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Weak and Strong Reinforcement Number For a Graph

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Abstract: Let G=(V(G),E(G)) be a graph.A set of vertices S in a graph G is called to be a Smarandachely dominating k-set, if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a dominating set of G. The Smarandachely domination k-number $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely dominating k-set of G. S is called weak domination set if $deg(u) \leq deg(v)$ for every pair of $(u, v) \in V(G) - S$. The minimum cardinality of a weak domination set S is called weak domination number and denoted by $\gamma_w(G)$. In this paper we introduce the weak reinforcement number which is the minimum number of added edges to reduce the weak dominating number. We give some boundary of this new parameter and trees. Furthermore, some boundary of strong reinforcement number has been given for a given graph G and its complemented graph $\overline{G}$.

Key Words: Connectivity, Smarandachely dominating k-set,Smarandachely dominating k-number, strong or weak reinforcement number.

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§1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $S \subseteq V$ is a Smarandachely dominating k-set of G if every vertex v in $V - S$ there exists a vertex u in S such that u and v are adjacent in G. The Smarandachely domination k-number of G, denoted $\gamma_k(G)$ is the minimum cardinality of a Smarandachely dominating k-set of G [7]. The concept of dominationin graphs, with its many variations, is well studied in graph theory and also many kind of dominating k-numbers have been described. Strong domination(sd-set) and weak domination(sw-set) was introduced by Sampathkumar and Latha [2]. Let $uv \in E$. Then u and v dominate each other. Further, u strongly dominates [weakly dominates] v if $deg(u) \geq deg(v)$ [$deg(u) \leq deg(v)$]. A set $S \subseteq V$ is strong dominating set(sd-set) [weakly dominating set(sw-set)] if every vertex $v \in V - S$ is strongly dominated [weakly dominated] by some u in S. The strong domination number $\gamma_s(G)$ [weak domination number $\gamma_w(G)$] of G is the minimum cardinality of a Smarandachely dominating k-set S [5]. If Smarandachely domination k-number of G is small, then distance between each pair of vertices is small in G. This property is easily see that

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$\gamma_k(G) = \gamma_s(G) = \gamma_w(G) = 1$, where G is complete and the distance between each pairs is 1. If any edge could removed from graph G then the Smarandachely domination k-number of G increase. Fink et al.[4] introduced the bondage number of a graph in 1990. The bondage number $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with Smarandachely domination k-number grater than $\gamma_k(G)$ [1,4,5]. Strong and weak bondage number introduced by Ebadi et al. in 2009 [7]. If some edge added from graph G then the Smarandachely domination k-number of G could decrease. In 1990, Kok and Mynhardt [6] introduced the reinforcement number $r(G)$ of a graph G, which is the minimum number of extra edges whose addition to graph G results in a graph G' with $\gamma_k(G)$ $< \gamma_k(G')$. They defined $r(G) = 0$ if $\gamma_k(G) = 1$. In 1995, Ghoshol et al. introduced strong reinforcement number r_s , the cardinality of a smallest set F which satisfies $\gamma_s(G+F) < \gamma_s(G)$ where $F \subset E(\overline{G})$ [5]. In Figure1, $\gamma_k(G) = 2$,$\gamma_s(G) = 3$,$\gamma_w(G) = 4$, $r(G) = 2$ and $r_s(G) = 1$ for graph G.

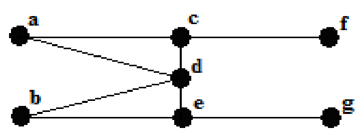

Figure1: Graph G

Cardinality of $\{c, e\}$-set equals to the $\gamma_k(G)$, cardinality of $\{c, d, e\}$-set equals to the $\gamma_s(G)$, cardinality of $\{a, b, f, g\}$-set equals to the $\gamma_w(G)$. Moreover, when we add two edge from vertex d to vertex f and g, $\gamma_k(G)$ decrease. Then,$r(G) = 2$. Similarly, when we add an edge from vertex c to vertex g, it is easy to see that $r_s(G) = 1$. In this paper, for $\Delta(G)$ and $\delta(G)$ denote the number of maximum and minimum degree, respectively.

§2 Weak reinforcement number

In this section we introduced a new reinforcement concept. Let F be a subset of $E(\overline{G})$. Weak reinforcement number r_w , the cardinality of smallest set F which satisfies$\gamma_w(G+F) < \gamma_w(G)$. Then here, some weak reinforcement number boundaries' are given and reinforcement numbers of basic graph are computed.

Theorem 2.1 *Let* G *be a connected graph, then* $1 \le r_w \le \frac{n.(n-1)}{2} - m$ *, where* $n = |V(G)|$ *and* $m = |E(G)|$ *for any graph* G*.*

Proof If $\Delta(G) = n-1$, then $r_w(G) = 0$ by definition. To dominate all vertices of a graph by a vertex which has minimum degree, it is necessary all vertices have $n-1$ degree, so the graph is a complete graph. For any graph G, we can add $\frac{n.(n-1)}{2} - m$ edges to make a complete graph and it's an upper boundary. Lower boundary is 1, because of star graph's structure. Consequently, when we add at least 1-edge and at most $\frac{n.(n-1)}{2} - m$, decrease $\gamma_w(G)$. □

Observation 2.1 If G is a complete graph then,$\gamma_w(G) = 1$.

Theorem 2.2 *If* $\gamma_w(G)$ *is 2, then* $r_w(G) = \frac{n.(n-1)}{2} - m$ *for any graph* G.

Proof Let weak domination number of a graph G be 2. We can decrease this number only 1. Due to the Observation 2.1 the graph G must be a complete. To make graph G complete must add $|E(\overline{G})|$ edges to graph, i.e. we must add $\frac{n.(n-1)}{2} - m$ edges. □

Lemma 2.1([6]) *The weak and strong domination number of* n-cycle *is*

$$\gamma_w(C_n) = \gamma_s(C_n) = \lceil \frac{n}{3} \rceil for, n \geq 3.$$

Theorem 2.3 *The weak reinforcement number of the* n-cycle *(with* $n \geq 7$ *and* $n \neq 9$*) is*

$$r_w(C_n) = \begin{cases} 2\,, & n = 1\ (mod3) \\ 4\,, & n = 2\ (mod3) \\ 6\,, & n = 0\ (mod3) \end{cases}$$

Proof From Lemma 2.1, the weak domination number of graph C_n is $\lceil \frac{n}{3} \rceil$. When $\gamma_w(G)$ is decreased, there arises 3 cases.

Case 1 If $n \equiv 1(mod3)$, the vertex which is taken to weak domination set, including itself dominates 3 vertices. In order for a vertex to dominate both itself and the other 3 vertices, to graph C_n two edges are added (see Figure 2).

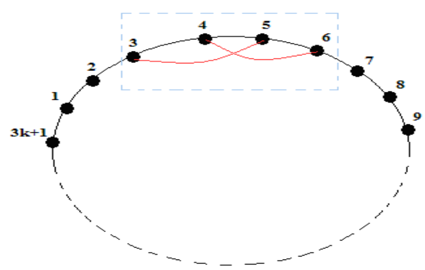

Figure 2

In conclusion, in the weak domination set there are vertices from K_4 structure in Figure 2 together with the $\frac{n-4}{3}$ vertices. Then,$\gamma_w(C_n + F) = \frac{n-4}{3} + 1 = \frac{n-1}{3}$. Since $\frac{n-1}{3} < \lceil \frac{n}{3} \rceil$, then $r_w = 2$.

Case 2 If $n \equiv 2(mod3)$, similar to Case1, by creating two K_4 structure, the proof is set. In conclusion, in the weak domination set there has been $\frac{n-8}{3}$+2 vertices. Then,$\gamma_w(C_n+F) = \frac{n-2}{3}$. Since $\frac{n-2}{3} < \lceil \frac{n}{3} \rceil$, then $r_w = 4$.

Case 3 If $n \equiv 0(mod3)$, when it is set similar to Case1,$r_w = 6$.

Combining Cases 1-3, the proof is complete. □

Theorem 2.4 *Values of weak reinforcement number of* C_4,C_5,C_6 *and* C_9 *are 2, 5, 9 and 7, respectively.*

Proof The weak reinforcement number of C_4,C_5 and C_6 are 2. It is easily seeing that from Theorem 2.2, $r_w(C_4) = 2$,$r_w(C_5) = 2$,$r_w(C_6) = 9$. Moreover, $\gamma_w(C_9) = 3$. To decrease this number,we must obtain a K_4 and K_5 from C_9 vertices.Then it's easily see that $r_w(C_9) = 7$. □

Lemma 2.2([4]) *The weak and strong domination number of the path of* order-n *is*

$$\gamma_s(P_n) = \lceil \frac{n}{3} \rceil, forn \geq 3,$$

$$\gamma_w(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil \ , & if\ n = 1\ (mod3), \\ \lceil \frac{n}{3} \rceil + 1\ , & otherwise \end{cases}$$

Theorem 2.5 *The weak reinforcement number of the* path of order-n *is*

$$r_w(P_n) = \begin{cases} 3\ , & n = 1\ (mod3) \\ 1\ , & otherwise. \end{cases}$$

Proof If $n = 3k$ and $n = 3k+2$ then $\gamma_w(P_n) = \lceil \frac{n}{3} \rceil + 1$. For these cases, we add an e_1-edge to two vertices, which has degree 1, then the graph be a C_n. $\gamma_w(C_n) > \gamma_w(C_n + e_1)$ since $\gamma_w(C_n)$ is $\lceil \frac{n}{3} \rceil$. For this reason, $r_w(P_n) = 1$. If $n = 3k+1$ then we add an edge to two vertices, which has degree 1, then the graph be a C_n. Then we add 2 more edges, likes Theorem 2.3, Case1. Since $\gamma_w(C_n) > \gamma_w(C_n + F)$, then $r_w(P_n) = 3$, where F is a set of added edges. □

Lemma 2.3([4]) *The weak and strong domination number of the wheel graph $W_{1,n}$ is*

$$\gamma_s(W_{1,n}) = 1\ ,\ \gamma_w(W_{1,n}) = \lceil \frac{n}{3} \rceil.$$

Theorem 2.6 *The weak reinforcement number of the wheel graph $W_{1,n}$ (with $n \geq 7$ and $n \neq 9$)*

$$r_w(W_{1,n}) = \begin{cases} 2\ , & n = 1\ (mod3) \\ 4\ , & n = 2\ (mod3) \\ 6\ , & n = 0\ (mod3) \end{cases}$$

Proof The proof is similar to that of Theorem 2.3. □

Theorem 2.7 *If $n = 4, 5, 6, 9$ then $r_w(W_{1,n})$ is 2,5,9 and 7, respectively.*

Proof The proof makes similar to that of Theorem 2.4. □

Lemma 2.4([5]) *The weak and strong domination number of the complete bipartite graph $K_{m,n}$ is*

$$\gamma_s(K_{m,n}) = \begin{cases} 2 & ,\ if\ 2 \leq m = n, \\ m & ,\ if\ 1 \leq m < n. \end{cases}$$

$$\gamma_w(K_{m,n}) = \begin{cases} 2 & ,\ if\ 2 \leq m = n, \\ n & ,\ if\ 1 \leq m < n. \end{cases}$$

Theorem 2.8 *The weak reinforcement number of complete bipartite graph $K_{m,n}$, where $m \leq n$ is*

$$r_w(K_{m,n}) = \begin{cases} m^2 - m & , \quad m = n \geq 2, \\ 1 & , \quad m < n. \end{cases}$$

Proof If $m = n$, then $\gamma_w(K_{m,n}) = 2$. Due to Theorem 2.2, the graph must be a complete while weak domination number decreasing. The edge number of graph K_{2m} is $\dfrac{2m(2m-1)}{2}$. The edge number of $K_{m,n}$ is m^2. So, r_w number is $m^2 - m$. If $m < n$ then $\gamma_w(K_{m,n}) = n$. When we add an edge between two vertices which have degree of m, we obtain the r_w number is 1. □

Result 2.1 *If m=1, then $r_w(K_{1,n}) = 1$, where $K_{1,n}$ is a star graph.*

Lemma 2.5([5]) *Define a support to be a vertex in a tree which adjacent to an end-vertex. Every tree T with $(n \geq 4)$ has at least one of the following characteristic.*

(*i*) *A support adjacent to at least 2 end-vertex;*
(*ii*) *A support is adjacent to a support of degree 2;*
(*iii*) *A vertex is adjacent to 2 support of degree 2;*
(*iv*) *A support of a leaf and the vertex adjacent to the support are both of degree 2.*

Theorem 2.9 *If any vertex of tree T is adjacent with two or more end-vertices, then $r_w(T) = 1$.*

Proof Let T has two or more end-vertices, which denote by $u_1, u_2, \ldots$. The u_i's belong to every minimum weak domination set of T. We add an e-edge between two vertices, then $\gamma_w(T) > \gamma_w(T + e)$. Hence, $r_w(T) = 1$. □

Theorem 2.10 *Let T be any tree order of n $(n > 3)$, then $r_w(T) \leq 3$.*

Proof It is easy to see that $r_w(T) = 0$ and $r_w(T) = 1$ for n=2 and n=3, respectively. Assume that $n > 3$. From Lemma 2.5, there are 4 cases. (see Figure3)

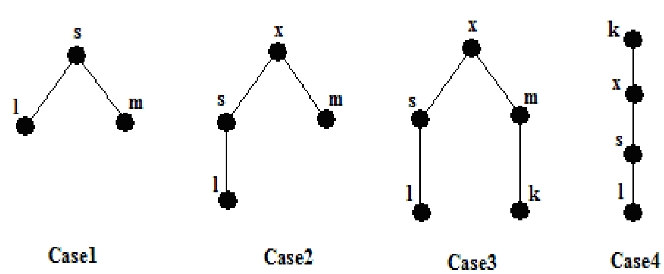

Figure3: End characteristic of trees

Case 1 Assume that supported vertex s is adjacent to two or more vertices. All end-vertices are in weak domination set. When we add an e-edge between any two end-vertices, $\gamma_w(T) > \gamma_w(T + e)$ is obtained. Hence, $r_w(T) = 1$.

Case 2 In this case two end-vertices are in weak domination set. We must obtain K_4 structure for weak dominate four vertices by a vertex. For this, worst case situation, we must add three edges. Hence,$r_w(T) \leq 3$.

Case 3 **and Case** 4 The proofs make similar to Case2. Consequently, $r_w(T) \leq 3$.

Combining Cases 1-4, the proof is complete. □

§3. Strong reinforcement number

In these section general results is given for strong reinforcement number and some boundaries of strong reinforcement number of any graph G and its complemented graph $\overline{G}$. In [5], Theorems 3.1-3.6 following are proved.

Theorem 3.1 *The strong reinforcement number of the* path of order-n *and* n-cycle *is*

$$r_s(P_n) = r_s(C_n) = i, where n \equiv i (mod 3).$$

Theorem 3.2 *The strong reinforcement number of multipicle complete graph is*

$$r_s(K_{m_1,m_2,\ldots,m_t}) = \begin{cases} 0 & , if\, m_1 = 1 \\ m_1 - 1 & , if\, m_1 \neq 1\, and\, m_1 = m_2 \\ 1 & , if\, m_1 \neq 1\, and\, m_1 \neq m_2 \end{cases}$$

Theorem 3.3 $r_s(G) \leq p - 1 - \Delta(G)$ *for any graph* G, *where* $p = |V(G)|$.

Theorem 3.4 *If* G *is any graph* G, *then* $r_s(G) = p - 1 - \Delta(G)$ *if and only if* $\gamma_s(G) = 2$.

Theorem 3.5 $r_s(G) \leq \Delta(G) + 1$, *for any graph* G *with* $\gamma_s(G) \geq 2$.

Theorem 3.6 $\gamma_s(G) + r_s(G) \leq p - \Delta(G) + 1$ *for any graph G, where* $p = |V(G)|$.

Theorem 3.7 *Let* G *be any graph* order of n *and* $\overline{G}$ *be a complemented graph of* G. *If graph* G *has at least one vertex which has degree 1, then* $\gamma_s(\overline{G}) = 2$ *and* $r_s(\overline{G}) = 1$.

Proof Let vertex u has degree 1. vertex u adjacent to *n-2* vertices in $\overline{G}$. Then taking vertex v in strong domination set where vertex v adjacent to vertex u. Hence,$\gamma_s(\overline{G}) = 2$ .From Theorem 3.4,$r_s(\overline{G}) = p - 1 - \Delta(\overline{G})$. Since $\Delta(\overline{G}) = n - 2$, it is easily see that $r_s(\overline{G}) = 1$. □

Theorem 3.8 *Let G be any graph* order of n *and* $\overline{G}$ *be a complemented graph of* G. *Then,* $r_s(\overline{G}) \leq \delta(G)$.

Proof It is obvious that $\Delta(\overline{G}) = n - \delta(G) - 1$ and $r_s(\overline{G}) \leq n - 1 - \Delta(\overline{G})$ from the Theorem 3.3. Whence,

$$r_s(\overline{G}) \leq n - 1 - (n - \delta(G) - 1), \quad r_s(\overline{G}) \leq \delta(G).$$

□

Theorem 3.9 *Let* G *be any graph* order of n *and* $\overline{G}$ *be a complemented graph of* G. *Then,* $r_s(G) + r_s(\overline{G}) \leq n + \delta(G) - (\Delta(G) + 1)$.

Proof It easily see that from Theorems 3.3 and 3.8. □

§4. Conclusion

The concept of domination in graph is very effective both in theoretical developments and applications. Also, domination set problem can be used to solve hierarchy problem, network flows and many combinatoric problems. If graph G has a small domination number then each pairs of vertex has small distance. So, in any graph if we want to decrease to domination number, we have to decrease distance between each pairs of vertex. More than thirty domination parameters have been investigated by different authors, and in this paper we have introduced the concept of domination. We called weak reinforcement number its. Then, we computed weak reinforcement number for some graph and some boundary of strong reinforcement number has been given for a given graph G and its complemented graph $\overline{G}$. Work on other domination parameters will be reported in forthcoming papers.

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