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Some Fixed Point Theorems in Fuzzy n-Normed Spaces

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Abstract: The main purpose of this paper is to study the existence of a fixed points in fuzzy n-normed spaces. we proved our main results, a fixed point theorem for a self mapping and a common fixed point theorem for a pair of weakly compatible mappings on fuzzy n-normed spaces. Also we gave some remarks on fuzzy n-normed spaces.

Key Words: Smarandache space, Pseudo-Euclidean space, fuzzy n-normed spaces, n-seminorm.

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§1. Introduction

A Pseudo-Euclidean space is a particular Smarandache space defined on a Euclidean space $\mathbb{R}^n$such that a straight line passing through a point p may turn an angle $\theta_p \geq 0$. If $\theta_p \geq 0$, then p is called a non-Euclidean point. Otherwise, a Euclidean point. In this paper, normed spaces are considered to be Euclidean, i.e., every point is Euclidean.In [7], S. Gähler introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [8,10,12-13]. In [8], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$- norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1,3,4,5,6,9,11]. In [14], A. Narayanan and S. Vijayabalaji have extend n-normed linear space to fuzzy n-normed linear space. In section 2, we quote some basic definitions, and we show that a fuzzy n-norm is closely related to an ascending system of n-seminorms. In section 3, we introduce a locally convex topology in a fuzzy n-normed space. In section 4, we consider finite dimensional fuzzy n-normed linear spaces. In section 5, we give some fixed point theorem in fuzzy $n-$ normed spaces.

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§2. Fuzzy n-norms and ascending families of n-seminorms

Let n be a positive integer, and let X be a real vector space of dimension at least n. We recall the definitions of an n-seminorm and a fuzzy n-norm [14].

Definition 2.1 *A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from X^n to $[0, \infty)$ is called an n-seminorm on X if it has the following four properties:*

(S1) $\|x_1, x_2, \ldots, x_n\| = 0$ *if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(S2) $\|x_1, x_2, \ldots, x_n\|$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

(S3) $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, , \ldots, x_{n-1}, x_n\|$ *for any real* c*;*

(S4) $\|x_1, \ldots, x_{n-1}, y + z\| \leqslant \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

An n-seminorm is called a n-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

Definition 2.1 *A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if :*

(F1) For all $t \leqslant 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$*;*

(F2) For all $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(F3) $N(x_1, x_2, \ldots, x_n, t)$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

(F4) For all $t > 0$ *and* $c \in \mathbb{R}$, $c \neq 0$,

$$N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});$$

(F5) For all $s, t \in \mathbb{R}$,

$$N(x_1, \ldots, x_{n-1}, y + z, s + t) \geq \min\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}z, t)\}.$$

(F6) $N(x_1, x_2, \ldots, x_n, t)$ *is a non-decreasing function of* $t \in \mathbb{R}$ *and*

$$\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

The following two theorems clarify the relationship between Definitions $2,1$ and 2.2.

Theorem 2.1 *Let N be a fuzzy n-norm on X. As in [14] define for $x_1, x_2, \ldots, x_n \in X$ and $\alpha \in (0, 1)$*

$$\|x_1, x_2, \ldots, x_n\|_\alpha := \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}. \tag{1}$$

Then the following statements hold.

(A1) For every $\alpha \in (0, 1)$, $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$ *is an n-seminorm on* X*;*

(A2) If $0 < \alpha < \beta < 1$ *and* $x_1, \ldots, x_n \in X$ *then*

$$\|x_1, x_2, \ldots, x_n\|_\alpha \leqslant \|x_1, x_2, \ldots, x_n\|_\beta;$$

(A3) If $x_1, x_2, \ldots, x_n \in X$ *are linearly independent then*

$$\lim_{\alpha \to 1-} \|x_1, x_2, \ldots, x_n\|_\alpha = \infty.$$

Proof (A1) and (A2) are shown in [14, Theorem 3.4]. Let $x_1, x_2, \ldots, x_n \in X$ be linearly independent, and $t > 0$ be given. We set $\beta := N(x_1, x_2, \ldots, x_n, t)$. It follows from (F2) that $\beta \in [0, 1)$. Then (F6) shows that, for $\alpha \in (\beta, 1)$,

$$\|x_1, x_2, \ldots, x_n\|_\alpha \geqslant t.$$

This proves (A3). □

We now prove a converse of Theorem 2.1.

Theorem 2.2 *Suppose we are given a family* $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$, $\alpha \in (0, 1)$, *of n-seminorms on X with properties (A2) and (A3). We define*

$$N(x_1, x_2, \ldots, x_n, t) := \inf\{\alpha \in (0, 1) : \|x_1, x_2, \ldots, x_n\|_\alpha \geqslant t\}. \tag{2}$$

where the infimum of the empty set is understood as 1. *Then N is a fuzzy n-norm on X.*

Proof (F1) holds because the values of an n-seminorm are nonnegative.

(F2): Let $t > 0$. If $x_1, \ldots, x_n$ are linearly dependent then $N(x_1, \ldots, x_n, t) = 1$ follows from property (S1) of an n-seminorm. If $x_1, \ldots, x_n$ are linearly independent then $N(x_1, \ldots, x_n, t) < 1$ follows from (A3).

(F3) is a consequence of property (S2) of an n-seminorm.

(F4) is a consequence of property (S3) of an n-seminorm.

(F5): Let $\alpha \in (0, 1)$ satisfy

$$\alpha < \min\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, s)\}. \tag{3}$$

It follows that $\|x_1, \ldots, x_{n-1}, y\|_\alpha < s$ and $\|x_1, \ldots, x_{n-1}, z\|_\alpha < t$. Then (S4) gives

$$\|x_1, \ldots, x_{n-1}, y + z\|_\alpha < s + t.$$

Using (A2) we find $N(x_1, \ldots, x_{n-1}, y+z, s+t) \geqslant \alpha$ and, since α is arbitrary in (3), (F5) follows.

(F6): Definition 2.2 shows that N is non-decreasing in t. Moreover, $\lim_{t\to\infty} N(x_1, \ldots, x_n, t) = 1$ because seminorms have finite values. □

It is easy to see that Theorems 2.1 and 2.2 establish a one-to-one correspondence between fuzzy n-norms with the additional property that the function $t \mapsto N(x_1, \ldots, x_n, t)$ is left-continuous for all $x_1, x_2, \ldots, x_n$ and families of n-seminorms with properties (A2), (A3) and the additional property that $\alpha \mapsto \|x_1, \ldots, x_n\|_\alpha$ is left-continuous for all $x_1, x_2, \ldots, x_n$.

Example 2.3([14,Example 3.3] Let $\|\bullet, \bullet, \ldots, \bullet\|$ be a n-norm on X. Define $N(x_1, x_2, \ldots, x_n, t)$ $= 0$ if $t \leqslant 0$ and, for $t > 0$,

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then the seminorms (2.1) are given by

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1-\alpha}\|x_1, x_2, \ldots, x_n\|.$$

§3. The locally convex topology generated by a fuzzy n-norm

In this section (X, N) is a fuzzy n-normed space, that is, X is real vector space and N is fuzzy n-norm on X. We form the family of n-seminorms $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$, $\alpha \in (0, 1)$, according to Theorem 2.1. This family generates a family $\mathcal{F}$ of seminorms

$$\|x_1, \ldots, x_{n-1}, \bullet\|_\alpha, \quad \text{where } x_1, \ldots, x_{n-1} \in X \text{ and } \alpha \in (0, 1).$$

The family $\mathcal{F}$ generates a locally convex topology on X; see [15, Def. (37.9)], that is, a basis of neighborhoods at the origin is given by

$$\{x \in X : p_i(x) \leqslant \epsilon_i \text{ for } i = 1, 2, \ldots, n\},$$

where $p_i \in \mathcal{F}$ and $\epsilon_i > 0$ for $i = 1, 2 \ldots, n$. We call this the locally convex topology generated by the fuzzy n-norm N.

Theorem 3.1 *The locally convex topology generated by a fuzzy n-norm is Hausdorff.*

Proof Given $x \in X$, $x \neq 0$, choose $x_1, \ldots, x_{n-1} \in X$ such that $x_1, \ldots, x_{n-1}, x$ are linearly independent. By Theorem 2.1(A3) we find $\alpha \in (0, 1)$ such that $\|x_1, \ldots, x_{n-1}, x\|_\alpha > 0$. The desired statement follows; see [15, Theorem 37.21]. □

Some topological notions can be expressed directly in terms of the fuzzy-norm N. For instance, we have the following result on convergence of sequences. We remark that the definition of convergence of sequences in a fuzzy n-normed space as given in [20, Definition 2.2] is meaningless.

Theorem 3.2 *Let $\{x_k\}$ be a sequence in X and $x \in X$. Then $\{x_k\}$ converges to x in the locally convex topology generated by N if and only if*

$$\lim_{k\to\infty} N(a_1, \ldots, a_{n-1}, x_k - x, t) = 1 \tag{4}$$

for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$.

Proof Suppose that $\{x_k\}$ converges to x in (X, N). Then, for every $\alpha \in (0, 1)$ and all $a_1, a_2, \ldots, a_{n-1} \in X$, there is K such that, for all $k \geqslant K$, $\|a_1, a_2, \ldots, a_{n-1}, x_k - x\|_\alpha < \epsilon$. The latter implies

$$N(a_1, a_2, \ldots, a_{n-1}, x_k - x, \epsilon) \geqslant \alpha.$$

Since $\alpha \in (0,1)$ and $\epsilon > 0$ are arbitrary we see that (4) holds. The converse is shown in a similar way. $\square$

In a similar way we obtain the following theorem.

Theorem 3.3 *Let $\{x_k\}$ be a sequence in X. Then $\{x_k\}$ is a Cauchy sequence in the locally convex topology generated by N if and only if*

$$\lim_{k,m\to\infty} N(a_1,\ldots,a_{n-1},x_k - x_m,t) = 1 \tag{5}$$

for all $a_1,\ldots,a_{n-1} \in X$ and all $t > 0$.

It should be noted that the locally convex topology generated by a fuzzy n-norm is not metrizable, in general. Therefore, in many cases it will be necessary to consider nets $\{x_i\}$ in place of sequences. Of course, Theorems 3.2 and 3.3 generalize in an obvious way to nets.

§4. Fuzzy n-norms on finite dimensional spaces

In this section (X, N) is a fuzzy n-normed space and X has finite dimension at least n. Since the locally convex topology generated by N is Hausdorff by Theorem 3.1 Tihonov's theorem [15, Theorem 23.1] implies that this locally convex topology is the only one on X. Therefore, all fuzzy n-norms on X are equivalent in the sense that they generate the same locally convex topology.

In the rest of this section we will give a direct proof of this fact (without using Tihonov's theorem). We will set $X = \mathbb{R}^d$ with $d \geqslant n$.

Lemma 4.1 *Every n-seminorm on $X = \mathbb{R}^d$ is continuous as a function on X^n with the euclidian topology.*

Proof For every $j = 1,2,\ldots,n$, let $\{x_{j,k}\}_{k=1}^{\infty}$ be a sequence in X converging to $x_j \in X$. Therefore, $\lim\limits_{k\to\infty} \|x_{j,k} - x_j\| = 0$, where $\|x\|$ denotes the euclidian norm of x. From property (S4) of an n-seminorm we get

$$|\|x_{1,k}, x_{2,k},\ldots,x_{n,k}\| - \|x_1, x_{2,k},\ldots,x_{n,k}\|| \le \|x_{1,k} - x_1, x_{2,k},\ldots,x_{n,k}\|.$$

Expressing every vector in the standard basis of $\mathbb{R}^d$ we see that there is a constant M such that

$$\|y_1, y_2,\ldots,\ y_n\| \le M\,\|y_1\|\ldots\|y_n\| \ \text{ for all } y_j \in X.$$

Therefore,

$$\lim_{k\to\infty} \|x_{1,k} - x_1,\ x_{2,k},\ldots,\ x_{n,k}\| = 0$$

and so

$$\lim_{k\to\infty} |\|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\| - \|x_1,\ x_{2,k},\ldots,\ x_{n,k}\|| = 0.$$

We continue this procedure until we reach

$$\lim_{k\to\infty} \|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\| = \|x_1,\ x_2,\ldots,\ x_n\|\,. \qquad \square$$

Lemma 4.2 *Let $(\mathbb{R}^d, N)$ be a fuzzy n-normed space. Then $\|x_1, x_2, \ldots, x_n\|_\alpha$ is an n-norm if $\alpha \in (0,1)$ is sufficiently close to 1.*

Proof We consider the compact set

$$S = \left\{(x_1,\ x_2, \ldots,\ x_n) \in \mathbb{R}^{dn} : x_1,\ x_2, \ldots,\ x_n \text{ is an orthonormal system in } \mathbb{R}^d\right\}.$$

For each $\alpha \in (0,1)$ consider the set

$$S_\alpha = \{(x_1, x_2, \ldots, x_n) \in S : \|x_1, x_2, \ldots, x_n\|_\alpha > 0\}.$$

By Lemma 4.1, S_α is an open subset of S. We now show that

$$S = \bigcup_{\alpha \in (0,1)} S_\alpha. \tag{6}$$

If $(x_1,\ x_2, \ldots,\ x_n) \in S$ then $(x_1,\ x_2, \ldots,\ x_n)$ is linearly independent and therefore there is β such that $N(x_1, x_2, \ldots, x_n, 1) < \beta < 1$. This implies that $\|x_1,\ x_2, \ldots,\ x_n\|_\beta \geq 1$ so (6) is proved. By compactness of S, we find $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$ such that

$$S = \bigcup_{i=1}^{m} S_{\alpha_i}.$$

Let $\alpha = \max\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m\}$. Then $\|x_1,\ x_2, \ldots,\ x_n\|_\alpha > 0$ for every $(x_1,\ x_2, \ldots,\ x_n) \in S$.

Let $x_1,\ x_2, \ldots,\ x_n \in X$ be linearly independent. Construct an orthonormal system $e_1,\ e_2, \ldots,\ e_n$ from $x_1,\ x_2, \ldots,\ x_n$ by the Gram-Schmidt method. Then there is $c > 0$ such that

$$\|x_1,\ x_2, \ldots,\ x_n\|_\alpha = c\,\|e_1,\ e_2, \ldots,\ e_n\|_\alpha > 0.$$

This proves the lemma. □

Theorem 4.1 *Let N be a fuzzy n-norm on $\mathbb{R}^d$, and let $\{x_k\}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$.*

(a) $\{x_k\}$ converges to x with respect to N if and only if $\{x_k\}$ converges to x in the euclidian topology.

(b) $\{x_k\}$ is a Cauchy sequence with respect to N if and only if $\{x_k\}$ is a Cauchy sequence in the euclidian metric.

Proof (a) Suppose $\{x_k\}$ converges to x with respect to euclidian topology. Let $a_1, a_2, \ldots, a_{n-1} \in X$. By Lemma 4.1, for every $\alpha \in (0,1)$,

$$\lim_{k\to\infty} \|a_1,\ a_2, \ldots,\ a_{n-1},\ x_k - x\|_\alpha = 0.$$

By definition of convergence in $(\mathbb{R}^d, N)$, we get that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. Conversely, suppose that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. By Lemma 4.2, there is $\alpha \in (0,1)$ such that $\|y_1, y_2, \ldots, y_n\|_\alpha$ is an n-norm. By definition, $\{x_k\}$ converges to x in the n-normed space $(\mathbb{R}^d, \|\cdot\|_\alpha)$. It is known from[8, Proposition 3.1] that this implies that $\{x_k\}$ converges to x with respect to euclidian topology.

(b) is proved in a similar way. □

Theorem 4.2 *A finite dimensional fuzzy n-normed space (X, N) is complete.*

Proof This follows directly from Theorem 3.4. □

§5. Some fixed point theorem in fuzzy $n-$ normed spaces

In this section we prove some fixed point theorems.

Definition 5.1 *A sequence a $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to be fuzzy n-convergent to $x^* \in X$ and denoted by $x_k \rightsquigarrow x^*$ as $k \to \infty$ if*

$$\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and x^* is called the fuzzy n-limit of $\{x_k\}$.

Remark 5.1 It is noted that if (X, N) is a fuzzy n-normed space then the fuzzy n-limit of a fuzzy n-convergent sequence is unique. Indeed, if $\{x_k\}$ is a fuzzy n-convergent sequence and suppose it converges to x^* and y^* in X. Then by definition $\lim\limits_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$ and $\lim\limits_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - y^*, t) = 1$ for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$. By $(N5)$, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, x - y, t) &= N(x_1, \cdots, x_{n-1}, x^* - x_k + x_k - y^*, t/2 + t/2) \\ &\geqslant \min\{N(x_1, \cdots, x_{n-1}, x^* - x_k, t/2), N(x_1, \cdots, x_{n-1}, x_k - y^*, t/2)\}. \end{aligned}$$

By letting $k \to \infty$, we obtain $N(x_1, \cdots, x_{n-1}, x^* - y^*, t) = 1$, which implies that $x^* = y^*$.

Definition 5.2 *A sequence $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to be fuzzy n-Cauchy sequence if*

$$\lim_{k,m\to\infty} N(x_1, \ldots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$.

Proposition 5.1 *In a fuzzy n-normed space (X, N), every fuzzy n-convergent sequence is a fuzzy n-Cauchy sequence.*

Proof Let $\{x_k\}$ be a fuzzy n-convergent sequence in X converging to $x^* \in X$. Then $\lim\limits_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$ for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$. By (N5),

$$\begin{aligned} &N(x_1, \cdots, x_{n-1}, x_k - x_m, t) \\ &\quad = N(x_1, \cdots, x_{n-1}, x_k - x^* + x^* - x_m, t/2 + t/2) \\ &\quad \geqslant \min\{N(x_1, \cdots, x_{n-1}, x_k - x^*, t/2), N(x_1, \cdots, x_{n-1}, x^* - x_m, t/2)\}. \end{aligned}$$

By letting $n, m \to \infty$, we get,

$$\lim_{k,m\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$, i.e., $\{x_k\}$ is a fuzzy n-Cauchy sequence. □

If every fuzzy n-Cauchy sequence in X converges to an $x^* \in X$, then (X, N) is called a complete fuzzy n-normed space. A complete fuzzy n-normed space is then called a fuzzy n-Banach space.

Theorem 5.1 *Let (X, N) be a fuzzy n-normed space. Let $f : X \to X$ be a map satisfies the condition:*
There exists a $\lambda \in (0,1)$ such that for all $x, x_1, \cdots, x_{n-1} \in X$ and for all $t > 0$, one has

$$N(x_1, \cdots, x_{n-1}, x, t) > 1 - t \quad \Rightarrow \quad N(x_1, \cdots, x_{n-1}, f(x), \lambda t) > 1 - \lambda t. \tag{7}$$

Then

(i) For any real number $\epsilon > 0$ there exists $k_0(\epsilon) \in \mathbb{N}$ such that $f^k(x) \rightsquigarrow \theta$.

(ii) f has at most a fixed point, that is the null vector of X. Moreover, if f is a linear mapping, f has exactly one fixed point.

Proof (i) Note that if f satisfies the condition (1), then for every $\epsilon \in (0,1)$, there exists a $k_0 = k_0(\epsilon)$ such that, for all $k \geqslant k_0$, and for every $x, x_1, \cdots, x_{n-1} \in X$

$$N(x_1, \cdots, x_{n-1}, f^k(x), \epsilon) > 1 - \epsilon$$

holds. Indeed, one has easily that

$$N(x_1, \cdots, x_{n-1}, x, 1 + \epsilon) > 1 - (1 + \epsilon).$$

Then by condition (1), for all $x, x_1, \cdots, x_{n-1} \in X$ and $k \geqslant 1$,

$$N(x_1, \cdots, x_{n-1}, f^k(x), \lambda^k(1 + \epsilon)) > 1 - \lambda^k(1 + \epsilon)$$

holds. Indeed, for each $\epsilon > 0$ there exists a $k = k_0$ implies that $\lambda^n(1 + \epsilon) \leqslant \epsilon$, from which, because of condition (N6), there exists a $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$,

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f^k(x), \epsilon) &\geqslant N(x_1, \cdots, x_{n-1}, f^k(x), \lambda^k(1 + \epsilon)) \\ &> 1 - \lambda^k(1 + \epsilon) \\ &\geqslant 1 - \epsilon. \end{aligned}$$

Since ϵ is an arbitrary, we have $f^k(x) \rightsquigarrow \theta$ as required.

(ii) Assume that $f(x) = x$. By applying part (i), for all $\epsilon \in (0,1)$ one has

$$N(x_1, \cdots, x_{n-1}, x, \epsilon) > 1 - \epsilon$$

for every $x_1, \cdots, x_{n-1} \in X$. This implies that

$$N(x_1, \cdots, x_{n-1}, x, 0+) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$, i.e., $x = \theta$. □

Lemma 5.1 *Let $\{x_k\}$ be a sequence in a fuzzy n-normed space (X, M). If for every $t > 0$, there exists a constant $\lambda \in (0, 1)$ such that*

$$N(x_1, \ldots, x_{n-1}, x_k - x_{k+1}, t) \geq N(x_1, \ldots, x_{n-1}, x_{k-1} - x_k, t/\lambda) \tag{8}$$

for all $x_1, \cdots, x_{n-1} \in X$, then $\{x_k\}$ is a fuzzy n-Cauchy sequence in X.

Proof Let $t > 0$ and $\lambda \in (0, 1)$. Then for $m \geqslant k$, by using (N5) and the inequality (1), we have

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_k - x_m, t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_k - x_{k+1}, (1-\lambda)t),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\}\\
&\qquad \cdots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\}
\end{aligned}$$

Also,

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_{k+1} - x_{k+2}, (1-\lambda)\lambda t),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+2} - x_m, \lambda^2 t)\}\\
&\qquad \cdots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+2} - x_m, \lambda^2 t)\}
\end{aligned}$$

By repeating these argument, we get

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_k - x_m, t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{m-1} - x_m, \lambda^{m-n-1} t)\}\\
&\qquad \cdots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{t}{\lambda^k})\}
\end{aligned}$$

Since $(1-\lambda)\frac{t}{\lambda^k} \leq \frac{t}{\lambda^k}$ and the property (F6), we conclude that

$$N(x_1, \ldots, x_{n-1}, x_k - x_m, t) \geq N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}).$$

Therefore, by letting $m \geqslant k \to \infty$, we get

$$\lim_{k,m\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$, i.e., $\{x_k\}$ is a fuzzy n-Cauchy sequence. $\square$

Definition 5.3 *A pair of maps (f, g) is called weakly compatible pair if they commute at coincidence point, i.e., $fx = gx$ implies $fgx = gfx$.*

Theorem 5.2 *Let (X, M) be a fuzzy n-normed space and let $f, g \colon X \to X$ satisfy the following conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) any one $f(X)$ or $g(X)$ is complete;

(iii) $N(x_1, \ldots, x_{n-1}, f(x)-f(y), t) \geqslant N(x_1, \ldots, x_{n-1}, g(x)-g(y), t/\lambda)$, *for all* $x, y, x_1, \cdots, x_{n-1} \in X$, $t > 0$, $\lambda \in (0, 1)$.

Then f and g have a unique common fixed point provided f and g are weakly compatible on X.

Proof Let $x_0 \in X$. By condition (i), we can find $x_1 \in X$ such that $f(x_0) = g(x_1) = y_1$. By induction, we can define a sequence y_k in X such that

$$y_{k+1} = f(x_k) = g(x_{k+1}),$$

$n = 0, 1, 2, \cdots$. We consider two cases:
Case I: If $y_r = y_{r+1}$ for some $r \in \mathbb{N}$, then

$$y_r = f(x_{r-1}) = f(x_r) = g(x_r) = g(x_{r+1}) = y_{r+1} = z$$

for some $z \in X$. Since $f(x_r) = g(x_r)$ and f, g are weakly compatible, we have $f(z) = fg(x_r) = gf(x_r) = g(z)$. By condition (iii), for all $x_1, \cdots, x_{n-1} \in X$ and for all $t > 0$, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(z) - z, t) &= N(x_1, \cdots, x_{n-1}, f(z) - f(x_r), t) \\ &\geqslant N(x_1, \cdots, x_{n-1}, g(z) - g(x_r), t/\lambda) \\ &\geqslant \cdots \geq N(x_1, \cdots, x_{n-1}, g(z) - g(x_r), t/\lambda^k). \end{aligned}$$

Clearly, the righthand side of the inequality approaches 1 as $k \to \infty$ for every $x_1, \ldots, x_{n-1} \in X$ and $t > 0$. Hence, $N(x_1, \cdots, x_{n-1}, f(z) - z, t) = 1$. This implies that $f(z) = z = g(z)$, i.e., z is a common fixed point of f and g.
Case II $y_k \neq y_{k+1}$, for each $k = 0, 1, 2, \cdots$. Then, by condition (ii) again, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, y_k - y_{k+1}, t) &= N(x_1, \cdots, x_{n-1}, g(x_k) - g(x_{k+1}), t) \\ &= N(x_1, \cdots, x_{n-1}, f(x_{k-1}) - f(x_k), t) \\ &\geq N(x_1, \cdots, x_{n-1}, g(x_{k-1}) - g(x_k), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, y_{k-1} - y_k, t) \end{aligned}$$

Then, by Lemma 5.1, $\{y_k\}$ is a Cauchy sequence (with respect to fuzzy n-norm) in X. Since $g(X)$ is complete, there exists $w \in g(X)$ such that

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} g(x_k) = w.$$

Also, since $w \in g(X)$, we can find a $p \in X$ such that $g(p) = w$. Note that

$$w = g(p) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} f(x_k).$$

Thus, by (iii), we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(p) - g(p), t) &= \lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, f(p) - f(x_k), t) \\ &\geq \lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, g(p) - g(x_k), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, g(p) - w, t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, w - w, t/\lambda), \end{aligned}$$

which implies that $w = f(p) = g(p)$ is a common fixed point of f and g. Furthermore, f and g are weakly compatible maps, we have

$$f(w) = fg(w) = gf(w) = g(w).$$

But than, by (iii),

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(w) - w, t) &= N(x_1, \cdots, x_{n-1}, f(w) - f(p), t) \\ &\geq N(x_1, \cdots, x_{n-1}, g(w) - g(p), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, f(w) - f(p), t/\lambda) \\ &\geq \cdots \geq N(x_1, \cdots, x_{n-1}, g(w) - g(p), t/\lambda^k). \end{aligned}$$

Clearly, the expression on the righthand side approaches 1 as $k \to \infty$ for every $x_1, \ldots, x_{n-1} \in X$ and $t > 0$, which implies that $f(w) = w$. Therefore, w is a common fixed point of f and g. The uniqueness of fixed point is immediate from condition (iii). □

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