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Degree Equitable Sets in a Graph

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Abstract: Let $G = (V, E)$ be a graph. A subset S of V is called a *Smarandachely degree equitable k-set* for any integer k, $0 \leq k \leq \Delta(G)$ if the degrees of any two vertices in S differ by at most k. It is obvious that $S = V(G)$ if $k = \Delta(G)$. A Smarandachely degree equitable 1-set is usually called a *degree equitable set*. The degree equitable number $D_e(G)$, the lower degree equitable number $d_e(G)$, the independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$ are defined by

$D_e(G) = \max\{|S| : S$ is a degree equitable set in $G\}$,
$d_e(G) = \min\{|S| : S$ is a maximal degree equitable set in $G\}$,
$D_{ie}(G) = \max\{|S| : S$ is an independent and degree equitable set in $G\}$ and
$d_{ie}(G) = \min\{|S| : S$ is a maximal independent and degree equitable set in $G\}$.

In this paper we initiate a study of these four parameters on Smarandachely degree equitable 1-sets.

Key Words: Smarandachely degree equitable k-set, degree equitable set, degree equitable number, lower, degree equitable number, independent degree equitable number, lower independent degree equitable number.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1]. For any graph G, the set $D(G)$ of all distinct degrees of the vertices of G is called the degree set of G. In this paper we introduce four graph theoretic parameters which just depend on the basic concept of vertex degrees. We need the following definitions and theorems.

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Definition 1.1 *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. The corona $G_1 \circ G_2$ is defined to be the graph obtained by taking n_1 copies of G_2 and joining the i^{th} vertex of G_1 to all the vertices of the i^{th} copy of G_2.*

Definition 1.2 *A set S of vertices is said to be an independent set if no two vertices in S are adjacent. The maximum number of vertices in an independent set of a graph G is called the independence number of G and is denoted by $\beta_0(G)$.*

Definition 1.3 *A dominating set S of a graph G is called an independent dominating set of G if S is independent in G. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set.*

Definition 1.4 *Let $\mathcal{F}$ be a family of nonempty subsets of a set S. The intersection graph $\Omega(\mathcal{F})$ is the graph whose vertex set is $\mathcal{F}$ and two distinct elements $A, B \in \mathcal{F}$ are adjacent in $\Omega(\mathcal{F})$ if $A \cap B \neq \emptyset$.*

Definition 1.5 *A graph G is called a block graph if each block of G is a complete subgraph.*

Definition 1.6 *A split graph is a graph $G = (V, E)$ whose vertices can be partitioned into two sets V' and V'', where the vertices in V' form a complete graph and the vertices in V'' are independent.*

Definition 1.7 *A clique in G is a complete subgraph of G. The maximum order of a clique in G is called the clique number of G and is denoted by $\omega(G)$ or simply ω.*

Theorem 1.8([1], Page 59) *Let T be a non-trivial tree with $\Delta(T) = k$ and let n_i be the number of vertices of degree i in T, $1 \leq i \leq k$. Then $n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (k-2)n_k + 2$.*

Theorem 1.9([1], Page 130) *Let G be a maximal planar graph of order $n \geq 4$ and let n_i denote the number of vertices of degree i in G, $3 \leq i \leq k = \Delta(G)$. Then $3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + \cdots + (k-6)n_k + 12$.*

Theorem 1.10([2]) *Given a graph G and a positive integer $k \leq |V|$, the problem of determining whether G contains an independent set of cardinality at least k is NP-complete even when G is restricted to cubic planar graphs.*

§2. Degree Equitable Sets

In social network theory one studies the relationships that exist on the members of a group. The people in such a group are called actors, relationships among the actors is usually defined in terms of a dichotomous property. A social network graph is a graph in which the vertices represent the actors and an edge between the two actors indicates the property under consideration holds between the corresponding actors. In the social network graph the degree of a vertex v gives a measure of influence the corresponding actor has within the group. Hence identifying the maximum number of actors who have almost equal influence within the group is a significant problem. This motivates the following definition of degree equitable sets.

Definition 2.1 *Let $G = (V, E)$ be a graph. A subset S of V is called a degree equitable set if the degrees of any two vertices in S differ by at most one. The maximum cardinality of a degree equitable set in G is called the degree equitable number of G and is denoted by $D_e(G)$. The minimum cardinality of a maximal degree equitable set in G is called the lower degree equitable number of G and is denoted by $d_e(G)$.*

Observation 2.2 If S is a degree equitable set in G, then any subset of S is degree equitable, so that degree equitableness is a hereditary property. Hence a degree equitable set S is maximal if and only if S is 1-maximal, or equivalently $S \cup \{v\}$ is not a degree equitable set for all $v \in V - S$. Thus a degree equitable set S is maximal if and only if for every $v \in V - S$, there exists $u \in S$ such that $|deg\ u - deg\ v| \geq 2$.

Example 2.3

1. For the complete bipartite graph $K_{r,s}$, we have

$$D_e(K_{r,s}) = \begin{cases} \max\{r, s\} & \text{if } |r - s| \geq 2 \\ r + s & \text{otherwise.} \end{cases}$$

$$d_e(K_{r,s}) = \begin{cases} \min\{r, s\} & \text{if } |r - s| \geq 2 \\ r + s & \text{otherwise.} \end{cases}$$

2. For the wheel W_n on n-vertices, we have

$$D_e(W_n) = \begin{cases} n & \text{if } n = 4 \text{ or } 5 \\ n - 1 & \text{otherwise.} \end{cases}$$

$$d_e(W_n) = \begin{cases} n & \text{if } n = 4 \text{ or } 5 \\ 1 & \text{otherwise.} \end{cases}$$

3. If G is any connected graph, then for the corona $H = G \circ K_1$, $|S_1(H)| \geq |V(G)| = \frac{|V(H)|}{2}$ and hence $D_e(H) = |S_1(H)|$.

Observation 2.4 If G_1 and G_2 are two graphs with same degree sequence, then $D_e(G_1) = D_e(G_2)$ and $d_e(G_1) = d_e(G_2)$. Further a subset S of V is degree equitable in G if and only if it is degree equitable in the complement $\overline{G}$ and hence $D_e(G) = D_e(\overline{G})$ and $d_e(G) = d_e(\overline{G})$.

Observation 2.5 Clearly $1 \leq d_e(G) \leq D_e(G) \leq n$ and $D_e(G) = d_e(G) = n$ if and only if either $D(G) = \{k\}$ or $D(G) = \{k, k+1\}$ for some non-negative integer k. Also $D_e(G) = 1$ if and only if $G = K_1$ and $d_e(G) = 1$ if and only if there exists a vertex $u \in V(G)$ such that $deg\ u = k$ and $|deg\ u - deg\ v| \geq 2$ for all $v \in V - \{u\}$.

Observation 2.6 For any integer i with $\delta \leq i \leq \Delta - 1$, let $S_i = \{v \in V : deg\ v = i \text{ or } i + 1\}$. Clearly a nonempty subset A of V is a maximal degree equitable set if and only if $A = S_i$ for some i. Hence $D_e(G) = \max\{|S_i| : \delta \leq i \leq \Delta - 1\}$ and $d_e(G) = \min\{|S_i| : \delta \leq i \leq \Delta - 1 \text{ and } S_i \neq \emptyset\}$. Since the degrees of the vertices of G and the sets S_i, $\delta \leq i \leq \Delta - 1$, can be determined in linear time, it follows that $D_e(G)$ and $d_e(G)$ can be computed in linear time.

Observation 2.7 Let n and k be positive integers with $k \leq n$. Then there exists a graph G of order n with $d_e(G) = k$. If $k < \frac{n}{2}$, we take G to be the graph obtained from the path $P = (v_1, v_2, \ldots, v_k)$ and the complete graph K_{n-k} by joining v_1 to a vertex of K_{n-k}. If $k \geq \frac{n}{2}$, we take G to be the graph obtained from the cycle C_k by attaching exactly one leaf at $n-k$ vertices of C_k.

Theorem 2.8 *Let G be a non-trivial graph on n vertices. Then $2 \leq D_e(G) \leq n$ and $D_e(G) = 2$ if and only if $G = K_2$ or $\overline{K_2}$.*

Proof The inequalities are trivial.

Suppose $D_e(G) = 2$. Let $D(G) = \{d_1, d_2, \ldots, d_k\}$, where $d_1 < d_2 < \ldots < d_k$. Clearly $k \leq n-1$ and there exist at most two vertices with degree $d_i, 1 \leq i \leq k$. Let $d_{i_1} \in D(G)$ be such that exactly two vertices have degree d_{i_1}. Since $D_e(G) = 2$, it follows that $d_{i_1} - 1, d_{i_1} + 1 \notin D(G)$ if $i_1 < k$ and $d_k - 1 \notin D(G)$ if $i_1 = k$. Hence by Pigeonhole principle, there exists $d_{i_2} \in D(G) - \{d_{i_1}\}$ such that exactly two vertices of G have degree d_{i_2}. Continuing this process we get for each $d_i \in D(G)$, there exist exactly two vertices with degree d_i and $|d_i - d_j| \geq 2$ if $i \neq j$. Hence the degree sequence of G is given by $\Pi_1 = (1, 1, 3, 3, 5, 5, \ldots, n-1, n-1)$ or $\Pi_2 = (0, 0, 2, 2, 4, 4, \ldots, n-2, n-2)$. Hence it follows that $n = 2$ and $G = K_2$ or $\overline{K_2}$. □

Theorem 2.9 *If a and b are positive integers with $a \leq b$, then there exists a graph G with $d_e(G) = a$ and $D_e(G) = b$, except when $a = 1$ and $b = 2$.*

Proof If $a = b$, then for any regular graph G of order a, we have $d_e(G) = D_e(G) = a$. Hence we assume that $a < b$. If $b \geq a + 2$, then for the graph G consisting of a copy of K_a and a copy of K_b along with a unique edge joining a vertex of K_a to a vertex of K_b, we have $d_e(G) = a$ and $D_e(G) = b$. If $b = a + 1$ and $a > 3$, then for the graph G consisting of the cycle C_a and the complete graph K_b with an edge joining a vertex of C_a to a vertex of K_b, we have $d_e(G) = a$ and $D_e(G) = b$. For the graphs G_1 and G_2 given in Fig.1, we have $d_e(G_1) = 3$ and $D_e(G_1) = 4$ and $d_e(G_2) = 2, D_e(G_2) = 3$. Also it follows from Theorem 2.8 that there is no graph G with $d_e(G) = 1$ and $D_e(G) = 2$. □

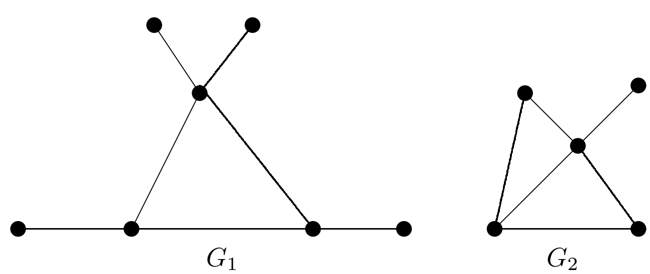

Fig.1

Proposition 2.10 *For a tree T, $D_e(T) = |S_1(T)| = |\{v \in V : deg\ v = 1$ or $2\}|$.*

Proof Let n_i denote the number of vertices of degree i in T where $1 \leq i \leq \Delta$. Clearly $|S_i(T)|$

$= n_i + n_{i+1}$, where $1 \leq i \leq \Delta - 1$. By Theorem 1.8, $n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta + 2$. Hence $|S_1(T)| \geq |S_i(T)| + 2$, for all i, $2 \leq i \leq \Delta - 1$, so that $D_e(T) = |S_1(T)|$. □

Proposition 2.11 *Let G be a maximal planar graph with $\delta(G) = 5$. Then $D_e(G) = |S_5(G)|$.*

Proof It follows from Theorem 1.9 that $n_5 = n_7 + 2n_8 + 3n_9 + \cdots + (\Delta - 6)n_\Delta + 12$ and hence $D_e(G) = |S_5(G)|$. □

Proposition 2.12 *For any unicyclic graph G with cycle C, $D_e(G) = |S_1(G)|$.*

Proof If $G= C$, then $D_e(G) = |V(G)| = |S_1(G)|$. Suppose $G \neq C$. Let $e = uv$ be any edge of C and let $T = G - e$. It follows from Proposition 2.10 that, $D_e(T) = |S_1(T)|$ and $|S_1(T)| \geq |S_i(T)| + 2$, for all $i = 2, 3, \ldots, \Delta - 1$.

Clearly, $|S_i(T)| - 2 \leq |S_i(G)| \leq |S_i(T)| + 2$. If $|S_1(T)| = |S_1(G)|$, then $|S_1(G)| = |S_1(T)| \geq |S_i(T)| + 2 \geq |S_i(G)|$, for all $i = 2, 3, \ldots, \Delta - 1$. Suppose $|S_1(G)| \neq |S_1(T)|$. Then the vertices u and v have degree either 2 or 3 and at least one of the vertices have degree 3 in G. Let $deg\ u = k_1$ and $deg\ v = k_2$.

Case 1. $k_1 = 3$ and $k_2 = 2$.

Then $|S_1(G)| = |S_1(T)| - 1$, $|S_2(G)| = |S_2(T)| + 1$, $|S_3(G)| = |S_3(T)| + 1$ and $|S_i(G)| = |S_i(T)|$, for all $i \geq 4$. Hence $|S_1(G)| = |S_1(T)| - 1 \geq |S_i(T)| + 2 - 1 \geq |S_i(T)| + 1 \geq |S_i(G)|$, for all $i = 2, 3, \ldots, \Delta - 1$.

Case 2. $k_1 = k_2 = 3$.

Then $|S_1(G)| = |S_1(T)| - 2, |S_2(G)| = |S_2(T)|, |S_3(G)| = |S_3(T)| + 2$ and $|S_i(G)| = |S_i(T)|$, for all $i \geq 4$. We claim that $|S_1(G)| \geq |S_i(G)|$ for all $i = 2, 3, \ldots, \Delta - 1$. Since $|S_1(G)| = |S_1(T)| - 2$, $|S_1(T)| \geq |S_i(T)| + 2$, for all $i = 2, 3, \ldots, \Delta - 1$ and $|S_i(G)| = |S_i(T)|$ for all $i \neq 3$, it follows that $|S_1(G)| \geq |S_i(G)|$ if $i \neq 3$. We now prove that $|S_1(G)| \geq |S_3(G)|$. Let n_i denote the number of vertices of degree i in G, $1 \leq i \leq \Delta$. Since G is unicyclic, $n_1 + 2n_2 + 3n_3 + \cdots + \Delta n_\Delta = 2n$. Also $n_1 + n_2 + \cdots + n_\Delta = n$. Hence it follows that $n_1 = n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta$. Since $|S_3(G)| = n_3 + n_4$ it follows that $n_1 > |S_3(G)|$ and hence $|S_1(G)| > |S_3(G)|$. Thus $|S_1(G)| \geq |S_i(G)|$ for all $i = 2, 3, \ldots, \Delta - 1$ and hence $D_e(G) = |S_1(G)|$. □

The study of the effect of the removal of a vertex or an edge on any graph theoretic parameter has interesting applications in the context of a network since the removal of a vertex can be interpreted as a faulty component in the network, and the removal of an edge can be interpreted as the failure of a link joining two elements of the network.

We now proceed to investigate the effect of the removal of a vertex on $D_e(G)$.

Observation 2.13 On the removal of a vertex, $D_e(G)$ may increase arbitrarily or decrease arbitrarily or remain unaltered. For the complete bipartite graph $G = K_{r,r+2}$ with bipartition $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_{r+2}\}$, $D_e(G) = r + 2$ and

$$D_e(G - v) = \begin{cases} 2r + 1 & \text{if } v \in Y \\ r + 2 & \text{if } v \in X. \end{cases}$$

Also for the graph $G = K_{r,r+1}$ with bipartition $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_{r+1}\}$, $D_e(G) = 2r + 1$ and $D_e(G - v) = r + 1$ for all $v \in X$.

Hence the vertex set of G can be partitioned into three sets (not necessarily nonempty) as follows.

$V^0 = \{v \in V : D_e(G) = D_e(G - v)\}$,

$V^+ = \{v \in V : D_e(G) < D_e(G - v)\}$ and

$V^- = \{v \in V : D_e(G) > D_e(G - v)\}$.

Example 2.14

1. For any regular graph G, we have $V = V^-$ and $V^0 = V^+ = \emptyset$.

2. There exist graphs for which all the sets V^0, V^+ and V^- are nonempty. For the graph G given in Fig.2, $D_e(G) = 6$, $V^0 = \{6, 5, 3, 2, 1\}$, $V^+ = \{4\}$ and $V^- = \{8, 9, 7\}$.

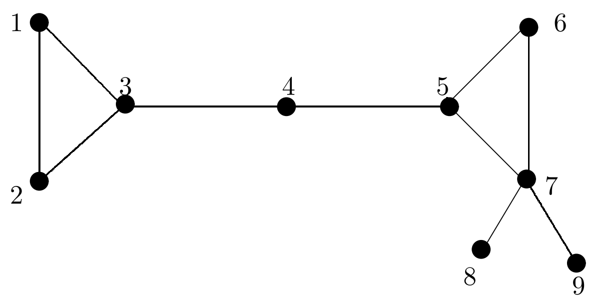

Fig.2 G

We now proceed to determine the sets V^0, V^+ and V^- for trees and unicyclic graphs. We need the following lemma.

Lemma 2.15 *Let G be a disconnected graph in which every component is either a tree or a unicyclic graph. Then $D_e(G) = \max\{|S_0(G)|, |S_1(G)|\}$.*

Proof Let d_0, d_1 and d_2 denote respectively the number of vertices of degree zero, one and two in G. Then $|S_0(G)| = d_0 + d_1$ and $|S_1(G)| = d_1 + d_2$. Hence $|S_0(G)| \geq |S_1(G)|$ if $d_0 \geq d_2$ and $|S_0(G)| < |S_1(G)|$ if $d_0 < d_2$. Also it follows from Proposition 2.10 and Proposition 2.12 that $|S_1(G)| \geq |S_i(G)|$ for all $i \geq 2$. Hence $D_e(G) = \max\{|S_0(G)|, |S_1(G)|\}$. □

Theorem 2.16 *Let G be a tree or a unicyclic graph and let $v \in V(G)$. Let $N(v) = \{w_1, w_2, \ldots, w_k\}$. Let k_1, k_2 and k_3 denote respectively the number of vertices in $N(v)$ with degrees 1,2 and 3 respectively. Let m_2 denote the number of vertices of degree 2 in G.*

(a) If deg $v = 1$, then $v \in V^0$ if and only if deg $w_1 = 3$ and $v \in V^-$ otherwise.

(b) If deg $v = 2$, then $v \in V^+$ if deg $w_1 =$ deg $w_2 = 3$, $v \in V^0$ if deg $w_1 = 2$ and deg $w_2 = 3$ or deg $w_1 = 3$ and deg $w_2 \geq 4$ and in all other cases $v \in V^-$.

(c) If deg $v \geq 3$, then $v \in V^-$ if $m_2 > k_2$ and $k_1 > k_3$, $v \in V^+$ if $k_3 > k_1$ and in all other cases $v \in V^0$.

Proof We prove the theorem for a tree T. The proof for unicyclic graphs is similar.

a) Suppose $deg\ v = 1$. Then $T_1 = T - v$ is also a tree. Further,

$$|S_1(T_1)| = \begin{cases} |S_1(T)| & \text{if } deg\ w_1 = 3 \\ |S_1(T)| - 1 & \text{otherwise} \end{cases}$$

Hence it follows from Proposition 2.10 that $v \in V^0$ if and only if $deg\ w_1 = 3$ and $v \in V^-$ otherwise.

b) Let $deg\ v = 2$. Then $F = T - v$ is a forest with two components T_1 and T_2.

If $deg\ w_1 = deg\ w_2 = 3$, then $|S_1(F)| = |S_1(T)| + 1$. Also by Lemma 2.15, $D_e(F) = |S_1(F)| > |S_1(T)| = D_e(T)$. Hence $v \in V^+$.

If $deg\ w_1 = 2$ and $deg\ w_2 = 3$ or $deg\ w_1 = 3$ and $deg\ w_2 \geq 4$, then $|S_1(F)| = |S_1(T)|$. Hence $D_e(F) = |S_1(F)| = |S_1(T)| = D_e(T)$, so that $v \in V^0$.

If $deg\ w_1 = deg\ w_2 = 1$, then $T = K_{1,2}$ and hence $D_e(F) = 2$ and $D_e(T) = 3$. If $deg\ w_1 = 1$ and $deg\ w_2 = 2$, then $|S_0(F)| = k_1 < |S_1(T)|$ and $|S_1(F)| = |S_1(T)| - 2$. Hence $D_e(F) = \max\{|S_0(F)|, |S_1(F)|\} < D_e(T)$. If $deg\ w_1 = deg\ w_2 = 2$, then $D_e(F) = |S_1(F)| = |S_1(T)| - 1$. If $deg\ w_1 = 2$ and $deg\ w_2 \geq 4$ or if $deg\ w_1 \geq 4$ and $deg\ w_2 \geq 4$, then $D_e(F) = |S_1(F)| = |S_1(T)| - 1 = D_e(T) - 1$. Hence in all cases $D_e(F) < D_e(T)$, so that $v \in V^-$.

c) Let $deg\ v \geq 3$.

In this case F is a forest with k components, where $k = deg\ v$. Then $|S_0(F)| = |S_1(T)| - m_2 + k_2$ and $|S_1(F)| = |S_1(T)| - k_1 + k_3$. Since $m_2 \geq k_2$, we have $|S_0(F)| \leq |S_1(T)|$. Now, if $m_2 > k_2$ and $k_1 > k_3$ then $|S_0(F)| < |S_1(T)|$ and $|S_1(F)| < |S_1(T)|$. Hence $D_e(F) < D_e(T)$ so that $v \in V^-$. If $k_1 < k_3$, then $|S_1(F)| > |S_1(T)|$. Hence $D_e(F) = \max\{|S_0(F)|, |S_1(F)|\} = |S_1(F)| > |S_1(T)| > D_e(T)$, so that $v \in V^+$. If $m_2 = k_2$ and $k_1 > k_3$, then $|S_0(F)| = |S_1(T)|$ and $|S_1(F)| < |S_1(T)|$. If $k_1 = k_3$ then $|S_1(F)| = |S_1(T)|$. Thus in both cases, $D_e(F) = |S_1(T)| = D_e(T)$ and hence $v \in V^0$. □

We now proceed to investigate the effect of the removal of an edge on $D_e(G)$. Let $e = uv \in E(G)$ and let $H = G - e$. Since $d_H(u) = d_G(u) - 1$, $d_H(v) = d_G(v) - 1$ and $d_H(w) = d_G(w)$, for all $w \in V - \{u, v\}$, it follows that $D_e(G) - 2 \leq D_e(G - e) \leq D_e(G) + 2$. Hence the edge set of G can be partitioned into five subsets as follows.

$E^{-2} = \{e \in E : D_e(G) = D_e(G - e) + 2\}$,

$E^{-1} = \{e \in E : D_e(G) = D_e(G - e) + 1\}$,

$E^0 = \{e \in E : D_e(G) = D_e(G - e)\}$,

$E^1 = \{e \in E : D_e(G) = D_e(G - e) - 1\}$ and

$E^2 = \{e \in E : D_e(G) = D_e(G - e) - 2\}$.

The following examples illustrate that all five types of edges can exist.

Example 2.17

1. For the graph G_1 given in Fig.3, $D_e(G_1) = 5, D_e(G_1 - e_1) = 4, D_e(G_1 - e_2) = 3$ and $D_e(G_1 - e_3) = 5$. Hence $e_3 \in E^0, e_1 \in E^{-1}$ and $e_2 \in E^{-2}$.

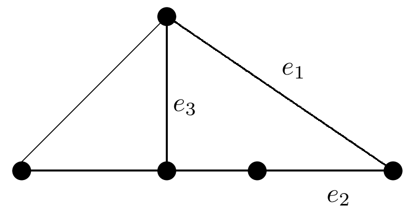

Fig.3 G_1

2. For the graph G_2 given in Fig.4, $D_e(G_2) = 4$ and $D_e(G_2 - e) = 6$, so that $e \in E^2$.

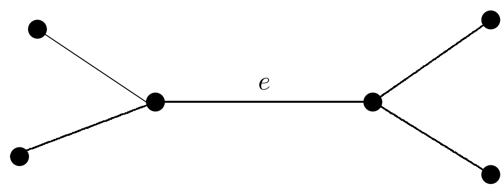

Fig.4 G_2

3. For the graph G_3 given in Fig.5, $D_e(G_3) = 6$ and $D_e(G_3 - e) = 7$, so that $e \in E^1$.

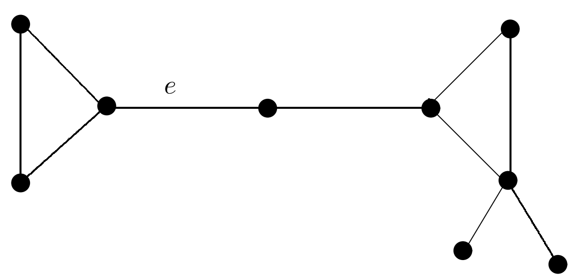

Fig.5 G_3

Theorem 2.18 *Let $T \neq K_2$ be a tree and let $e = uv$ be an edge of T.*

(a) If either u or v is a leaf, then $e \in E^0$ if T has no vertex of degree 2 and $e \in E^{-1}$ otherwise.

(b) If deg $u \geq 4$ and deg $v \geq 4$, then $e \in E^0$.

(c) If deg $u \geq 4$ and deg $v = 2$, then $e \in E^0$.

(d) If deg $u \geq 4$ and deg $v = 3$, then $e \in E^1$.

(e) If deg $u =$ deg $v = 3$, then $e \in E^2$.

(f) If deg $u = 3$ and deg $v = 2$, then $e \in E^1$.

(g) If deg $u =$ deg $v = 2$, then $e \in E^0$.

Proof Let $T_1 = T - uv$. Clearly T_1 is a forest with exactly two components. Suppose v is a leaf. If $deg\ u = 3$ and T has no vertex of degree 2, then $S_0(T_1) \subseteq S_1(T)$ and $|S_1(T_1)| = |S_1(T)|$. Hence $D_e(T_1) = D_e(T)$, so that $e \in E^0$. If T has a vertex of degree 2, then $S_0(T_1) \subsetneq S_1(T)$ and $|S_1(T_1)| = |S_1(T)| - 1$, so that $D_e(T_1) = D_e(T) - 1$ and $e \in E^{-1}$.

Now, suppose $deg\ u \geq 2$ and $deg\ v \geq 2$, so that $|S_0(T_1)| \leq |S_1(T)|$. Now if (b) or (c) holds then $|S_1(T_1)| = |S_1(T)|$. If (d) or (f) holds then $|S_1(T_1)| = |S_1(T)| + 1$ and if (e) holds, $|S_1(T_1)| = |S_1(T)| + 2$. Hence the result follows. □

We now consider the effect of removal of a vertex or an edge on the lower degree equitable number $d_e(G)$.

Observation 2.19 On the removal of a vertex, $d_e(G)$ may increase arbitrarily or decrease arbitrarily or remain unaltered. For the complete bipartite graph $G = K_{r,r+2}$ with bipartition $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_{r+2}\}$, $d_e(G) = r$ and

$$d_e(G - v) = \begin{cases} 2r + 1 & \text{if } v \in Y \\ r - 1 & \text{if } v \in X. \end{cases}$$

This shows that $d_e(G)$ may increase arbitrarily on vertex removal.

Also for the bistar $G = B(n_1, n_2)$ with $|n_1 - n_2| = 1$, $d_e(G) = 2$ and $d_e(G - v) = 2$, where v is any leaf of G, so that $d_e(G)$ remains unaltered.

The following example shows that $d_e(G)$ may decrease arbitrarily on vertex removal. Let G_1 be a 4-regular graph on n_1 vertices and let G_2 be a 6-regular graph on n_2 vertices where $n_1 < n_2$. Let G_3 be a $n_1 + 1$-regular graph on n_3 vertices where $n_3 > n_1 + n_2$. Let G be the graph obtained from G_1, G_2 and G_3 as follows.

Add a new vertex v and join it to all vertices of G_1. Remove two disjoint edges x_1y_1 and x_2y_2 from G_3 and remove an edge x_3y_3 from G_2 and add the edges vx_1, vy_1, x_3x_2 and y_3y_2. Clearly $D(G) = (5, 6, n_1 + 1, n_1 + 2)$. Also $|S_5| = n_1 + n_2$ and $|S_{n_1+1}| = n_3$. Since $n_3 > n_1 + n_2$ it follows that $d_e(G) = n_1 + n_2$. Now, $D(G - \{v\}) = \{4, 6, n_1, n_1 + 1\}$. Also $|S_4| = n_1, |S_6| = n_2$ and $|S_{n_1}| = n_3$. Hence $d_e(G - v) = n_1$.

Theorem 2.20 *Given a positive integer k, there exist graphs G_1 and G_2 such that $d_e(G_1) - d_e(G_1 - e) = k$ and $d_e(G_2 - e) - d_e(G_2) = k$.*

Proof Let $G_1 = P_{k+3} = (v_1, v_2, \ldots, v_{k+3})$. Then $d_e(G_1) = k + 3$ and $d_e(G_1 - v_1v_2) = 3$ and hence $d_e(G_1) - d_e(G_1 - e) = k$. Let H be the complete bipartite graph, $K_{k+4,k+8}$ with bipartition $X = \{x_1, x_2, \ldots, x_{k+4}\}$ and $Y = \{y_1, y_2, \ldots, y_{k+8}\}$. Let G_2 be the graph obtained from H by adding the edges y_1y_2, y_2y_3, y_3y_4. Then $d_e(G_2) = 4$, $d_e(G_2 - y_2y_3) = k + 4$ and hence $d_e(G_2 - e) - d_e(G_2) = k$. □

Hence for $d_e(G)$ the vertex set $V(G)$ and the edge set $E(G)$ can be partitioned into subsets V_0, V_+, V_- and E_0, E_+, E_- as follows.

$V_- = \{v \in V : d_e(G) > d_e(G - v)\}$,

$V_0 = \{v \in V : d_e(G) = d_e(G - v)\}$,

$V_+ = \{v \in V : d_e(G) < d_e(G - v)\}$,

$E_- = \{e \in E : d_e(G) > d_e(G - e)\}$,

$E_0 = \{e \in E : d_e(G) = d_e(G - e)\}$ and

$E_+ = \{e \in E : d_e(G) < d_e(G - e)\}$.

Example 2.21

1. For the complete graph K_n, we have $V = V_-$.

2. For the corona of the cycle $C_n \circ K_1$, we have $V = V_0$.

3. For the graph $G = K_{1,3}$, we have $V = V_+$ and $E = E_+$.

4. For any regular graph G we have $E = E_0$.

5. For the graph G given in Fig.6, we have $d_e(G) = 12$ and $d_e(G-e) = 10$ for every $e \in E(G)$ and hence $E = E_-$.

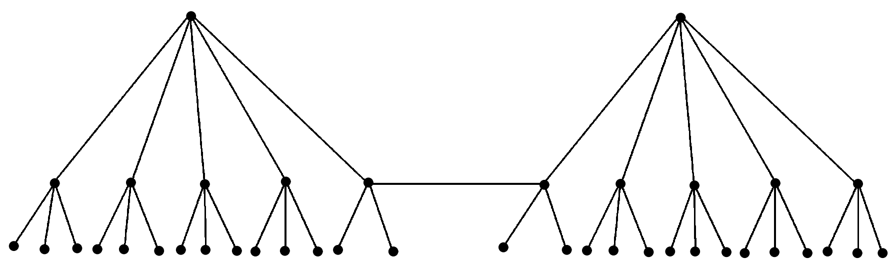

Fig.6 G

The following are some interesting problems for further investigation.

Problem 2.22

1. *Characterize graphs for which* $V = V_-$.
2. *Characterize graphs for which* $V = V_0$.
3. *Characterize graphs for which* $V = V_+$.
4. *Characterize graphs for which* $E = E_-$.
5. *Characterize graphs for which* $E = E_0$.
6. *Characterize graphs for which* $E = E_+$.
7. *Characterize vertices and edges in different classes.*

§3. Independent Degree Equitable Sets

In this section we consider subsets which are both degree equitable and independent. We introduce the concepts of independent degree equitable number and the lower independent degree equitable number, and present some basic results on these parameters.

Definition 3.1 *The independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$ of a graph G are defined by $D_{ie}(G) = \max\{|S| : S \subseteq V$ and S is an independent and degree equitable set in $G\}$ and $d_{ie}(G) = \min\{|S| : S$ is a maximal independent and degree equitable set in $G\}$.*

Example 3.2

1. For the complete bipartite graph $K_{m,n}$, we have $D_{ie}(K_{m,n}) = \max\{m, n\}$ and $d_{ie}(G) = \min\{m, n\}$.
2. For the wheel W_n on n-vertices, we have $D_{ie}(W_n) = \beta_0(C_{n-1})$ and $d_{ie}(W_n) = 1$.

Observation 3.3 Let $H_i = \langle S_i \rangle$. Then $D_{ie}(G) = max\{\beta_o(H_i) : \delta \le i \le \Delta - 1\}$ and $d_{ie}(G) = min\{i(H_i) : \delta \le i \le \Delta - 1\}$.

Observation 3.4 For any graph G, we have $d_{ie}(G) \le D_{ie}(G) \le \beta_0(G)$. Also, since $D_{ie}(K_{a,b}) = \max\{a, b\}$ and $d_{ie}(K_{a,b}) = \min\{a, b\}$, the difference between the parameters $D_{ie}(G)$ and $d_{ie}(G)$ can be made arbitrarily large.

Observation 3.5 For any regular graph, we have $D_{ie}(G) = \beta_0(G)$. By Theorem 1.10 the computation of $\beta_0(G)$ is NP-complete even for cubic planar graphs. Hence it follows that the computation of $D_{ie}(G)$ is NP-complete.

Observation 3.6 The difference between $\beta_0(G)$ and $D_{ie}(G)$ can also be made arbitrarily large. If $G_i = K_{2i+1}$, $i = 1, 2, \ldots, k+1$ and G is the graph obtained from $G_1, G_2, \ldots, G_{k+1}$ by joining a vertex of G_i to a vertex of G_{i+1},where $1 \le i \le k$, then $D_{ie}(G) = 1$ and $\beta_0(G) = k+1$. Hence $\beta_0(G) - D_{ie}(G) = k$.

Observation 3.7 For any connected graph G, $D_{ie}(G) = n - 1$ if and only if $G \cong K_{1,n-1}$.

Observation 3.8 Let G be a graph with $\beta_0(G) = n-2$. If A is any β_0-set in G, then $deg\ v = 1$ or 2 for all $v \in A$ and hence $D_{ie}(G) = \beta_0(G) = n - 2$.

Proposition 3.9 *For any connected graph G, $d_{ie}(G) = 1$ if and only if either $\Delta = n - 1$ or for any two nonadjacent vertices $u, v \in V(G)$, $|deg\ u - deg\ v| \ge 2$.*

Proof Suppose $d_{ie}(G) = 1$ and $\Delta(G) < n-1$. Let u and v be any two nonadjacent vertices in G. Since $d_{ie}(G) = 1$, $\{u, v\}$ is not a degree equitable set and hence $|deg\ u - deg\ v| \ge 2$. The converse is obvious. □

Proposition 3.10 *For any connected graph G, $D_{ie}(G) = 1$ if and only if $G \cong K_n$ or for any two nonadjacent vertices $u, v \in V(G)$, $|deg\ u - deg\ v| \ge 2$.*

Proof Suppose $D_{ie}(G) = 1$. If $G \ne K_n$, let u and v be any two nonadjacent vertices in G. Since $D_{ie}(G) = 1$, $\{u, v\}$ is not a degree equitable set and hence $|deg\ u - deg\ v| \ge 2$. The converse is obvious. □

Observation 3.11 Every independent set of a graph G is degree equitable if and only if for any two nonadjacent vertices u and v, $|deg\ u - deg\ v| \le 1$.

Theorem 3.12 *In a tree T every independent set is degree equitable if and only if T is a star or a path.*

Proof Let T be a tree. Suppose every independent set in T is degree equitable. If all the vertices of T are of degree 1 or 2 then T is a path. If there exists a vertex v with $deg\ v > 2$, then all the leaves of T are adjacent to v and hence T is a star. The converse is obvious. □

Theorem 3.13 *Let G be a unicyclic graph with cycle C. Then every independent set of G is degree equitable if and only if $G = C$ or the graph obtained from the cycle C_3 by attaching at least one leaf at a vertex or the graph obtained from a cycle $C_k, k \geq 4$ by attaching exactly one leaf at a vertex.*

Proof Let G be a unicyclic graph with cycle C. Suppose every independent set in G is degree equitable. If all the vertices of G are of degree 2, then $G = C$. Suppose there exists a vertex v on C with $deg\ v > 2$. Then $\delta = 1$. If there exists a leaf w which is not adjacent to v, then $\{w, v\}$ is an independent set in G and is not degree equitable, which is a contradiction. Thus every leaf of G is adjacent to v and hence all the vertices of C other than v are of degree 2. Also if the length of the cycle C is at least 4 and $deg\ v \geq 4$, then $\{v, x\}$ where x is any vertex on C which is not adjacent to v is an independent set which is not degree equitable. Hence G is isomorphic to one of the graphs given in the theorem. The converse is obvious. □

We now consider the effect of removal of a vertex or an edge on the independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$.

Observation 3.14

1. For the complete graph K_n, $D_{ie}(K_n) = d_{ie}(K_n) = 1$ and $D_{ie}(K_n - v) = d_{ie}(K_n - v) = 1$ for all $v \in V(K_n)$.

2. For the wheel $G = W_n$ on n vertices, we have $d_{ie}(G) = 1$. Further if v is the central vertex of G, then $G - v$ is the cycle C_{n-1} and hence $d_{ie}(G - v) = \left\lfloor \frac{n-1}{2} \right\rfloor$. This shows that the lower independent degree equitable number may increase arbitrarily on vertex removal.

3. For the graph G obtained from a copy of K_5 and a copy K_6 by joining a vertex u of K_5 with a vertex v of K_6, we have $d_{ie}(G) = D_{ie}(G) = 2$. Also $d_{ie}(G - w) = D_{ie}(G - w) = 1$ for any $w \in V(K_5) - \{u\}$.

4. For the graph G obtained from a copy of K_5 and a copy of K_7 by joining a vertex u of K_5 with a vertex v of K_7, we have $d_{ie}(G) = D_{ie}(G) = 1$. Also $D_{ie}(G - w) = d_{ie}(G - w) = 2$ for any $w \in V(K_7) - \{v\}$.

Thus the independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$ may increase or decrease or remain same on removal of a vertex. Hence the vertex set $V(G)$ can be partitioned into subsets as follows.

$V^{(-)} = \{v \in V : D_{ie}(G) > D_{ie}(G - v)\}$,

$V^{(0)} = \{v \in V : D_{ie}(G) = D_{ie}(G - v)\}$,

$V^{(+)} = \{v \in V : D_{ie}(G) < D_{ie}(G-v)\}$,
$V_{(-)} = \{v \in V : d_{ie}(G) > d_{ie}(G-v)\}$,
$V_{(0)} = \{v \in V : d_{ie}(G) = d_{ie}(G-v)\}$ and
$V_{(+)} = \{v \in V : d_{ie}(G) < d_{ie}(G-v)\}$.

The following theorem shows that on removal of an edge, $D_{ie}(G)$ can decrease by at most 1 and increase by at most 2.

Theorem 3.15 *Let G be a graph. Let $e = uv \in E(G)$. Then $D_{ie}(G) - 1 \leq D_{ie}(G-e) \leq D_{ie}(G) + 2$.*

Proof Let S be an independent degree equitable set in G with $|S| = D_{ie}(G)$. Then at most one of the vertices u, v belong to S. If $u \notin S$ and $v \notin S$, then S is an independent degree equitable set in $G - e$ and if $u \in S$, $v \notin S$, then $S - \{u\}$ is an independent degree equitable set in $G - e$. Hence $D_{ie}(G-e) \geq D_{ie}(G) - 1$.

Now, let S be an independent degree equitable set in $G - e$ with $|S| = D_{ie}(G-e)$. If both u and v are in S, then $S - \{u, v\}$ is an independent degree equitable set in G. If $u \in S$ and $v \notin S$, then $S - \{u\}$ is an independent degree equitable set in G. If both u and v are not in S, then S is an independent degree equitable set in G. Hence $D_{ie}(G) \geq D_{ie}(G-e) - 2$. □

Observation 3.16

1. For the complete graph K_n, $n \geq 3$, we have $D_{ie}(G) = d_{ie}(K_n) = 1$ and $D_{ie}(K_n - e) = 2$, $d_{ie}(K_n - e) = 1$ for any edge $e \in E(K_n)$.

2. For the path $P_n = (v_1, v_2, \ldots, v_n)$ we have $D_{ie}(P_n) = d_{ie}(P_n) = \left\lceil \frac{n}{2} \right\rceil$. Also $d_{ie}(P_n - v_1v_2) = 3$ and

$$D_{ie}(P_n - v_1v_2) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \text{ is odd} \\ \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n \text{ is even.} \end{cases}$$

3. For the corona $G = K_3 \circ K_1$, we have $d_{ie}(G) = 1$ and $d_{ie}(G-e) = 2$ for any edge $e \in E(K_3)$.

4. For the graph G given in Fig.7, $D_{ie}(G) = 6$ and $D_{ie}(G-e) = 5$.

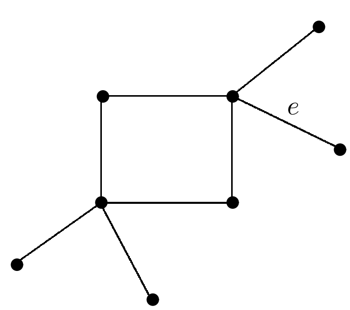

Fig.7 G

5. Let H be a split graph with split partition X, Y such that X is independent, $\langle Y \rangle$ is complete, $|Y| \geq |X| + 3$ and $D(X) = \{|Y| - 4, |Y| - 3\}$ and at least two vertices in X have degree $|Y| - 3$. Let $G = H + uv$ where $u, v \in X$ and $deg\ u = deg\ v = |Y| - 3$. Then $D_{ie}(G) = |X| - 2$ and $D_{ie}(G - uv) = D_{ie}(H) = |X|$.

Thus the independent degree equitable number $D_{ie}(G)$ and the lower independent degree equitable number $d_{ie}(G)$ may increase or decrease or remain same on removal of an edge. Hence for $D_{ie}(G)$ the edge set $E(G)$ can be partitioned into 4 subsets as follows.

$E^{(-1)} = \{e \in E : D_{ie}(G - e) = D_{ie}(G) - 1\}$,

$E^{(0)} = \{e \in E : D_{ie}(G - e) = D_{ie}(G)\}$,

$E^{(1)} = \{e \in E : D_{ie}(G - e) = D_{ie}(G) + 1\}$ and

$E^{(2)} = \{e \in E : D_{ie}(G - e) = D_{ie}(G) + 2\}$.

Hence for $d_{ie}(G)$ the edge set $E(G)$ can be partitioned into 3 subsets as follows.

$E_{(-)} = \{e \in E : d_{ie}(G) > d_{ie}(G - e)\}$,

$E_{(0)} = \{e \in E : d_{ie}(G) = d_{ie}(G - e)\}$ and

$E_{(+)} = \{e \in E : d_{ie}(G) < d_{ie}(G - e)\}$.

Example 3.17

1. *For the complete graph* K_n *where* $n \geq 3$, *we have* $V = V_{(0)} = V^{(0)}$ *and* $E = E_{(0)}$.
2. *For the complete graph* K_2, *we have* $E = E_{(+)}$.
3. *For any odd cycle* C_{2n+1} *where* $n \geq 2$, *we have* $E = E^{(1)}$.
4. *For any even cycle* C_{2n}, *we have* $E = E^{(0)}$.

Problem 3.18

1. *Characterize graphs for which* $V = V^{(0)}$.
2. *Characterize graphs for which* $V = V_{(0)}$.
3. *Characterize graphs for which* $E = E^{(0)}$.
4. *Characterize graphs for which* $E = E_{(0)}$.
5. *Characterize graphs for which* $E = E^{(1)}$.
6. *Characterize graphs for which* $E = E_{(+)}$.
7. *Characterize vertices and edges in different classes.*

§4. Degree Equitable Graphs

Given a graph $G = (V, E)$, we define another graph G^{de} using the concept of degree equitableness and present some basic results.

Definition 4.1 *Let $G = (V, E)$ be a graph. The degree equitable graph of G, denoted by G^{de} is defined as follows.*

$V(G^{de}) = V(G)$ and two vertices u and v are adjacent in G^{de} if and only if $|deg\ u - deg\ v| \leq 1$.

Observation 4.2 For any maximal degree equitable set S_i in G, the induced subgraph $\langle S_i \rangle$ of G^{de} is a clique in G^{de} and hence it follows that the clique number $\omega(G^{de})$ is equal to the degree equitable number $D_e(G)$.

Theorem 4.3 *Let G be any graph. Then the number of edges in G^{de} is given by $\sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} - \sum_{i=\delta}^{\Delta-1} \binom{|S_i \cap S_{i+1}|}{2}$.*

Proof Each $\langle S_i \rangle$ is complete in G^{de} and hence the subgraph $\langle S_i \rangle$ has $\binom{|S_i|}{2}$ edges. Also the edges in the subgraph $\langle S_{i+1} \cap S_i \rangle$ are counted twice in $\sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2}$. Hence the number of edges in $G^{de} = \sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} - \sum_{i=\delta}^{\Delta-1} \binom{|S_i \cap S_{i+1}|}{2}$. □

Theorem 4.4 *Let G be any graph. Then the following are equivalent.*

(i) *G^{de} is connected.*

(ii) *$D(G) = \{\delta, \delta+1, \ldots, \Delta\}$.*

(iii) *The intersection graph H of the set of all maximal degree equitable sets of G is a path.*

Proof Suppose G^{de} is connected. If there exists an integer i such that $i, i+2 \in D(G)$ and $i+1 \notin D(G)$, then $S_i \cap S_{i+1} = \emptyset$ and no edge in G^{de} joins a vertex of S_i and a vertex of S_{i+1}. Now, $V_1 = S_\delta \cup S_{\delta+1} \cup \cdots \cup S_i$ and $V_2 = S_{i+1} \cup \cdots \cup S_{\Delta-1}$ forms a partition of V and no edge of G^{de} joins a vertex of V_1 and a vertex of V_2. Hence G^{de} is disconnected, which is a contradiction. Hence $D(G) = \{\delta, \delta+1, \ldots, \Delta\}$, so that (i) implies (ii).

Now, if $D(G) = \{\delta, \delta+1, \ldots, \Delta\}$, then $S_i \cap S_{i+1} \neq \emptyset$ and $S_i \cap S_j = \emptyset$ if $|i - j| \geq 2$. Hence H is a path, so that (ii) implies (iii).

Now, suppose H is a path. Then $S_i \cap S_{i+1} \neq \emptyset$ and since $\langle S_i \rangle$ is a complete graph in G^{de}, it follows that G^{de} is connected. Thus (iii) implies (i). □

Theorem 4.5 *Let G be a connected graph. Then G^{de} is a connected block graph if and only if $|S_i \cap S_{i+1}| = 1$ for every i, $\delta \leq i \leq \Delta - 1$.*

proof The induced subgraph $\langle S_i \rangle$ of G^{de} is complete and each $\langle S_i \rangle$ is a block in G^{de} if and only if $|S_i \cap S_{i+1}| = 1$. □

Definition 4.6 *A graph H is called a degree equitable graph if there exists a graph G such that H is isomorphic to G^{de}.*

Example 4.7 Any complete graph K_n is a degree equitable graph, since $K_n = G^{de}$ for any regular graph G.

Theorem 4.8 *Any triangle free graph H is not a degree equitable graph.*

Proof Suppose $H = G^{de}$ for some graph G. Then $D_e(G) = 2$. Hence it follows from Theorem 2.8 that $G = K_2$ or $\overline{K_2}$, which is a contradiction. Hence any triangle free graph is not a degree equitable graph. □

Problem 4.9 *Characterize degree equitable graphs.*

Conclusion and Scope. In this paper we have introduced the concept of degree equitable sets. The concept of degree equitableness can be combined with any other graph theoretic property concerning subsets of V. For example one can consider concepts such as degree equitable dominating sets or degree equitable connected sets and study the existence of such sets in graphs.

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References

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs,* CRC (4^{th} edition), 2005.
[2] M.R. Garey, D.S. Johnson and L. Stockmeyer, Some simplified NP-Complete graph Problems, *Theor. Comput. Sci.,* **1**(1976), 237-267.