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On Functions Preserving Convergence of Series in Fuzzy n-Normed Spaces

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Abstract: The purpose of this paper is to introduce finite convergence sequences and functions preserving convergence of series in fuzzy n-normed spaces.

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§1. Introduction

A *Pseudo-Euclidean space* is a particular Smarandache space defined on a Euclidean space $\mathbf{R}^n$ such that a straight line passing through a point p may turn an angle $\theta_p \geq 0$. If $\theta_p > 0$, then p is called a non-Euclidean point. Otherwise, a Euclidean point. In this paper, normed spaces are considered to be Euclidean, i.e., every point is Euclidean. In [7], S. Gähler introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [9,12,14,15]. In [9], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$- norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1,2,4,5,6,11,13,18]. In [16], A. Narayanan and S. Vijayabalaji have extended the n-normed linear space to fuzzy n-normed linear space and in [20] the authors have studied the completeness of fuzzy n-normed spaces.

The main purpose of this paper is to study the results concerning infinite series (see, [3,17,19,21]) in fuzzy n-normed spaces. In section 2, we quote some basic definitions of fuzzy n-normed spaces. In section 3, we consider the absolutely convergent series in fuzzy n- normed spaces and obtain some results on it. In section 4, we study the property of finite convergence sequences in fuzzy n-normed spaces. In the last section we introduce and study the concept of

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function preserving convergence of series in fuzzy n-norm spaces and obtain some results.

§2. Preliminaries

Let n be a positive integer, and let X be a real vector space of dimension at least n. We recall the definitions of an n-seminorm and a fuzzy n-norm [16].

Definition 2.1 *A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from X^n to $[0, \infty)$ is called an n-seminorm on X if it has the following four properties:*

(S1) $\|x_1, x_2, \ldots, x_n\| = 0$ if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(S2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

(S3) $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, , \ldots, x_{n-1}, x_n\|$ for any real c;

(S4) $\|x_1, \ldots, x_{n-1}, y + z\| \leqslant \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

An n-seminorm is called a n-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

Definition 2.2 *A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if:*

(F1) For all $t \leqslant 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$;

(F2) For all $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(F3) $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

(F4) For all $t > 0$ and $c \in \mathbb{R}$, $c \neq 0$,

$$N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});$$

(F5) For all $s, t \in \mathbb{R}$,

$$N(x_1, \ldots, x_{n-1}, y + z, s + t) \geq \min \left\{ N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, t) \right\}.$$

(F6) $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and

$$\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

The pair (X, N) will be called a *fuzzy $n-$normed space.*

Theorem 2.1 *Let $\mathcal{A}$ be the family of all finite and nonempty subsets of fuzzy n-normed space (X, N) and $A \in \mathcal{A}$. Then the system of neighborhoods*

$$\mathcal{B} = \{B(t, r, A) \colon t > 0,\ 0 < r < 1,\ A \in \mathcal{A}\}$$

where $B(t,r,A) = \{x \in X : N(a_1, \cdots, a_{n-1}, x, t) > 1 - r,\ a_1, \cdots, a_{n-1} \in A\}$ *is a base of the null vector* θ*, for a linear topology on* X*, named* N*-topology generated by the fuzzy* n*-norm* N*.*

Proof We omit the proof since it is similar to the proof of Theorem 3.6 in [8]. □

Definition 2.3 *A sequence* $\{x_k\}$ *in a fuzzy* $n-$ *normed space* (X,N) *is said to converge to* x *if given* $r > 0,\ t > 0,\ 0 < r < 1,$ *there exists an integer* $n_0 \in \mathbf{N}$ *such that* $N(x_1, x_2, \ldots, x_{n-1}, x_k - x, t) > 1 - r$ *for all* $k \geq n_0$*.*

Definition 2.4 *A sequence* $\{x_k\}$ *in a fuzzy* $n-$ *normed space* (X,N) *is said to be Cauchy sequence if given* $\epsilon > 0,\ t > 0,\ 0 < \epsilon < 1,$ *there exists an integer* $n_0 \in \mathbf{N}$ *such that* $N(x_1, x_2, \ldots, x_{n-1}, x_m - x_k, t) > 1 - \epsilon$ *for all* $m,\ k \geq n_0$*.*

Theorem 2.1([13]) *Let* N *be a fuzzy* $n-$ *norm on* X*. Define for* $x_1, x_2, \ldots, x_n \in X$ *and* $\alpha \in (0,1)$

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\ \}.$$

Then the following statements hold.

(A_1) for every $\alpha \in (0,1)$, $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$ *is an* $n-$ *seminorm on* X;

(A_2) If $0 < \alpha < \beta < 1$ *and* $x_1, x_2, \ldots, x_n \in X$ *then*

$$\|x_1, x_2, \ldots, x_n\|_\alpha \leq \|x_1, x_2, \ldots, x_n\|_\beta.$$

Example 2.3[10, Example 2.3] Let $\|\bullet, \bullet, \ldots, \bullet\|$ be a n-norm on X. Then define $N(x_1, x_2, \ldots, x_n, t) = 0$ if $t \leq 0$ and, for $t > 0$,

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then the seminorms (2.1) are given by

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1-\alpha}\|x_1, x_2, \ldots, x_n\|.$$

§3. Absolutely Convergent Series in Fuzzy $n-$Normed Spaces

In this section we introduce the notion of the absolutely convergent series in a fuzzy n-normed space (X,N) and give some results on it.

Definition 3.1 *The series* $\sum\limits_{k=1}^{\infty} x_k$ *is called absolutely convergent in* (X,N) *if*

$$\sum_{k=1}^{\infty} \|a_1, \ldots,\ a_{n-1},\ x_k\|_\alpha < \infty$$

for all $a_1, \ldots,\ a_{n-1} \in X$ *and all* $\alpha \in (0,1)$*.*

Using the definition of $\|...\|_\alpha$ the following lemma shows that we can express this condition directly in terms of N.

Lemma 3.1 *The series* $\sum_{k=1}^{\infty} x_k$ *is absolutely convergent in* (X, N) *if, for every* $a_1, ..., a_{n-1} \in X$ *and every* $\alpha \in (0,1)$ *there are* $t_k \geq 0$ *such that* $\sum_{k=1}^{\infty} t_k < \infty$ *and* $N(a_1, \ldots, a_{n-1}, x_k, t_k) \geq \alpha$ *for all* k.

proof Let $\sum_{k=1}^{\infty} x_k$ be absolutely convergent in (X, N). Then

$$\sum_{k=1}^{\infty} \|a_1, ..., a_{n-1}, x_k\|_\alpha < \infty$$

for every $a_1, ..., a_{n-1} \in X$ and every $\alpha \in (0,1)$. Let $a_1, ..., a_{n-1} \in X$ and $\alpha \in (0,1)$. For every k there is $t_k \geq 0$ such that $N(a_1, \ldots, a_{n-1}, x_k, t_k) \geq \alpha$ and

$$t_k < \|a_1, ..., a_{n-1}, x_k\|_\alpha + \frac{1}{2^k}.$$

Then

$$\sum_{k=1}^{\infty} t_k < \sum_{k=1}^{\infty} \|a_1, ..., a_{n-1}, x_k\|_\alpha + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

The other direction is even easier to show. □

Definition 3.2 *A fuzzy* $n-$*normed space* (X, N) *is said to be sequentially complete if every Cauchy sequence in it is convergent.*

Lemma 3.2 *Let* (X, N) *be sequentially complete, then every absolutely convergent series* $\sum_{k=1}^{\infty} x_k$ *converges and*

$$\left\| a_1, ..., a_{n-1}, \sum_{k=1}^{\infty} x_k \right\|_\alpha \leq \sum_{k=1}^{\infty} \|a_1, ..., a_{n-1}, x_k\|_\alpha$$

for every $a_1, ..., a_{n-1} \in X$ *and every* $\alpha \in (0,1)$.

Proof Let $\sum_{k=1}^{\infty} x_k$ be an infinite series such that $\sum_{k=1}^{\infty} \|a_1, ..., a_{n-1}, x_k\|_\alpha < \infty$ for every $a_1, ..., a_{n-1} \in X$ and every $\alpha \in (0,1)$. Let $y_n = \sum_{k=1}^{n} x_k$ be a partial sum of the series. Let $a_1, ..., a_{n-1} \in X$, $\alpha \in (0,1)$ and $\epsilon > 0$. There is N such that $\sum_{k=N+1}^{\infty} \|a_1, ..., a_{n-1}, x_k\|_\alpha < \epsilon$.

Then, for $n > m \geq N$,

$$\begin{aligned} \left| \|a_1, ..., a_{n-1}, y_n\|_\alpha - \|a_1, ..., a_{n-1}, y_m\|_\alpha \right| &\leq \|a_1, ..., a_{n-1}, y_n - y_m\|_\alpha \\ &\leq \sum_{k=m+1}^{n} \|a_1, ..., a_{n-1}, x_k\|_\alpha \\ &\leq \sum_{k=N+1}^{\infty} \|a_1, ..., a_{n-1}, x_k\|_\alpha \\ &< \epsilon. \end{aligned}$$

This is shows that $\{y_n\}$ is a Cauchy sequence in (X, N) . But since (X, N) is sequentially complete, the sequence $\{y_n\}$ converges and so the series $\sum_{k=1}^{\infty} x_k$ converges. □

Definition 3.3 *Let I be any denumerable set . We say that the family $(x_\alpha)_{\alpha \in I}$ of elements in a complete fuzzy $n-$normed space (X, N) is absolutely summable, if for a bijection Ψ of* ***N(t****he set of all natural numbers***)** *onto I the series $\sum_{n=1}^{\infty} x_{\Psi(n)}$ is absolutely convergent.*

The following result may not be surprising but the proof requires some care.

Theorem 3.1 *Let $(x_\alpha)_{\alpha \in I}$ be an absolutely summable family of elements in a sequentially complete fuzzy $n-$ normed space (X, N). Let (B_n) be an infinite sequence of a non-empty subset of A, such that $A = \bigcup_n B_n$, $B_i \cap B_j = \varnothing$ for $i \neq j$, then if $z_n = \sum_{\alpha \in B_n} x_\alpha$, the series $\sum_{n=0}^{\infty} z_n$ is absolutely convergent and $\sum_{n=0}^{\infty} z_n = \sum_{\alpha \in I} x_\alpha$.*

Proof It is easy to see that this is true for finite disjoint unions $I = \bigcup_{n=1}^{N} B_n$. Now consider the disjoint unions $I = \bigcup_{n=1}^{\infty} B_n$. By Lemma 3.2

$$\begin{aligned} \sum_{n=1}^{\infty} \|a_1, ..., a_{n-1}, z_n\|_\alpha &\leq \sum_{n=1}^{\infty} \sum_{i \in B_n} \|a_1, ..., a_{n-1}, x_i\|_\alpha \\ &= \sum_{i \in I} \|a_1, ..., a_{n-1}, x_i\|_\alpha < \infty \end{aligned}$$

for every $a_1, ..., a_{n-1} \in X$, and every $\alpha \in (0, 1)$. Therefore, $\sum_{n=0}^{\infty} z_n$ is absolutely convergent. Let $y = \sum_{i \in I} x_i$, $z = \sum_{n=1}^{\infty} z_n$. Let $\epsilon > 0$, $a_1, ..., a_{n-1} \in X$ and $\alpha \in (0, 1)$. There is a finite set $J \subset I$ such that

$$\sum_{i \notin J} \|a_1, ..., a_{n-1}, x_i\|_\alpha < \frac{\epsilon}{2}.$$

Choose N large enough such that $B = \bigcup_{n=1}^{N} B_n \supset J$ and

$$\left\| a_1, ..., a_{n-1}, z - \sum_{n=1}^{N} z_n \right\|_\alpha < \frac{\epsilon}{2}.$$

Then

$$\left\|a_1, ..., \ a_{n-1}, \ y - \sum_{i \in B} x_i \right\|_\alpha < \frac{\epsilon}{2}.$$

By the first part of the proof

$$\sum_{n=1}^{N} z_n = \sum_{i \in B} x.$$

Therefore, $\|a_1, ..., \ a_{n-1}, \ y - z\|_\alpha < \epsilon$. This is true for all ϵ so $\|a_1, ..., \ a_{n-1}, \ y - z\|_\alpha = 0$. This is true for all $a_1, ..., \ a_{n-1} \in X$, $\alpha \in (0, \ 1)$ and $(X, \ N)$ is Hausdorff see [8, Theorem 3.1]. Hence $y = z$. □

Definition 3.4 *Let (X^*, N) be the dual of fuzzy n-normed space (X, N). A linear functional $f \colon X^* \to K$ where K is a scalar field of X is said to be bounded linear operator if there exists a $\lambda > 0$ such that*

$$\|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha \leq \lambda \|a_1, \cdots, a_{n-1}, x_k\|_\alpha,$$

for all $a_1, \cdots, a_{n-1} \in X$ and all $\alpha \in (0, 1)$.

Definition 3.5 *The series $\sum_{k=1}^{\infty} x_k$ is said to be weakly absolutely convergent in (X, N) if*

$$\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha < \infty$$

for all $f \in X^*$, all $a_1, \cdots, a_{n-1} \in X$ and all $\alpha \in (0, 1)$.

Theorem 3.2 *Let the series $\sum_{k=1}^{\infty} x_k$ be weakly absolutely convergence in (X, N). Then there exists a constant $\lambda > 0$ such that*

$$\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha \leq \lambda \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha$$

Proof Let $\{e_r\}_{r=1}^{\infty}$ be a standard basis of the space (X, N). Define continuous operators $S_r \colon X^* \to X$ where $r \in \mathbb{Z}$ by the formula $S_r(f) = \sum_{k=1}^{r} f(x_k) e_k$, we have

$$\|a_1, \cdots, a_{n-1}, S_r(f)\|_\alpha = \sum_{k=1}^{r} \|a_1, \cdots, a_{n-1}, f(x_k) e_k\|_\alpha.$$

Since for any fixed $f \in X^*$, the numbers $\|a_1, \cdots, a_{n-1}, S_r(f)\|_\alpha$ are bounded by $\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha$, by Banach-Steinhaus theorem, we have

$$\sup_r \|a_1, \cdots, a_{n-1}, S_r(f)\|_\alpha = \lambda < \infty.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha &= \sup_r \|a_1, \cdots, a_{n-1}, S_r(f)\|_\alpha \\ &\leq \lambda \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha. \end{aligned}$$

§4. Finite Convergent Sequences in Fuzzy $n-$Normed Spaces

In this section our principal goal is to show that every sequence having finite convergent property is Cauchy and every Cauchy sequence has a subsequence which has finite convergent property in every metrizable fuzzy $n-$normed space (X, N).

Definition 4.1 *A sequence $\{x_k\}$ in a fuzzy $n-$normed space (X, N) is said to have finite convergent property if*

$$\sum_{j=1}^{\infty} \|a_1, ..., \ a_{n-1}, \ x_j - x_{j-1}\|_{\alpha} < \infty$$

for all $a_1, ..., \ a_{n-1} \in X$ and all $\alpha \in (0, 1)$.

Definition 4.2 *A fuzzy $n-$ normed space (X, N) is said to be metrizable, if there is a metric d which generates the topology of the space.*

Theorem 4.1 *Let (X, N) be a metrizable fuzzy $n-$normed space, then every sequence having finite convergent propperty is Cauchy and every Cauchy sequence has a subsequence which has finite convergent property.*

proof Since X is metrizable, there is a sequence $\left\{\|a_{1,r}, ..., \ a_{n-1,r}, \ x\|_{\alpha_r}\right\}$ for all $a_{1,r}, ..., \ a_{n-1,r} \in X$ and all $\alpha_r \in (0, 1)$ generating the topology of X. We choose an increasing sequence $\{m_{k,1}\}$ such that

$$\sum_{k=1}^{\infty} \left\|a_{1,1}, ..., \ a_{n-1,1}, \ x_{m_{k+1,1}} - x_{m_{k,1}}\right\|_{\alpha_1} < \infty$$

where $a_{1,1}, ..., \ a_{n-1,1} \in X$ and $\alpha_1 \in (0, 1)$. Then we choose a subsequence $m_{k,2}$ of $m_{k,1}$ such that

$$\sum_{k=1}^{\infty} \left\|a_{1,2}, ..., \ a_{n-1,2}, \ x_{m_{k+1,2}} - x_{m_{k,2}}\right\|_{\alpha_2} < \infty$$

where $a_{1,2}, ..., \ a_{n-1,2} \in X$ and $\alpha_2 \in (0, 1)$. Continuing in this way we construct recursively sequences $m_{k,r}$ such that $m_{k,r+1}$ is a subsequence of $m_{k,r}$ and such that

$$\sum_{k=1}^{\infty} \left\|a_{1,r}, ..., \ a_{n-1,r}, \ x_{m_{k+1,r}} - x_{m_{k,r}}\right\|_{\alpha_r} < \infty$$

for all $a_{1,r}, ..., \ a_{n-1,r} \in X$ and all $\alpha_r \in (0, 1)$. Now consider the diagonal sequence $m_k = m_{k,k}$. Let $r \in \mathbb{N}$. The sequence $\{m_k\}_{k=r}^{\infty}$ is a subsequence of $\{m_{k,r}\}_{k=r}^{\infty}$. Let $k \geq r$. There are pairs of integers (u, v), $u < v$ such that $m_k = m_{u,r}$ and $m_{k+1} = m_{v,r}$. Then by the triangle inequality

$$\left\|a_{1,r}, ..., \ a_{n-1,r}, \ x_{m_{k+1}} - x_{m_k}\right\|_{\alpha_r} \leq \sum_{i=u}^{v-1} \left\|a_{1,r}, ..., \ a_{n-1,r}, \ x_{m_{i+1,r}} - x_{m_{i,r}}\right\|_{\alpha_r}$$

and therefore,

$$\sum_{k=r}^{\infty} \left\|a_1, ..., \ a_{n-1}, \ x_{m_{k+1}} - x_{m_k}\right\|_{\alpha} \leq \sum_{j=r}^{\infty} \left\|a_1, ..., \ a_{n-1}, \ x_{m_{j+1,r}} - x_{m_{j,r}}\right\|_{\alpha}$$

for all $a_1, ..., \ a_{n-1} \in X$ and all $\alpha \in (0,1)$. The statement of the theorem follows. $\square$

The above theorem shows that many Cauchy sequence has a subsequence which has finite convergent. Therefore, it is natural to ask for an example of Cauchy sequence has a subsequence which has not finite convergent property.

Example 4.2 We consider the set S consisting of all convergent real sequences. Let X be the space of all functions $f : S \to \mathbb{R}$ equipped with the topology of pointwise convergence. This topology is generated by

$$\|f_{1,s}, ..., \ f_{n-1,s}, \ f\|_{\alpha_s} = |f(s)|,$$

for all $f_{1,s}, ..., \ f_{n-1,s}, f \in X$ and all $\alpha_s \in (0,1)$, where $s \in S$. Then consider the sequence $f_n \in X$ defined by $f_n(s) = s_n$ where $s = (s_n) \in S$. The sequence f_n is a Cauchy sequence in X but there is no subsequence f_{n_k} such that

$$\sum_{k=1}^{\infty} \left\|f_{1,s}, ..., \ f_{n-1,s}, \ f_{n_{k+1}} - f_{n_k}\right\|_{\alpha_s} < \infty$$

for all $s \in S$. We see this as follows. If $n_1 < n_2 < n_3 < ...$ is a sequence then define $s_n = (-1)^k \ \frac{1}{k}$ for $n_k \le n < n_{k+1}$. Then $s = (s_n) \in S$ but

$$\sum_{k=1}^{\infty} \left\|f_{1,s}, ..., \ f_{n-1,s}, \ f_{n_{k+1}} - f_{n_k}\right\|_{\alpha_s} = \sum_{k=1}^{\infty} \left|s_{n_{k+1}} - s_{n_k}\right| \ge \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

§5. Functions Preserving Convergence of Series in Fuzzy $n-$Normed Spaces

In this section we shall introduce the functions $f\colon X \to X$ that preserve convergence of series in fuzzy $n-$normed spaces. Our work is an extension of functions $f\colon \mathbb{R} \to \mathbb{R}$ that preserve convergence of series studied in [19] and [3].

We read in Cauchy's condition in (X, N) as follows: the series $\sum_{k=1}^{\infty} x_k$ converges if and only if for every $\epsilon > 0$ there is an N so that for all $n \ge m \ge N$,

$$\|a_1 \cdots, a_{n-1}, \sum_{k=m}^{n} x_k\| < \epsilon,$$

where $a_1 \cdots, a_{n-1} \in X$.

Definition 5.1 *A function $f\colon X \times X \to X$ is said to be additive in fuzzy n-normed space (X, N) if*

$$\|a_1, \cdots, a_{n-1}, f(x,y)\|_\alpha = \|a_1, \cdots, a_{n-1}, f(x)\|_\alpha + \|a_1, \cdots, a_{n-1}, f(y)\|_\alpha,$$

for each $x, y \in X$, $a_1, \cdots, a_{n-1} \in X$ and for all $\alpha \in (0,1)$.

Definition 5.2 *A function $f\colon X \to X$ is convergence preserving (abbreviated CP) in (X, N) if for every convergent series $\sum_{k=1}^{\infty} x_k$, the series $\sum_{k=1}^{\infty} f(x_k)$ is also convergent, i.e., for every $a_1, \cdots, a_{n-1} \in X$,*

$$\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, f(x_k)\|_\alpha < \infty$$

whenever $\sum_{k=1}^{\infty} \|a_1, \cdots, a_{n-1}, x_k\|_\alpha < \infty$.

Theorem 5.1 *Let* (X, N) *be a fuzzy* n*-normed space and* $f\colon X \to X$ *be an additive and continuous function in the neighborhood* $B(t, r, A)$*. Then the function* f *is* CP *of infinite series in* (X, N)*.*

Proof Assume that f is additive and continuous in $B(\alpha, \delta, A) = \{x \in X\colon \|a_1, \cdots, a_{n-1}, x\|_\alpha < \delta\}$, where $a_1, \cdots, a_{n-1} \in A$ and $\delta > 0$. From additivity of f in $B(\alpha, \delta, A)$ implies that $f(0) = 0$. Let $\sum_{k=1}^{\infty} x_k$ be a absolute convergent series and $x_k \in X$ $(k = 1, 2, 3, \cdots)$. We show that $\sum_{k=1}^{\infty} f(x_k)$ is also absolute convergent.

By Cauchy condition for convergence of series, there exists a $k \in \mathbb{N}$ such that for every $p \in \mathbb{N}$

$$\|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} x_j\|_\alpha < \frac{\delta}{2}.$$

From this we have

$$\|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{\infty} x_j\|_\alpha < \frac{\delta}{2}.$$

By the additivity of f in $B(\alpha, \delta, A)$, we get

$$\|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{k+p} x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} f(x_j)\|_\alpha < \frac{\delta}{2}.$$

Now, let $y_p = \sum_{j=k+1}^{k+p} x_j$ for $p = 1, 2, 3, \cdots$ and $y = \sum_{j=k+1}^{\infty} x_j$ belong to the neighborhood $B(\alpha, \delta, A)$. The function f is continuous in $B(\alpha, \delta, A)$, i.e., $f(y_p) \to f(y)$ because $y_p \to y$ for $p \to \infty$. Hence

$$\lim_{p\to\infty} \|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{k+p} x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, f(\sum_{j=k+1}^{\infty} x_j)\|_\alpha.$$

This implies

$$\lim_{p\to\infty} \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{k+p} f(x_j)\|_\alpha = \|a_1, \cdots, a_{n-1}, \sum_{j=k+1}^{\infty} f(x_j)\|_\alpha$$

and this guarantee the convergence of the series $\sum_{j=k+1}^{\infty} f(x_j)$ and therefore the series $\sum_{j=1}^{\infty} f(x_j)$ must also be convergent in X, i.e., the function f is CP infinite series in (X, N). $\square$

References

[1] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, *J. Fuzzy Math.*, **11** (2003), no. 3, 687-705.

[2] S.C. Chang and J. N. Mordesen, Fuzzy linear operators and fuzzy normed linear spaces, *Bull. Calcutta Math. Soc.*, **86** (1994), no. 5, 429-436.

[3] M. Dindos, I. Martisovits and T. Salat, Remarks on infinite series in linear normed spaces, http://tatra.mat.savba.sk/Full/19/04DINDOS.ps

[4] C. Felbin, Finite- dimensional fuzzy normed linear space, *Fuzzy Sets and Systems*, **48** (1992), no. 2, 239-248.

[5] C. Felbin, The completion of a fuzzy normed linear space, *J. Math. Anal. Appl.*, **174** (1993), no. 2, 428-440.

[6] C. Felbin, Finite dimensional fuzzy normed linear space. II., *J. Anal.*, **7** (1999), 117-131.

[7] S. Gähler, Untersuchungen uber verallgemeinerte m-metrische Raume, I, II, III., *Math. Nachr.*, **40** (1969), 165-189.

[8] I. Golet, On generalized fuzzy normed spaces, *Int. Math. Forum*, **4**(2009)no. 25, 1237-1242.

[9] H. Gunawan and M. Mashadi, On n-normed spaces, *Int. J. Math. Math. Sci.*, **27** (2001), no. 10, 631-639.

10 Hans Volkmer and Sayed Elagan, Some remarks on fuzzy n-normed spaces, *J. Topology. Appl.*, (under review).

[11] A. K. Katsaras, Fuzzy topological vector spaces. II., *Fuzzy Sets and Systems*, **12** (1984), no. 2, 143-154.

[12] S. S. Kim and Y. J. Cho, Strict convexity in linear $n-$ normed spaces, *Demonstratio Math.*,. **29** (1996), no. 4, 739-744.

[13] S. V, Krish and K. K. M. Sarma, Separation of fuzzy normed linear spaces, *Fuzzy Sets and Systems*, **63** (1994), no. 2, 207-217.

[14] R. Malceski, Strong n-convex n-normed spaces, *Math. Bilten*, No. **21** (1997), 81-102.

[15] A. Misiak, n-inner product spaces, *Math. Nachr.*, **140** (1989), 299-319.

[16] Al. Narayanan and S. Vijayabalaji, Fuzzy $n-$ normed linear spaces, *Int. J. Math. Math. Sci.*, **27** (2005), no. 24, 3963-3977.

[17] R. Rado, A theorem on infinite series, *J. Lond. Math. Soc.*, **35** (1960), 273-276.

[18] G. S. Rhie, B. M. Choi, and D. S. Kim, On the completeness of fuzzy normed linear spaces, *Math. Japan.*, **45** (1997), no. 1, 33-37.

[19] A. Smith, Convergence preserving function: an alternative discussion, *Amer. Math. Monthly*, **96** (1991), 831-833.

[20] S. Vijayabalaji and N. Thilligovindan, Complete fuzzy n-normed space, *J. Fund. Sciences*, 3 (2007), 119-126 (available online at www.ibnusina.utm.my/jfs)

[21] G. Wildenberg, Convergence preserving functions, *Amer. Math. Monthly*, **95** (1988), 542-544.