D\rangle_G$ having graphical property $\mathscr{P}$. If $d(u, D) \leq g$ for $u \in U - D$ but $d(v, D) > g$ for $v \in V - D$, such a vertex subset D is called a *Smarandachely distance-g paired-*(U, V) *dominating* $\mathscr{P}$*-set*. Particularly, if $U = V(G)$, $V = \emptyset$ and $\mathscr{P}$=perfect matching, i.e., a Smarandachely distance-g paired-$(V(G), \emptyset)$ dominating $\mathscr{P}$-set D is called a *distance-g paired dominating set*. The minimum cardinality among all the distance-g paired dominating sets for graph G is the distance-g paired domination number, denoted by $\gamma_p^g(G)$. A set $S \subseteq V$, of vertices in G is called a *distance-g total dominating set* if $N^g(S) = V(G)$. The distance$-g$ total domination number $\gamma_t^g(G)$ of G is the minimum cardinality among all the distance $-g$ total dominating sets in G and the corresponding distance$-g$ total dominating set is called a γ_t^g-set. A set $D \subseteq V$, of vertices in G is said to be *distance-g connected dominating set* if every vertex in $V(G) - D$ is within distance g of a vertex in D and the induced subgraph $< D >$ is $g-$ connected (If $x \in N_g[y]$ for all $x, y \in D$, then x and y are $g-$connected). The minimum cardinality of a distance $-g$ connected dominating set for a graph G is the distance $-g$ connected domination number, denoted by $\gamma_c^g(G)$. A set $D \subseteq V$ is called a *distance-g efficient dominating set* if for every vertex $v \in V$, $|N^g[v] \cap D| = 1$.

The concept of domination for circulant graphs has been studied by various authors and one can refer to [1,6-8] and Rani [9-11] obtained the various domination numbers including total, connected and independent domination numbers for Cayley graphs on Z_n. Paired domination was introduced by Haynes and Slater. In 2008, Joanna Raczek [2] generalized the paired domination and investigated properties of the distance paired domination number of a path, cycle and some non-trivial trees. Raczek also proved that distance$-g$ paired domination problem is NP-complete. Haoli Wang et al. [3] obtained distance$-g$ paired domination number of circulant graphs for a particular kind of generating set. In this paper, we attempt to find the sharp upper bounds for distance$-g$ paired domination number for circulant graphs for a general generating set. The distance version of domination have a strong background of applications. For instance, efficient construction of distance$-g$ dominating sets can be applied in the context of distributed data structure, where it is proposed that distance$-g$ dominating sets can be selected for locating copies of a distributed directory. Also it is useful for efficient selection of network centers for server placement.

Throughout this paper, n is a fixed positive integer, $\Gamma = \mathbb{Z}_n$, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n - a_k, n - a_{k-1}, \ldots, n - a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$, $A_1 = \{a_1, a_2, \ldots, a_k\}$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$ and $d = \max_{1 \leq i \leq k} \{d_i\}$.

§2. Distance-g Domination

In this section, we obtain upper bounds for the distance$-g$ domination number and distance$-g$ efficient domination number. Also whenever the equality occurs we give the corresponding sets.

Theorem 2.1 *Let $n(\geq 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n - a_k, n - a_{k-1}, \ldots, n - a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$, and $G = Cir(n, A)$. If $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d = \max_{1 \leq i \leq k} \{d_i\}$, then $\gamma^g(G) \leq d \lceil \frac{n}{2ga_k + d} \rceil$.*

Proof Let $x = 2ga_k + d$ and $\ell = \lceil \frac{n}{x} \rceil$. Consider the set $D = \{0, 1, \ldots, d-1, x, x+1, \ldots, x+d-1, 2x, 2x+1, \ldots, 2x+d-1, \ldots, (\ell-1)x, (\ell-1)x+1, \ldots, (\ell-1)x+d-1\}$. Note that $|D| = d\ell$ and $ra_i \in N^g[a_i]$, for $1 \leq r \leq g$. Let $v \in V(G)$. By division algorithm, one can write $v = ix + j$ for some i with $0 \leq i \leq \ell - 1$ and $0 \leq j \leq x - 1$. We have the following cases:

Case i Suppose $0 \leq i \leq \ell - 1$ and $0 \leq j \leq ga_k + d - 1$.

SubCase i When $0 \leq j < a_1$, then by the definition of d, $v \in D \subseteq N^g[D]$.

SubCase ii When $a_1 \leq j \leq ga_k + d - 1$, one can write $j = ra_m + t$, for some integers r, m, t with $1 \leq r \leq g$, $1 \leq m \leq k$ and $0 \leq t \leq d - 1$ and so $v = ix + t + ra_m$ where as $ix + t \in D$. Since $ra_m \in N^g[a_m]$, we get $v \in N^g[\{ix, ix+1, \ldots, ix+(d-1)\}] \subseteq N^g[D]$.

Case ii Suppose $0 \leq i \leq \ell - 2$ and $ga_k + d \leq j \leq 2ga_k + d - 1$. Choose an integer h with $1 \leq h \leq ga_k$ such that $v + h = (i+1)x$. One can write $h = ra_m - t$, for some integers r, m, t with $1 \leq r \leq g$, $1 \leq m \leq k$ and $0 \leq t \leq d - 1$ and hence $v + ra_m = (i+1)x + t$, which means that $v \in N^g[\{(i+1)x, (i+1)x+1, \ldots, (i+1)x+(d-1)\}] \subseteq N^g[D]$.

Case iii Suppose $i = \ell - 1$ and $ga_k + d \leq j \leq 2ga_k + d - 1$. As mentioned earlier, one can choose an integer h with $1 \leq h \leq ga_k$ such that $v + h = 0$. Write $h = ra_m - t$ with $1 \leq r \leq m$, $1 \leq m \leq k$ and $0 \leq t \leq d - 1$, which means that $v \in N^g[\{0, 1, 2, \ldots, d-1\}] \subseteq N^g[D]$. Thus D is a distance$-g$ dominating set of G. □

Theorem 2.2 *Let $n(\geq 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{d, 2d, \ldots, kd, n - kd, n - (k-1)d, \ldots, n - d\}$ and $G = Cir(n, A)$. If $d(1 + 2gk)$ divides n, then $\gamma^g(G) = \frac{n}{1 + 2gk}$. In this case, $Cir(n, A)$ has a distance-g efficient dominating set.*

Proof In the notation of the Theorem 2.1, $a_i = id$ for all $1 \leq i \leq k$ and so $d_i = d$. By Theorem 2.1, $D = \{0, 1, \ldots, d-1, x, x+1, \ldots, x+(d-1), 2x, 2x+1, \ldots, 2x+(d-1), \ldots (\ell-1)x, (\ell-1)x+1, \ldots, (\ell-1)x+(d-1)\}$ is a distance-g dominating set and hence $\gamma^g(G) \leq d(\frac{n}{d(1+2gk)}) = \frac{n}{1+2gk}$. Let $n = \ell(d(1+2gk))$. Since $|N^g[v]| = 2gk + 1$, for all $v \in V(G)$, $|D| = \ell d$ and $|N^g[u] \cap N^g[v]| = \emptyset$ for any two distinct vertices $u, v \in D$, we have $\gamma^g(G) = \frac{n}{1+2gk}$. From this, one can conclude that D is a distance-g efficient dominating set in G.

§3. Distance-g Paired Domination, Distance-g Connected Domination and Distance-g Total Domination

In this section, we obtain upper bounds for the distance$-g$ paired domination number, distance$-g$ connected domination number and distance$-g$ total domination number. Also whenever the equality occurs we give the corresponding sets.

Theorem 3.1 *Let $n(\geq 3)$ be a positive integer, $m = \lfloor\frac{n}{2}\rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n-a_k, n-a_{k-1}, \ldots, n-a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$, and $G = Cir(n, A)$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d = \max_{1\leq i\leq k}\{d_i\}$. If $(2g+1)a_k + d$ divides n, then $\gamma_p^g(G) \leq 2d(\frac{n}{(2g+1)a_k+d})$.*

Proof Let $x = (2g+1)a_k + d$, $\ell = \frac{n}{x}$ and $D_p = \{0, 1, \ldots, d-1, a_k, a_k+1, \ldots, a_k+(d-1), x, x+1, \ldots, x+(d-1), a_k+x, a_k+x+1, \ldots, a_k+x+(d-1), \ldots, (\ell-1)x, (\ell-1)x+1, \ldots, (\ell-1)x+(d-1), a_k+(\ell-1)x, a_k+(\ell-1)x+1, \ldots, a_k+(\ell-1)x+(d-1)\}$. Note that $|D_p| = 2d\ell$ and $ra_i \in N^g[a_i]$ for $1 \leq r \leq g$. Let $v \in V(G)$. By division algorithm, one can write $v = ix + j$ for some i, j with $0 \leq i \leq \ell - 1$ and $0 \leq j \leq x - 1$. We have the following cases:

Case i Suppose $0 \leq i \leq \ell - 1$ and $0 \leq j \leq ga_k + (d-1)$.

SubCase i If $0 \leq j < a_1$ then by the definition of d, $v \in N^g[D_p]$.

SubCase ii When $a_1 \leq j \leq ga_k + d - 1$, one can write $j = ra_m + t$, for $1 \leq r \leq g$, $1 \leq m \leq k$ and $0 \leq t \leq d-1$, then $v = ix + ra_m + t$ and so $v \in N^g[\{ix, ix+1, \ldots, ix+(d-1)\}] \subseteq N^g[D_p]$.

Case ii Suppose $0 \leq i \leq \ell - 1$ and $ga_k + d \leq j \leq ga_k + a_k + d - 1$. In this case v can be written as $v = ix + ga_k + h$ where $d \leq h \leq a_k + (d-1)$. By the property of vertex transitivity and by case(i), we have $v \in N^g[\{ix + a_k, ix + a_k + 1, \ldots, ix + a_k + (d-1)\}] \subseteq N^g[D_p]$.

Case iii Suppose $0 \leq i \leq \ell - 1$ and $ga_k + a_k + d \leq j \leq 2ga_k + a_k + d - 1$.

SubCase i Suppose $0 \leq i \leq \ell - 2$. In this case v can be written as $v = (i+1)x + (j-x)$ for some i, j such that $0 \leq i \leq \ell - 2$ and $-ga_k \leq j - x \leq 0$. Thus $v + (x-j) = (i+1)x$ and $0 \leq x - j \leq ga_k$. Hence by case (i), we have $v \in N^g[\{(i+1)x, (i+1)x+1, \ldots, (i+1)x+(d-1)\}] \subseteq N^g[D_p]$.

SubCase ii Suppose $i = \ell - 1$. Then $v \in N^g[\{0, 1, \ldots, d-1\}] \subseteq N^g[D_p]$. Thus D_p is a distance-g dominating set of G. let $D' = \{0, 1, \ldots, d-1, x, x+1, \ldots, x+(d-1), \ldots, (\ell-1)x, (\ell-1)x+1, \ldots, (\ell-1)x+(d-1)\}$. It is note that $D' \subseteq D_p$ and for all $u \in D'$, there exists $v = u + a_k \in D_p$ such that u and v are adjacent in $< D_p >$. Hence $< D_p >$ has a perfect matching and D_p is a distance-g paired dominating set. □

Lemma 3.2 *let $n(\geq 3)$ be a positive integer, $m = \lfloor\frac{n}{2}\rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n-a_k, n-a_{k-1}, \ldots, n-a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$ and $G = Cir(n, A)$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d = \max_{1\leq i\leq k}\{d_i\}$. Then $\gamma_t^g(G) \leq 2d\lceil\frac{n}{(2g+1)a_k+d}\rceil$.*

Proof Let $\ell = \lceil \frac{n}{(2g+1)a_k+d} \rceil$ and let $x = d+(2g+1)a_k$. Then $n = (\ell-1)x+j$ for some $0 \le j \le x-1$. As in the proof of Theorem 2.1, one can prove that $D_t = \{0,1,\ldots,d-1,a_k,a_k+1,\ldots,a_k+(d-1),x,x+1,\ldots,x+(d-1),a_k+x,a_k+x+1,\ldots,a_k+x+(d-1),\ldots,(\ell-1)x,(\ell-1)x+1,\ldots,(\ell-1)x+(d-1),a_k+(\ell-1)x,a_k+(\ell-1)x+1,\ldots,a_k+(\ell-1)x+(d-1)\}$, is a distance-$g$ dominating set. Also note that, for every $z \in D_t$ there exists another adjacent vertex $z+a_k$ or $z-a_k \in D_t$. Thus D_t is a distance-g total dominating set. □

Now we obtain some equality for the distance g-paired domination number in certain classes of circulant graphs.

Corollary 3.3 *Let $n(\ge 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \le k \le m$ and g is a fixed positive integer such that $1 \le g \le m$. Let $A = \{1,2,\ldots,k,n-k,\ldots,n-1\} \subset \mathbb{Z}_n$ and $G = Cir(n,A)$. Then $\gamma_p^g(G) = 2(\frac{n}{(2g+1)k+1})$.*

Proof Take $a_k = k$ in the statement of Theorem 3.1. As $d = 1$ and by Theorem 3.1, one can easily prove $D = \{0,k,x,x+k,\ldots,(\ell-1)x,(\ell-1)x+k\}$ is a distance-g paired dominating set and hence $\gamma_p^g(G) \le 2(\frac{n}{(2g+1)k+1})$. Also,since any two adjacent vertices in D can dominate at most $(2g+1)k+1$ distinct vertices of $G, \gamma_p^g(G) \ge 2(\frac{n}{(2g+1)k+1})$. □

Remark 3.4 Joanna Raczek [2] has proved $\gamma_p^g(C_n) = 2\lceil \frac{n}{2g+2} \rceil$, for $n \ge 3$. This can be obtained by taking $a_k = 1$ and $d = 1$ in Theorem 3.1. Also, Haoli Wang et al. [3] have obtained the distance-g paired domination number for $Cir(n, A = \{1,k\})$ for $k = 2,3$ and 4.

Remark 3.5 The upper bound obtained for distance-g paired domination number matches with the distance-g total domination number. i.e., $\gamma_t^g(G) \le 2d\lceil \frac{n}{(2g+1)a_k+d} \rceil$. In general, for $Cir(n,A)$, the distance-g paired domination number is not equal to distance-g total domination, for all g.

Lemma 3.6 *Let $n(\ge 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \le k \le m$ and g is a fixed positive integer such that $1 \le g \le m$. Let $A = \{a_1,a_2,\ldots,a_k,n-a_k,n-a_{k-1},\ldots,n-a_1\} \subset \mathbb{Z}_n$ with $1 \le a_1 < a_2 < \ldots < a_k \le m$, and $G = Cir(n,A)$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \le i \le k$, $d = \max_{1 \le i \le k}\{d_i\}$, then $\gamma_c^g(G) \le d(1+\lceil \frac{n-(d+2ga_k)}{(d-1)+ga_k} \rceil)$.*

Proof Let $\ell = \lceil \frac{n-(d+2ga_k)}{(d-1)+ga_k} \rceil$ and $D_c = \{0,1,\ldots,d-1,d-1+ga_k,d-1+ga_k+1,\ldots,d-1+ga_k+d-1,d-1+2ga_k,d-1+2ga_k+1,\ldots,d-1+2ga_k+d-1,\ldots,2(d-1+ga_k),2(d-1+ga_k)+1,\ldots,\ell(d-1+ga_k)+d-1,\ell(d-1+ga_k),\ell(d-1+ga_k)+1,\ldots,\ell(d-1+ga_k)+d-1\}$. As in the proof of Theorem 2.1, we can prove D_c is a distance $-g$ dominating set. Since $1 \in A$ and $ra_i \in N^g[a_i]$ for $1 \le r \le g$, $0+j, d-1+ga_k+j, 2(d-1+ga_k)+j, \ldots, \ell(d-1+ga_k)+j$ are $-g$ connected in the induced subgraph $< D_c >$ for each j with $0 \le j \le d-1$. Thus D_c is a distance $-g$ connected dominating set for G with $|D_c| = d(1+\lceil \frac{n-(d+2ga_k)}{(d-1)+ga_k} \rceil)$. □

Remark 3.7 From the above lemma, by replacing $g = 1$, we get the usual connected domination number. i.e., when $g = 1$, $\gamma_c(G) \leq d(1 + \lceil \frac{n-(d+2a_k)}{(d-1)+a_k} \rceil)$.

References

[1] I. J. Dejter and O. Serra, Efficient dominating sets in Cayley graphs, *Discrete Appl.Math.*, 129(2003), 319-328.

[2] Joanna Raczek, Distance paired domination number of graphs, *Discrete Math.* 308(2008), 2473-2483.

[3] Haoli Wang, Xirong Xu, Yuansheng Yang, Guoqing Wang and Kai Lu, On the Distance Paired-Domination of Circulant Graphs, *Malaysian Mathematical Society*, To appear.

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] S. Lakshmivarahan and S. K. Dhall, *Parallel Computing*, 25(1999), 1877-1906.

[6] J. Lee, Independent perfect domination sets in Cayley graphs, *J.Graph Theory*, 37, No.4 (2001), 213-219.

[7] N. Obradovic, J. Peters and Goran Ruzic, Efficient domination in circulant graphs with two chord lengths, *Information Processing Letters*, 102(2007), 253-258.

[8] Jia Huang and Jun-Ming Xu, The bondage numbers and efficient dominations of vertex-transitive graphs, *Discrete Mathematics*, 308 (2008), 571-582.

[9] T. Tamizh Chelvam and I. Rani, Dominating sets in Cayley graphs on $\mathbb{Z}_n$, *Tamkang Journal of Mathematics* 37, No.4(2007), 341-345.

[10] T. Tamizh Chelvam and I. Rani, Independent Domination Number of Cayley graphs on $\mathbb{Z}_n$, *J.Combin.Math.Combin.Comput.* 69(2009), 251-255.

[11] T. Tamizh Chelvam and I. Rani, Total and Connected domination numbers for Cayley graphs on $\mathbb{Z}_n$, *Advanced Studies in Contemporary Mathematics* 20(2010), 57-61.