International J.Math. Combin. Vol.3 (2010), 57-63

A Result of Ramanujan and Brahmagupta Polynomials Described by a Matrix Identity

R. Rangarajan

(Department of Mathematics, Mysore University, Manasa Gangotri, Mysore-570 006, India)

Email: rajra63@gmail.com

Abstract: In the present paper, the following result of Ramanujan [2] is shown to be contained as special case of a matrix identity in two parameters [3]: If a , b , c , d are real numbers such that $a\,d \;-\; b\,c \;=\; 0$, then

$(a + b + c)^2 + (b + c + d)^2 + (a - d)^2 = (c + d + a)^2 + (d + a + b)^2 + (b - c)^2.$

$(a + b + c)^4 + (b + c + d)^4 + (a - d)^4 = (c + d + a)^4 + (d + a + b)^4 + (b - c)^4.$

Combinatorial properties of the two pairs of Brahmagupta polynomials defined by the matrix identities in one and two parameters are also described.

Key Words: Results of Ramanujan, matrix identity, Brahmagupta polynomials, combinatorial properties.

AMS(2000): 01A32, 11B37, 11B39

§1. Introduction

E.R. Suryanarayan [4] has described the following matrix identity:

$$\begin{bmatrix} x_n & y_n \\ t\,y_n & x_n \end{bmatrix} = \begin{bmatrix} x & y \\ t\,y & x \end{bmatrix}^n \tag{1}$$

with $x_0 \;=\; 1,\; y_0 \;=\; 0,\; n = 0, 1, 2, \cdots$. The identity (1) is the starting point to define a pair of homogeneous polynomials $\{x_n(x, y; t), y_n(x, y; t)\}$ of degree n in two real variables x, y and a real parameter $t \neq 0$ such that $x^2 \;-\; ty^2 \neq 0$ called Brahmagupta Polynomials. An extensive list of properties of Brahmagupta polynomials is given in [4].

R.Rangarajan, Rangaswamy and E.R. Suryanarayan [3] have extended the matrix identity (1) in the following way: Let $\mathbf{B}^{(\mathbf{s},\mathbf{t})}$ denote the set of matrices of the form

$$B = \begin{bmatrix} x & y \\ ty & x + sy \end{bmatrix} \tag{2}$$

[1]Received July 28, 2010. Accepted September 10, 2010.

where t and s are two parameters and x and y are two real variables subjected to the condition that $x^2 + s\,x\,y - t\,y^2 \neq 0$. Define B to be the extended matrix in two parameters. It is easy to check that in $\mathbf{B^{(s,t)}}$ the commutative law for multiplication holds. As a result,the following extended matrix identity in two parameters holds:

$$\begin{bmatrix} x & y \\ ty & x+sy \end{bmatrix}^n = \begin{bmatrix} x_n(x,y,s,t) & y_n(x,y,s,t) \\ ty_n(x,y,s,t) & x_n(x,y,s,t)+sy_n(x,y,s,t) \end{bmatrix} \tag{3}$$

It is very interesting to note that, if $s = t = y = 1$ and $x = 0$, then (3) takes the form:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} \tag{4}$$

where F_n is the n^{th} Fibonacci number

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The extended matrix identity (3) defines the pair $(x_n(x,y,s,t)\ ,\ y_n(x,y,s,t))$ of Brahmagupta polynomials in two parameters. An extensive list of properties of Brahmagupta polynomials in two parameters is given in [3].

In [1] an innovative matrix identity wherein each matrix has a determinant of the form $x^2 + y^2 + z^2$ is proposed to view Ramanujan result in the power 2. But the identity does not work in the power 4. However, the paper provided us a good motivation to seek an appropriate matrix identity in two parameters to view both the results of Ramanujan.

§2. A pair of results of Ramanujan

One of the remarkable results of Ramanujan, appearing on the page 385 of his note books [2] is stated as follows: If a, b, c, d are real numbers such that $ad = bc$, then

$$(a+b+c)^2 + (b+c+d)^2 + (a-d)^2 = (c+d+a)^2 + (d+a+b)^2 + (b-c)^2 \tag{5}$$

$$(a+b+c)^4 + (b+c+d)^4 + (a-d)^4 = (c+d+a)^4 + (d+a+b)^4 + (b-c)^4 \tag{6}$$

For example, if $a = 6, b = 3, c = 2$ and $d = 1$, then $11^2 + 6^2 + 5^2 = 9^2 + 10^2 + 1^2$ and $11^4 + 6^4 + 5^4 = 9^4 + 10^4 + 1^4$. Writing

$$x_1 = a+b+c,\ y_1 = b+c+d,\ z_1 = c+d+a,\ w_1 = d+a+b$$

the results (5) and (6) become

$$x_1^2 + y_1^2 + (x_1 - y_1)^2 = z_1^2 + w_1^2 + (z_1 - w_1)^2 \tag{7}$$

$$x_1^4 + y_1^4 + (x_1 - y_1)^4 = z_1^4 + w_1^4 + (z_1 - w_1)^4 \tag{8}$$

where x_1, y_1, z_1, w_1 are real numbers such that $x_1^2 + y_1^2 - x_1y_1 = z_1^2 + w_1^2 - z_1w_1$.
It is straightforward to workout

$$\begin{aligned} a &= \frac{1}{3}\ x_1 - \frac{2}{3}\ y_1 + \frac{1}{3}\ z_1 + \frac{1}{3}\ w_1, \\ b &= \frac{1}{3}\ x_1 + \frac{1}{3}\ y_1 - \frac{2}{3}\ z_1 + \frac{1}{3}\ w_1, \\ c &= \frac{1}{3}\ x_1 + \frac{1}{3}\ y_1 + \frac{1}{3}\ z_1 - \frac{2}{3}\ w_1, \\ d &= -\frac{2}{3}\ x_1 + \frac{1}{3}\ y_1 + \frac{1}{3}\ z_1 + \frac{1}{3}\ w_1 \end{aligned}$$

and hence $ad = bc$ is equivalent to

$$x_1^2 + y_1^2 - x_1y_1 = z_1^2 + w_1^2 - z_1w_1.$$

Now, it is very easy to verify the Ramanujan results because on expanding the last terms and simplifying both the sides of (7) and (8) one obtains:

$$2(x_1^2 + y_1^2 - x_1y_1) = 2(z_1^2 + w_1^2 - z_1w_1) \tag{9}$$

$$2(x_1^2 + y_1^2 - x_1y_1)^2 = 2(z_1^2 + w_1^2 - z_1w_1)^2 \tag{10}$$

By varying the choices for a, b, c, d one obtains infinitely many solutions of (5) and (6).
The main purpose of this paper is to generate infinite quadruple sequences of solutions $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$ to (7) and (8) starting from just one set $\{x_1, y_1, z_1, w_1\}$ of positive integers such that $x_n^2 + y_n^2 - x_ny_n = z_n^2 + w_n^2 - z_nw_n \neq 0$, using a suitable extended matrix in two parameters (2) wherein each matrix has a determinant of the form

$$x_1^2 + y_1^2 - x_1y_1 = \frac{1}{2}(x_1^2 + y_1^2 + (x_1 - y_1)^2).$$

This new idea enables us to construct a pair of two variable homogeneous polynomials of degree n which are useful to evaluate $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$.

The required extended matrix identity in two parameters: In order to achieve our objective, we shall consider the set of all the matrices appearing in the identity (3) with $s = t = -1$:

$$A(x, y) = \begin{pmatrix} x & y \\ -y & x - y \end{pmatrix} \tag{11}$$

where x and y are any two real numbers such that $x^2 + y^2 - xy \neq 0$. Clearly $A(x, y) \in GL_2(\mathbb{R})$, general linear group of all 2 by 2 invertible matrices. Let $\mathbb{A}_{(x,y)}$ be the set of all matrices of the form (11) where x and y are any two real numbers such that $x^2 + y^2 - xy \neq 0$.
Let $A(x_1, y_1)$ and $A(x_2, y_2)$ be any two matrices in $\mathbb{A}_{(x,y)}$. Then we shall show that $A(x_3, y_3) = A(x_1, y_1)A(x_2, y_2)$ is also in $\mathbb{A}_{(x,y)}$.

$$A(x_3, y_3) = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 - y_1 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 - y_2 \end{pmatrix}$$

$$= \begin{pmatrix} (x_1x_2 - y_1y_2) & (x_1y_2 + y_1x_2 - y_1y_2) \\ -(x_1y_2 + y_1x_2 - y_1y_2) & (x_1x_2 - y_1y_2) - (x_1y_2 + y_1x_2 - y_1y_2) \end{pmatrix}$$

where $x_3 = x_1x_2 - y_1y_2$ and $y_3 = (x_1y_2 + y_1x_2 - y_1y_2)$ are again real numbers and $x_3^2 + y_3^2 - x_3y_3 = (x_1^2 + y_1^2 - x_1y_1)(x_2^2 + y_2^2 - x_2y_2) \neq 0$. Moreover,

$$A(x_1, y_1)A(x_2, y_2) = A(x_2, y_2)A(x_1, y_1).$$

Hence $\mathbb{A}_{(x,y)}$ is a commutative matrix subgroup of $GL_2(\mathbb{R})$. In this matrix subgroup, Ramanujan result deduced in (9) and (10) can be restated as follows:

$$2det[A(x_1, y_1)] = 2 \ \ det[A(z_1, w_1)] \tag{12}$$

$$2\{det[A(x_1, y_1)]\}^2 = 2 \ \ \{det[A(z_1, w_1)]\}^2 \tag{13}$$

Now, the infinite quadruple solutions $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$ can be computed as follows:

$$A(x_n, y_n) = [A(x_1, y_1)]^n \tag{14}$$

$$A(z_n, w_n) = [A(z_1, w_1)]^n \tag{15}$$

Using the standard theorem on product of determinants, it is straight forward to workout

$$2 \ \ det[A(x_n, y_n)] = 2 \ \ det[A(z_n, w_n)] \tag{16}$$

$$2 \ \ \{det[A(x_n, y_n)]\}^2 = 2 \ \ \{det[A(z_n, w_n)]\}^2 \tag{17}$$

In order to workout (14) and (15), we shall use the following eigen relations:

$$\begin{pmatrix} x & y \\ -y & x - y \end{pmatrix}^n = \frac{1}{\omega^2 - \omega} \begin{pmatrix} 1 & 1 \\ \omega & \omega^2 \end{pmatrix} \begin{pmatrix} x + \omega y & 0 \\ 0 & x + \omega^2 y \end{pmatrix}^n \begin{pmatrix} \omega^2 & -1 \\ -\omega & 1 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$ is the cube root of unity. As a result, $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$ have the following binet forms:

$$x_n = \frac{-\omega^2(x_1 + \omega y_1)^n + \omega(x_1 + \omega^2 y_1)^n}{\omega - \omega^2} \tag{18}$$

$$y_n = \frac{(x_1 + \omega y_1)^n - (x_1 + \omega^2 y_1)^n}{\omega - \omega^2} \tag{19}$$

$$z_n = \frac{-\omega^2(z_1 + \omega w_1)^n + \omega(z_1 + \omega^2 w_1)^n}{\omega - \omega^2} \tag{20}$$

$$w_n = \frac{(z_1 + \omega w_1)^n - (z_1 + \omega^2 w_1)^n}{\omega - \omega^2} \tag{21}$$

Also, it is interesting to workout the following binary recurrence relations for $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$:

$$x_{n+1} = (2x_1 - y_1) \ \ x_n - (x_1^2 + y_1^2 - x_1y_1) \ \ x_{n-1}, x_0 = 1, x_1 = a + b + c \tag{22}$$

$$y_{n+1} = (2x_1 - y_1) \ \ y_n - (x_1^2 + y_1^2 - x_1y_1) \ \ y_{n-1}, y_0 = 0, y_1 = b + c + d \tag{23}$$

$$z_{n+1} = (2z_1 - w_1)\ \ z_n - (z_1^2 + w_1^2 - z_1 w_1)\ \ z_{n-1}, z_0 = 1, z_1 = c + d + a \tag{24}$$

$$w_{n+1} = (2z_1 - w_1)\ \ w_n - (z_1^2 + w_1^2 - z_1 w_1)\ \ w_{n-1}, w_0 = 0, w_1 = d + a + b \tag{25}$$

where a, b, c, d are any four real numbers such that $ad = bc$.

A pair of evaluating polynomials: The binet forms (18) – (21) define a Pair of Evaluating Polynomials,namely, $P_n(x, y)$ and $Q_n(x, y)$ given by

$$P_n(x, y) = \frac{-\omega^2(x + \omega y)^n + \omega(x + \omega^2 y)^n}{\omega - \omega^2} \tag{26}$$

$$Q_n(x, y) = \frac{(x + \omega y)^n - (x + \omega^2 y)^n}{\omega - \omega^2} \tag{27}$$

So that one can evaluate

$$P_n(x_1, y_1) = x_n, Q_n(x_1, y_1) = y_n, P_n(z_1, w_1) = z_n, Q_n(z_1, w_1) = w_n.$$

It is also a quite convenient method for computing $(P_n(x, y), Q_n(x, y))$ using the following extended matrix identity:

$$\begin{pmatrix} P_n(x,y) & Q_n(x,y) \\ -Q_n(x,y) & P_n(x,y) - Q_n(x,y) \end{pmatrix} = \begin{pmatrix} x & y \\ -y & x - y \end{pmatrix}^n$$

§3. Combinatorial properties of Brahmagupta Polynomials

The Brahmagupta polynomials in one parameter exhibit the following combinatorial properties:

Theorem 1([4]) *The Brahmagupta polynomials in one parameter have the following binet forms :*

$$\left.\begin{aligned} x_n &= \frac{1}{2}\left[(x + y\sqrt{t})^n + (x - y\sqrt{t})^n\right] \\ y_n &= \frac{1}{2\sqrt{t}}\left[(x + y\sqrt{t})^n - (x - y\sqrt{t})^n\right] \end{aligned}\right\}. \tag{28}$$

They satisfy the following three -term recurrences :

$$\left.\begin{aligned} x_{n+1} &= 2\,x\,x_n - (x^2 - ty^2)\,x_{n-1}\,,\ x_0 = 1,\ x_1 = x \\ y_{n+1} &= 2\,x\,y_n - (x^2 - ty^2)\,y_{n-1}\,,\ y_0 = 0,\ y_1 = y \end{aligned}\right\}. \tag{29}$$

The Brahmagupta polynomials in two parameters exhibit the following similar combinatorial properties:

Theorem 2([3]) $\left(x_n + \dfrac{s}{2} y_n\right)$ *and* y_n *have the following binet forms:*

$$\left.\begin{array}{l} \left(x_n + \frac{s}{2} y_n\right) = \frac{1}{2}\left[(x+\lambda_+ y)^n + (x+\lambda_- y)^n\right] \\ \\ y_n = \frac{1}{2\sqrt{(s^2/4)+t}}\left[(x+\lambda_+ y)^n - (x+\lambda_- y)^n\right] \end{array}\right\} \qquad (30)$$

where $\lambda_\pm = \dfrac{s}{2} \pm \sqrt{\dfrac{s^2}{4} + t}$.
As a consequence, the Brahmagupta polynomials in two parameters satisfy the following three -term recurrences:

$$\left.\begin{array}{rcl} x_{n+1} &=& (2x+sy)x_n - \left(x^2 + sxy - ty^2\right) x_{n-1}, x_0 = 1 \ , \ x_1 = x \\ \\ y_{n+1} &=& (2x+sy)y_n - \left(x^2 + sxy - ty^2\right) y_{n-1}, y_0 = 0, y_1 = y \end{array}\right\}. \qquad (31)$$

The first few Brahmagupta polynomials in two parameters are:

$$x_0 = 1,\ x_1 = x,\ x_2 = x^2 + ty^2,\ x_3 = x^3 + 3txy^2 + sty^3,$$

$$x_4 = x^4 + 4stx^3y + 6tx^2y^2 + stxy^3 + (t+s^2)y^4, \cdots;$$

$$y_0 = 0,\ y_1 = y,\ y_2 = 2xy + sy^2,\ y_3 = 3x^2y + 3sxy^2 + (t+s^2)y^3,$$

$$y_4 = 4x^3y + 6sx^2y^2 + 4(t+s^2)xy^3 + s(2t+s^2)y^4, \cdots.$$

In [4], as a consequence of Theorem 1. it is shown that Brahmagupta polynomials are polynomial solutions of t − *Cauchy's* - *Reimann* equations:

$$\left.\begin{array}{rcccl} \dfrac{\partial x_n}{\partial x} &=& \dfrac{\partial y_n}{\partial y} &=& n\ x_{n-1} \\ \\ \dfrac{\partial x_n}{\partial y} &=& t\ \dfrac{\partial y_n}{\partial y} &=& n\ t\ y_{n-1} \end{array}\right\}. \qquad (32)$$

As a further consequence, x_n and y_n are shown to satisfy the wave equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right) U = 0. \qquad (33)$$

The corresponding extended result is the following theorem :

Theorem 3 *The polynomials* $x_n(x, y, s, t)$ *and* $y_n(x, y, s, t)$ *satisfy the following second order linear partial differential equations :*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{s}{t}\frac{\partial^2}{\partial x \partial y} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right) U = 0. \qquad (34)$$

Proof Partial differentiation of (30) yields,

$$\frac{\partial}{\partial x} \quad (x_n + \frac{s}{2}y_n) = \left(-\frac{s}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) y_n = n\left(x_{n-1} + \frac{s}{2}y_{n-1}\right) \tag{35}$$

$$\frac{\partial}{\partial y}\left(x_n + \frac{s}{2}y_n\right) = n\left[\frac{s}{2}\left(x_{n-1} + \frac{s}{2}y_{n-1}\right) + \left(\frac{s^2}{4} + t\right) y_{n-1}\right] \tag{36}$$

$$\frac{\partial y_n}{\partial x} = n y_{n-1} \tag{37}$$

So we may simplify the above as follows-

$$\frac{\partial x_n}{\partial x} = -\left(s\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) y_n \tag{38}$$

$$\frac{\partial x_n}{\partial y} = -\frac{s}{2}\frac{\partial y_n}{\partial y} + \frac{s}{2}\left(-\frac{s}{2}\frac{\partial y_n}{\partial x} + \frac{\partial y_n}{\partial y}\right) + \left(\frac{s^2}{4} + t\right)\frac{\partial y_n}{\partial x} = t\frac{\partial y_n}{\partial x}$$

They naturally lead to

$$t\frac{\partial^2 y_n}{\partial x^2} + \frac{\partial}{\partial y}\left(s\ \frac{\partial}{\partial x} - \ \frac{\partial}{\partial y}\right) y_n = 0 \tag{39}$$

which is same as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{s}{t}\ \frac{\partial^2}{\partial x\,\partial y} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right)\ y_n = 0 \tag{40}$$

Also, the Partial differential equation for x_n may be derived as follows-

$$\frac{\partial x_n}{\partial x} + \frac{s}{t}\frac{\partial x_n}{\partial y} = \frac{\partial y_n}{\partial y} \tag{41}$$

$$\frac{1}{t}\frac{\partial x_n}{\partial y} = \frac{\partial y_n}{\partial x} \tag{42}$$

As a direct consequence, x_n satisfies the following Partial differential equation-

$$\left(\frac{\partial^2}{\partial x^2} + \frac{s}{t}\ \frac{\partial^2}{\partial x\,\partial y} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right)\ x_n = 0 \tag{43}$$

□

References

[1] R. Beauregard and E.R. Suryanarayan , A Ramanujan Result Viewed From Matrix Algebra, *The College Mathematics Journal*, 33.3 212-214 (2002).

[2] S. Ramanujan , "*Note Books* (2 Volumes)", Tata Institute of Fundamental Research, Mumbai 1957.

[3] R. Rangarajan , Rangaswamy and E.R. Suryanarayan, Properties of Brahmagupta Polynomials with two Parameters, *Proceedings of the Third Annual Conference of the Society For Special Functions and their Applications*, BHU, Varanasi March 4-6, 2002. Editors R.S.Pathak, A.K. Agarwal and M.A. Pathan. Society For Special Functions and their Applications, 57-69 (2003).

[4] E.R. Suryanarayan, The Brahmagupta polynomials, *Fibbonacci Quaterly*, **34.1** 30-39 (1996).