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# Biharmonic Slant Helices According to Bishop Frame in $\mathbb{E}^3$

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**Abstract**: In this paper, we study biharmonic slant helices in $\mathbb{E}^3$. We give some characterizations for biharmonic slant helices with Bishop frame in $\mathbb{E}^3$.

**Key Words**: Slant helix, biharmonic curve, bishop frame.

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## §1. Introduction

In 1964, J. Eells and J.H. Sampson introduced the notion of poly-harmonic maps as a natural generalization of harmonic maps [1].

Firstly, harmonic maps $f:(M,g)\longrightarrow(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\mathrm{trace}\nabla df. \tag{1.2}$$

Secondly, as suggested by Eells and Sampson in [1], we can define the bienergy of a map $f$ by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [3], showing that the Euler–Lagrange equation associated to $E_2$ is

$$\tau_2(f)=-\mathcal{J}^f(\tau(f))=-\Delta\tau(f)-\mathrm{trace}R^N(df,\tau(f))\,df=0 \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of $f$. The equation $\tau_2(f)=0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

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## §2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbb{E}^3$ are briefly presented.

The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. $\gamma$ is called a unit speed curve if velocity vector $v$ of $\gamma$ satisfies $\|a\| = 1$.

Denote by $\{T, N, B\}$ the moving Frenet–Serret frame along the curve $\gamma$ in the space $\mathbb{E}^3$. For an arbitrary curve $\gamma$ with first and second curvature, $\kappa$ and $\tau$ in the space $\mathbb{E}^3$, the following Frenet-Serret formulae is given

$$\begin{aligned}\mathbf{T}' &= \kappa\mathbf{N},\\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B},\\ \mathbf{B}' &= -\tau\mathbf{N},\end{aligned}$$

where

$$\begin{aligned}\langle\mathbf{T},\mathbf{T}\rangle &= \langle\mathbf{N},\mathbf{N}\rangle = \langle\mathbf{B},\mathbf{B}\rangle = 1,\\ \langle\mathbf{T},\mathbf{N}\rangle &= \langle\mathbf{T},\mathbf{B}\rangle = \langle\mathbf{N},\mathbf{B}\rangle = 0.\end{aligned}$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|\mathbf{T}(s)\|$ and $\tau(s) = -\left\langle \mathbf{N}, \mathbf{B}'\right\rangle$.

Torsion of the curve $\gamma$ is given by the aid of the mixed product

$$\tau(s) = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.$$

In the rest of the paper, we suppose everywhere $\kappa(s) \neq 0$ and $\tau(s) \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\mathbf{T}' = k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \mathbf{M}_1' = -k_1\mathbf{T}, \mathbf{M}_2' = -k_2\mathbf{T}. \tag{2.1}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_1\}$ as Bishop trihedra and $k_1$ and $k_2$ as Bishop curvatures. The relation matrix may be expressed as

$$\begin{aligned}&\mathbf{T} = \mathbf{T},\\ &\mathbf{N} = \cos\theta\,(s)\,\mathbf{M}_1 + \sin\theta\,(s)\,\mathbf{M}_2,\\ &\mathbf{B} = -\sin\theta\,(s)\,\mathbf{M}_1 + \cos\theta\,(s)\,\mathbf{M}_2,\end{aligned}$$

where $\theta(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. Here, Bishop curvatures are defined by

$$k_1 = \kappa(s) \cos\theta(s), \qquad k_2 = \kappa(s) \sin\theta(s).$$

On the other hand, we get

$$\begin{aligned} \mathbf{T} &= \mathbf{T}, \\ \mathbf{M}_1 &= \cos\theta(s)\,\mathbf{N} - \sin\theta(s)\,\mathbf{B}, \\ \mathbf{M}_2 &= \sin\theta(s)\,\mathbf{N} + \cos\theta(s)\,\mathbf{B}. \end{aligned}$$

## §3. Biharmonic curves in $\mathbb{E}^3$

Biharmonic equation for the curve $\gamma$ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\,\mathbf{T} = 0, \tag{3.1}$$

that is, $\gamma$ is called a biharmonic curve if it is a solution of the equation (3.1).

**Theorem** 3.1 *$\gamma : I \longrightarrow \mathbb{E}^3$ is a unit speed biharmonic curve if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= C, \\ k_1'' - k_1^3 - k_1 k_2^2 &= 0, \\ k_2'' - k_2^3 - k_2 k_1^2 &= 0, \end{aligned} \tag{3.2}$$

*where $C$ is non-zero constant of integration.*

*Proof* Using the bishop equations (2.1) and biharmonic equation (3.1), we obtain

$$(-3k_1'k_1 - 3k_2'k_2)\mathbf{T} + (k_1'' - k_1^3 - k_1k_2^2)\mathbf{M}_1 + (k_2'' - k_2^3 - k_2k_1^2)\mathbf{M}_2 - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\,\mathbf{T} = \mathbf{0}. \tag{3.3}$$

In $\mathbb{E}^3$, the Riemannian curvature is zero, we have

$$(-3k_1'k_1 - 3k_2'k_2)\mathbf{T} + (k_1'' - k_1^3 - k_1k_2^2)\mathbf{M}_1 + (k_2'' - k_2^3 - k_2k_1^2)\mathbf{M}_2 = 0. \tag{3.4}$$

By (3.4), we see that $\gamma$ is a unit speed biharmonic curve if and only if

$$\begin{aligned} -3k_1'k_1 - 3k_2'k_2 &= 0, \\ k_1'' - k_1^3 - k_1k_2^2 &= 0, \\ k_2'' - k_2^3 - k_2k_1^2 &= 0. \end{aligned} \tag{3.5}$$

These, together with (3.5), complete the proof of the theorem. □

**Corollary** 3.2 *$\gamma : I \longrightarrow \mathbb{E}^3$ is a unit speed biharmonic curve if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= C \neq 0, \\ k_1'' - Ck_1 &= 0, \\ k_2'' - Ck_2 &= 0, \end{aligned} \tag{3.6}$$

*where* $C$ *is constant of integration.*

**Theorem** 3.3 *Let* $\gamma : I \longrightarrow \mathbb{E}^3$ *is a unit speed biharmonic curve, then*

$$\begin{aligned} k_1^2(s) + k_2^2(s) &= C, \\ k_1(s) &= c_1 e^{\sqrt{C}s} + c_2 e^{-\sqrt{C}s}, \\ k_2(s) &= c_3 e^{\sqrt{C}s} + c_4 e^{-\sqrt{C}s}, \end{aligned} \tag{3.7}$$

*where* $C$, $c_1$, $c_2$, $c_3$, $c_4$ *are constants of integration.*

*Proof* Using (3.6), we have (3.7). □

**Corollary** 3.4 *If* $c_1 = c_3$ *and* $c_2 = c_4$, *then*

$$k_1(s) = k_2(s). \tag{3.8}$$

**Definition** 3.5 *A regular curve* $\gamma : I \longrightarrow \mathbb{E}^3$ *is called a slant helix provided the unit vector* $\mathbf{M}_1$ *of the curve* $\gamma$ *has constant angle* $\theta$ *with some fixed unit vector* $u$, *that is*

$$g(\mathbf{M}_1(s), u) = \cos\theta \text{ for all } s \in I. \tag{3.9}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

**Theorem 3.6** *Let* $\gamma : I \longrightarrow \mathbb{E}^3$ *be a unit speed curve with non-zero natural curvatures. Then,* $\gamma$ *is a slant helix if and only if*

$$\frac{k_1}{k_2} = \text{constant}. \tag{3.10}$$

*Proof* Differentiating (3.9) and by using the Bishop frame (2.1), we find

$$g(\nabla_{\mathbf{T}}\mathbf{M}_1, u) = g(k_1\mathbf{T}, u) = k_1 g(\mathbf{T}, u) = 0. \tag{3.11}$$

From (3.9), we get

$$g(\mathbf{T}, u) = 0.$$

Again differentiating from the last equality, we obtain

$$\begin{aligned} g(\nabla_{\mathbf{T}}\mathbf{T}, u) &= g(k_1\mathbf{M}_1 + k_2\mathbf{M}_2, u) \\ &= k_1 g(\mathbf{M}_1, u) + k_2 g(\mathbf{M}_2, u) \\ &= k_1\cos\theta + k_2\sin\theta = 0. \end{aligned}$$

Using above equation, we get

$$\frac{k_1}{k_2} = -\tan\theta = \text{constant}.$$

The converse statement is trivial. This completes the proof. □

**Theorem 3.7** . *Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit speed biharmonic slant helix with non-zero natural curvatures. Then,*

$$k_1 = \text{constant and } k_2 = \text{constant}. \tag{3.12}$$

*Proof* Suppose that $\gamma$ be a unit speed biharmonic slant helix. From (3.10) we have

$$k_1 = \sigma k_2. \tag{3.13}$$

where $\sigma$ is a constant.

On the other hand, using first equation of (3.6), we obtain that $k_2$ is a constant. Similarly, $k_1$ is a constant.

Hence, the proof is completed. □

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