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Combinatorial Field - An Introduction

Dedicated to Prof. Feng Tian on his 70th Birthday

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Abstract: A *combinatorial field* $\mathscr{W}_G$ is a multi-field underlying a graph G, established on a smoothly combinatorial manifold. This paper first presents a quick glance to its mathematical basis with motivation, such as those of *why the WORLD is combinatorial?* and *what is a topological or differentiable combinatorial manifold?* After then, we explain how to construct principal fiber bundles on combinatorial manifolds by the voltage assignment technique, and how to establish differential theory, for example, connections on combinatorial manifolds. We also show applications of combinatorial fields to other sciences in this paper.

Key Words: Combinatorial field, Smarandache multi-space, combinatorial manifold, WORLD, principal fiber bundle, gauge field.

AMS(2000): 51M15, 53B15, 53B40, 57N16, 83C05, 83F05.

§1. Why is the WORLD a Combinatorial One?

The multiplicity of the WORLD results in modern sciences overlap and hybrid, also implies its combinatorial structure. To see more clear, we present two meaningful proverbs following.

Proverb 1. *Ames Room*

An Ames room is a distorted room constructed so that from the front it appears to be an ordinary cubic-shaped room, with a back wall and two side walls parallel to each other and perpendicular to the horizontally level floor and ceiling. As a result of the optical illusion, a person standing in one corner appears to the observer to be a giant, while a person standing in the other corner appears to be a dwarf. The illusion is convincing enough that a person walking back and forth from the left corner to the right corner appears to grow or shrink. For details, see Fig.1.1 below.

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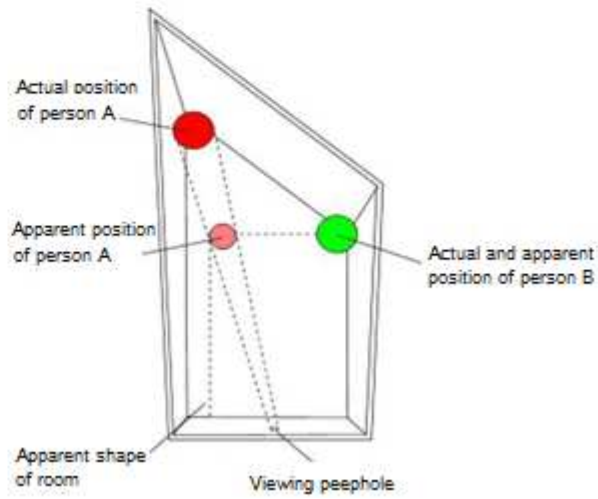


Fig.1.1

This proverb means that it is not all right by our visual sense for the multiplicity of world.

Proverb 2. *Blind men with an elephant*

In this proverb, there are six blind men were be asked to determine what an elephant looked like by feeling different parts of the elephant's body, seeing Fig.1.2 following. The man touched the elephant's leg, tail, trunk, ear, belly or tusk claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, respectively. They then entered into an endless argument and each of them insisted his view right.

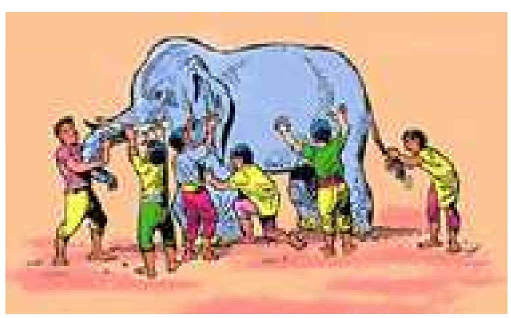


Fig.1.2

All of you are right! A wise man explains to them: *Why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.* Then

What is the meaning of Proverbs 1 *and* 2 *for understanding the structure of WORLD*?

The situation for one realizing behaviors of the WORLD is analogous to the observer in Ames room or these blind men in the second proverb. In fact, we can distinguish the WORLD by known or unknown parts simply, such as those shown in Fig.1.3.

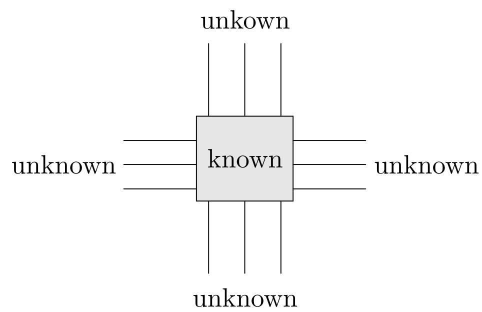


Fig.1.3

The laterality of human beings implies that one can only determines lateral feature of the WORLD by our technology. Whence, the WORLD should be the union of all characters determined by human beings, i.e., a Smarandache multi-space underlying a combinatorial structure in logic. Then *what can we say about the unknown part of the WORLD*? *Is it out order*? No! It must be in order for any thing having its own right for existing. Therefore, these is an underlying combinatorial structure in the WORLD by the *combinatorial notion*, shown in Fig.1.4.

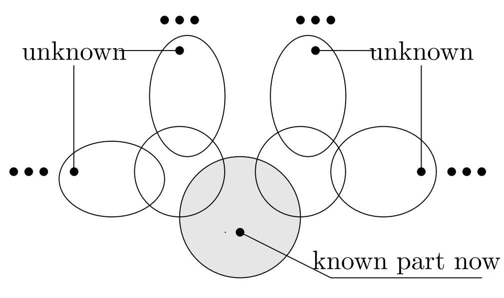


Fig.1.4

In fact, this combinatorial notion for the WORLD can be applied for all sciences. I presented this combinatorial notion in Chapter 5 of [8], then formally as the *CC conjecture for mathematics* in [11], which was reported at *the 2nd Conference on Combinatorics and Graph Theory of China* in 2006.

Combinatorial Conjecture *A mathematical science can be reconstructed from or made by combinatorialization.*

This conjecture opens an entirely way for advancing the modern sciences. It indeed means a deeply *combinatorial notion* on mathematical objects following for researchers.

(i) There is a combinatorial structure and finite rules for a classical mathematical system, which means one can make combinatorialization for all classical mathematical subjects.

(*ii*) One can generalizes a classical mathematical system by this combinatorial notion such that it is a particular case in this generalization.

(*iii*) One can make one combination of different branches in mathematics and find new results after then.

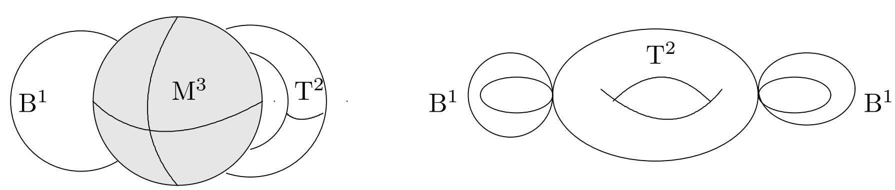
(*iv*) One can understand our WORLD by this combinatorial notion, establish combinatorial models for it and then find its behavior, and so on.

This combinatorial notion enables ones to establish a combinatorial model for the WORLD and develop modern sciences combinatorially. Whence, a science can not be ended if its combinatorialization has not completed yet.

§2. Topological Combinatorial Manifold

Now *how can we characterize these unknown parts in Fig.*1.4 *by mathematics*? Certainly, these unknown parts can be also considered to be fields. Today, we have known a best tool for understanding the known field, i.e., a topological or differentiable manifold in geometry ([1], [2]). So it is more natural to think each unknown part is itself a manifold. That is the motivation of combinatorial manifolds.

Loosely speaking, a combinatorial manifold is a combination of finite manifolds, such as those shown in Fig.2.1.



(a) (b)

Fig.2.1

In where (a) represents a combination of a 3-manifold, a torus and 1-manifold, and (b) a torus with 4 bouquets of 1-manifolds.

2.1 Euclidean Fan-Space

A *combinatorial Euclidean space* is a combinatorial system $\mathscr{C}_G$ of Euclidean spaces $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ underlying a connected graph G defined by

$$V(G) = \{\mathbf{R}^{n_1}, \mathbf{R}^{n_2}, \cdots, \mathbf{R}^{n_m}\},$$

$$E(G) = \{\ (\mathbf{R}^{n_i}, \mathbf{R}^{n_j}) \mid \mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} \neq \emptyset, 1 \leq i, j \leq m\},$$

denoted by $\mathscr{E}_G(n_1, \cdots, n_m)$ and abbreviated to $\mathscr{E}_G(r)$ if $n_1 = \cdots = n_m = r$, which enables us to view an Euclidean space $\mathbf{R}^n$ for $n \geq 4$. Whence it can be used for models of spacetime in physics.

A *combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ is the combinatorial Euclidean space $\mathscr{E}_{K_m}(n_1, \cdots, n_m)$ of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ such that for any integers $i, j,\ 1 \le i \ne j \le m$,

$$\mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} = \bigcap_{k=1}^{m} \mathbf{R}^{n_k}.$$

A combinatorial fan-space is in fact a *p-brane* with $p = \dim \bigcap\limits_{k=1}^{m} \mathbf{R}^{n_k}$ in *String Theory* ([21], [22]), seeing Fig.2.2 for details.

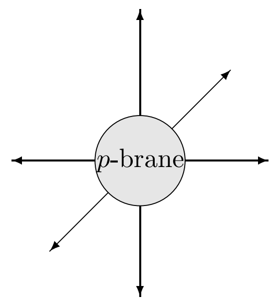


Fig.2.2

For $\forall p \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ we can present it by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x_{il} = \frac{x_l}{m}$ for $1 \le i \le m, 1 \le l \le \widehat{m}$,

$$[\overline{x}] = \left[\begin{array}{ccccccccc} x_{11} & \cdots & x_{1\widehat{m}} & x_{1(\widehat{m})+1)} & \cdots & x_{1n_1} & \cdots & 0 \\ x_{21} & \cdots & x_{2\widehat{m}} & x_{2(\widehat{m}+1)} & \cdots & x_{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x_{m1} & \cdots & x_{m\widehat{m}} & x_{m(\widehat{m}+1)} & \cdots & \cdots & x_{mn_m-1} & x_{mn_m} \end{array}\right].$$

Let $\mathscr{M}_{n\times s}$ denote all $n \times s$ matrixes for integers $n, s \ge 1$. We introduce the *inner product* $\langle (A), (B) \rangle$ for $(A), (B) \in \mathscr{M}_{n\times s}$ by

$$\langle (A), (B) \rangle = \sum_{i,j} a_{ij} b_{ij}.$$

Then we easily know that $\mathscr{M}_{n\times s}$ forms an Euclidean space under such product.

2.2 Topological Combinatorial Manifold

For a given integer sequence $0 < n_1 < n_2 < \cdots < n_m$, $m \ge 1$, a *combinatorial manifold* $\widetilde{M}$ is a *Hausdorff space* such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$, a combinatorial fan-space with

$$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\},$$

$$\bigcup_{p\in\widetilde{M}}\{n_1(p),n_2(p),\cdots,n_{s(p)}(p)\}=\{n_1,n_2,\cdots,n_m\},$$

denoted by $\widetilde{M}(n_1,n_2,\cdots,n_m)$ or $\widetilde{M}$ on the context and

$$\widetilde{\mathcal{A}}=\{(U_p,\varphi_p)|p\in\widetilde{M}(n_1,n_2,\cdots,n_m))\}$$

an atlas on $\widetilde{M}(n_1,n_2,\cdots,n_m)$.

A combinatorial manifold $\widetilde{M}$ is *finite* if it is just combined by finite manifolds with an underlying combinatorial structure G without one manifold contained in the union of others. Certainly, a finitely combinatorial manifold is indeed a combinatorial manifold. Examples of combinatorial manifolds can be seen in Fig.2.1.

For characterizing topological properties of combinatorial manifolds, we need to introduced the vertex-edge labeled graph. A *vertex-edge labeled graph* $G([1,k],[1,l])$ is a connected graph $G=(V,E)$ with two mappings

$$\tau_1:V\to\{1,2,\cdots,k\},\quad \tau_2:E\to\{1,2,\cdots,l\}$$

for integers $k,l\geq 1$. For example, two vertex-edge labeled graphs on K_4 are shown in Fig.2.3.

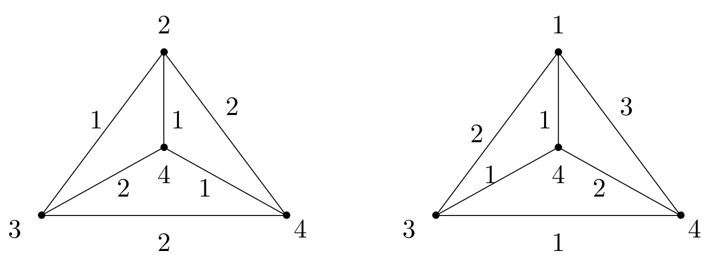


Fig.2.3

Let $\widetilde{M}(n_1,n_2,\cdots,n_m)$ be a finitely combinatorial manifold and $d,d\geq 1$ an integer. We construct a vertex-edge labeled graph $G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)]$ by

$$V(G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)])=V_1\bigcup V_2,$$

where $V_1=\{n_i-\text{manifolds } M^{n_i} \text{ in } \widetilde{M}(n_1,\cdots,n_m)|1\leq i\leq m\}$ and $V_2=\{$isolated intersection points $O_{M^{n_i},M^{n_j}}$ of M^{n_i},M^{n_j} in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ for $1\leq i,j\leq m\}$. Label n_i for each n_i-manifold in V_1 and 0 for each vertex in V_2 and

$$E(G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)])=E_1\bigcup E_2,$$

where $E_1=\{(M^{n_i},M^{n_j})$ labeled with $\dim(M^{n_i}\bigcap M^{n_j})\mid \dim(M^{n_i}\bigcap M^{n_j})\geq d,1\leq i,j\leq m\}$ and $E_2=\{(O_{M^{n_i},M^{n_j}},M^{n_i}),(O_{M^{n_i},M^{n_j}},M^{n_j})$ labeled with $0|M^{n_i}$ tangent M^{n_j} at the point $O_{M^{n_i},M^{n_j}}$ for $1\leq i,j\leq m\}$.

Now denote by $\mathcal{H}(n_1,n_2,\cdots,n_m)$ all finitely combinatorial manifolds $\widetilde{M}(n_1,n_2,\cdots,n_m)$ and $\mathcal{G}[0,n_m]$ all vertex-edge labeled graphs G^L with $\theta_L:V(G^L)\cup E(G^L)\to\{0,1,\cdots,n_m\}$ with conditions following hold.

(1)Each induced subgraph by vertices labeled with 1 in G is a union of complete graphs and vertices labeled with 0 can only be adjacent to vertices labeled with 1.

(2)For each edge $e = (u, v) \in E(G)$, $\tau_2(e) \le \min\{\tau_1(u), \tau_1(v)\}$.

Then we know a relation between sets $\mathcal{H}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}([0, n_m], [0, n_m])$ following.

Theorem 2.1 *Let $1 \le n_1 < n_2 < \cdots < n_m, m \ge 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ defines a vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}[0, n_m]$. Conversely, every vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}[0, n_m]$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ with a $1-1$ mapping $\theta : G([0, n_m]) \to \widetilde{M}$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$, $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v, w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0, n_m]))$ and $\forall (v, w) \in E(G([0, n_m]))$.*

2.4 Fundamental d-Group

For two points p, q in a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, if there is a sequence $B_1, B_2, \cdots, B_s$ of d-dimensional open balls with two conditions following hold.

(1) $B_i \subset \widetilde{M}(n_1, n_2, \cdots, n_m)$ for any integer $i, 1 \le i \le s$ and $p \in B_1$, $q \in B_s$;

(2) The dimensional number $\dim(B_i \bigcap B_{i+1}) \ge d$ for $\forall i, 1 \le i \le s - 1$.

Then points p, q are called d-dimensional connected in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and the sequence $B_1, B_2, \cdots, B_e$ a d-dimensional path connecting p and q, denoted by $P^d(p, q)$. If each pair p, q of points in the finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is d-dimensional connected, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called d-pathwise connected and say its connectivity$\ge d$.

Choose a graph with vertex set being manifolds labeled by its dimension and two manifold adjacent with a label of the dimension of the intersection if there is a d-path in this combinatorial manifold. Such graph is denoted by G^d. For example, these correspondent labeled graphs gotten from finitely combinatorial manifolds in Fig.2.1 are shown in Fig.2.4, in where $d = 1$ for (a) and (b), $d = 2$ for (c) and (d).

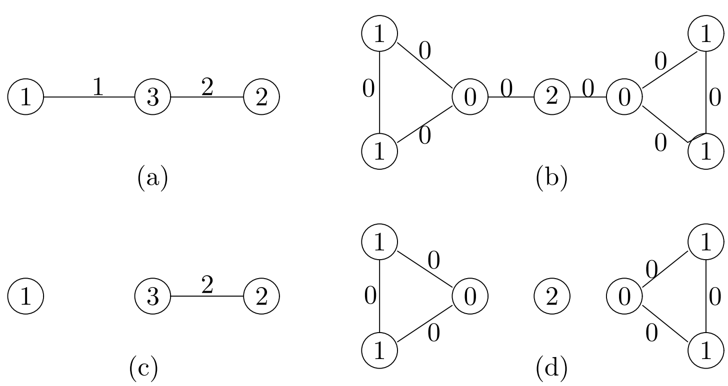


Fig.2.4

Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold of d-arcwise connectedness for an integer $d, 1 \le d \le n_1$ and $\forall x_0 \in \widetilde{M}(n_1, n_2, \cdots, n_m)$, a *fundamental d-group* at the point

x_0, denoted by $\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x_0)$ is defined to be a group generated by all homotopic classes of closed d-pathes based at x_0. If $d = 1$, then it is obvious that $\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x_0)$ is the common fundamental group of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ at the point x_0 ([18]). For some special graphs, their fundamental d-groups can be immediately gotten, for example, the d-dimensional graphs following.

A combinatorial Euclidean space $\mathscr{E}_G(\overbrace{d, d, \cdots, d}^{m})$ of $\mathbf{R}^d$ underlying a combinatorial structure $G, |G| = m$ is called a *d-dimensional graph*, denoted by $\widetilde{M}^d[G]$ if

(1) $\widetilde{M}^d[G] \setminus V(\widetilde{M}^d[G])$ is a disjoint union of a finite number of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open ball B^d;

(2) the boundary $\overline{e}_i - e_i$ of e_i consists of one or two vertices B^d, and each pair $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(\overline{B}^d, S^{d-1})$.

Then we get the next result by definition.

Theorem 2.2 $\pi^d(\widetilde{M}^d[G], x_0) \cong \pi_1(G, x_0)$, $x_0 \in G$.

Generally, we know the following result for fundamental d-groups of combinatorial manifolds ([13], [17]).

Theorem 2.3 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a d-connected finitely combinatorial manifold for an integer d, $1 \leq d \leq n_1$. If $\forall (M_1, M_2) \in E(G^L[\widetilde{M}(n_1, n_2, \cdots, n_m)])$, $M_1 \cap M_2$ is simply connected, then*

(1) *for $\forall x_0 \in G^d$, $M \in V(G^L[\widetilde{M}(n_1, n_2, \cdots, n_m)])$ and $x_{0M} \in M$,*

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x_0) \cong (\bigoplus_{M \in V(G^d)} \pi^d(M, x_{M0})) \bigoplus \pi(G^d, x_0),$$

where $G^d = G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ in which each edge (M_1, M_2) passing through a given point $x_{M_1 M_2} \in M_1 \cap M_2$, $\pi^d(M, x_{M0}), \pi(G^d, x_0)$ denote the fundamental d-groups of a manifold M and the graph G^d, respectively and

(2) *for $\forall x, y \in \widetilde{M}(n_1, n_2, \cdots, n_m)$,*

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) \cong \pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), y).$$

2.5 Homology Group

For a subspace A of a topological space S and an inclusion mapping $i : A \hookrightarrow S$, it is readily verified that the induced homomorphism $i_\sharp : C_p(A) \to C_p(S)$ is a monomorphism. Let $C_p(S, A)$ denote the quotient group $C_p(S)/C_p(A)$. Similarly, we define the *p-cycle group* and *p-boundary group* of (S, A) by ([19])

$$Z_p(S, A) = \mathrm{Ker}\partial_p = \{\ u \in C_p(S, A) \mid \partial_p(u) = 0\ \},$$

$$B_p(S, A) = \mathrm{Im}\partial_{p+1} = \partial_{p+1}(C_{p+1}(S, A)),$$

for any integer $p \geq 0$.It follows that $B_p(S,A) \subset Z_p(S,A)$ and the *pth relative homology group* $H_p(S,A)$ is defined to be

$$H_p(S,A) = Z_p(S,A)/B_p(S,A).$$

We know the following result.

Theorem 2.4 *Let $\widetilde{M}$ be a combinatorial manifold, $\widetilde{M}^d(G) \prec \widetilde{M}$ a d-dimensional graph with $E(\widetilde{M}^d(G)) = \{e_1, e_2, \cdots, e_m\}$ such that*

$$\widetilde{M} \setminus \widetilde{M}^d[G] = \bigcup_{i=2}^{k} \bigcup_{j=1}^{l_i} B_{i_j}.$$

Then the inclusion $(e_l, \dot{e}_l) \hookrightarrow (\widetilde{M}, \widetilde{M}^d(G))$ induces a monomorphism $H_p(e_l, \dot{e}_l) \to H_p(\widetilde{M}, \widetilde{M}^d(G))$ for $l = 1, 2 \cdots, m$ and

$$H_p(\widetilde{M}, \widetilde{M}^d(G)) \cong \begin{cases} \underbrace{\mathbf{Z} \oplus \cdots \mathbf{Z}}_{m}, & if\ p = d, \\ 0, & if\ p \neq d. \end{cases}$$

§3. Differentiable Combinatorial Manifolds

3.1 Definition

For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m$, a combinatorial C^h-differential manifold $(\widetilde{M}(n_1, \cdots, n_m); \widetilde{\mathcal{A}})$ is a finitely combinatorial manifold $\widetilde{M}(n_1, \cdots, n_m)$, $\widetilde{M}(n_1, \cdots, n_m) = \bigcup_{i \in I} U_i$, endowed with a atlas $\widetilde{\mathcal{A}} = \{(U_\alpha; \varphi_\alpha) | \alpha \in I\}$ on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ for an integer $h, h \geq 1$ with conditions following hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of $\widetilde{M}(n_1, n_2, \cdots, n_m)$.

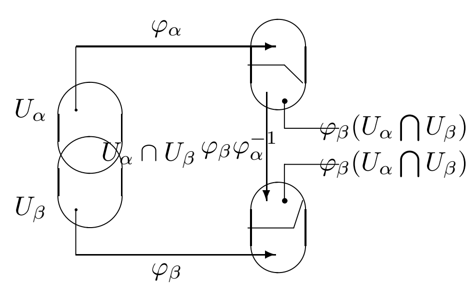


Fig.3.1

(2) For $\forall \alpha, \beta \in I$, local charts $(U_\alpha; \varphi_\alpha)$ and $(U_\beta; \varphi_\beta)$ are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\beta) \text{ and } \varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \bigcap U_\beta) \to \varphi_\alpha(U_\alpha)$$

are C^h-mappings, such as those shown in Fig.3.1.

(3) $\widetilde{\mathcal{A}}$ is maximal, i.e., if $(U;\varphi)$ is a local chart of $\widetilde{M}(n_1,n_2,\cdots,n_m)$ equivalent with one of local charts in $\widetilde{\mathcal{A}}$, then $(U;\varphi)\in\widetilde{\mathcal{A}}$.

Denote by $(\widetilde{M}(n_1,n_2,\cdots,n_m);\widetilde{\mathcal{A}})$ a combinatorial differential manifold. A finitely combinatorial manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$ is said to be *smooth* if it is endowed with a C^∞-differential structure. For the existence of combinatorial differential manifolds, we know the following result ([13],[17]).

Theorem 3.1 *Let $\widetilde{M}(n_1,\cdots,n_m)$ be a finitely combinatorial manifold and $d, 1\le d\le n_1$ an integer. If for $\forall M\in V(G^d[\widetilde{M}(n_1,\cdots,n_m)])$ is C^h-differential and*

$$\forall(M_1,M_2)\in E(G^d[\widetilde{M}(n_1,\cdots,n_m)])$$

there exist atlas

$$\mathcal{A}_1=\{(V_x;\varphi_x)|\forall x\in M_1\}\quad \mathcal{A}_2=\{(W_y;\psi_y)|\forall y\in M_2\}$$

such that $\varphi_x|_{V_x\bigcap W_y}=\psi_y|_{V_x\bigcap W_y}$ for $\forall x\in M_1, y\in M_2$, then there is a differential structures

$$\widetilde{\mathcal{A}}=\{(U_p;[\varpi_p])|\forall p\in\widetilde{M}(n_1,\cdots,n_m)\}$$

such that $(\widetilde{M}(n_1,\cdots,n_m);\widetilde{\mathcal{A}})$ is a combinatorial C^h-differential manifold.

3.2 Local Properties of Combinatorial Manifolds

Let $\widetilde{M}_1(n_1,\cdots,n_m)$, $\widetilde{M}_2(k_1,\cdots,k_l)$ be smoothly combinatorial manifolds and

$$f:\widetilde{M}_1(n_1,\cdots,n_m)\to\widetilde{M}_2(k_1,\cdots,k_l)$$

be a mapping, $p\in\widetilde{M}_1(n_1,n_2,\cdots,n_m)$. If there are local charts $(U_p;[\varpi_p])$ of p on $\widetilde{M}_1(n_1,n_2,\cdots,n_m)$ and $(V_{f(p)};[\omega_{f(p)}])$ of $f(p)$ with $f(U_p)\subset V_{f(p)}$ such that the composition mapping

$$\widetilde{f}=[\omega_{f(p)}]\circ f\circ[\varpi_p]^{-1}:[\varpi_p](U_p)\to[\omega_{f(p)}](V_{f(p)})$$

is a C^h-mapping, then f is called a C^h-mapping at the point p. If f is C^h at any point p of $\widetilde{M}_1(n_1,\cdots,n_m)$, then f is called a C^h-mapping. Denote by $\mathscr{X}_p$ all these C^∞-functions at a point $p\in\widetilde{M}(n_1,\cdots,n_m)$.

Now let $(\widetilde{M}(n_1,\cdots,n_m),\widetilde{\mathcal{A}})$ be a smoothly combinatorial manifold and $p\in\widetilde{M}(n_1,\cdots,n_m)$. A tangent vector $\overline{v}$ at p is a mapping $\overline{v}:\mathscr{X}_p\to\mathbf{R}$ with conditions following hold.

(1) $\forall g,h\in\mathscr{X}_p,\forall\lambda\in\mathbf{R},\ \overline{v}(h+\lambda h)=\overline{v}(g)+\lambda\overline{v}(h)$;
(2) $\forall g,h\in\mathscr{X}_p,\overline{v}(gh)=\overline{v}(g)h(p)+g(p)\overline{v}(h)$.

Let $\gamma:(-\epsilon,\epsilon)\to\widetilde{M}$ be a smooth curve on $\widetilde{M}$ and $p=\gamma(0)$. Then for $\forall f\in\mathscr{X}_p$, we usually define a mapping $\overline{v}:\mathscr{X}_p\to\mathbf{R}$ by

$$\overline{v}(f)=\frac{df(\gamma(t))}{dt}|_{t=0}.$$

We can easily verify such mappings $\overline{v}$ are tangent vectors at p.

Denote all tangent vectors at $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ by $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and define addition+and scalar multiplication·for $\forall \overline{u}, \overline{v} \in T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$, $\lambda \in \mathbf{R}$ and $f \in \mathscr{X}_p$ by

$$(\overline{u} + \overline{v})(f) = \overline{u}(f) + \overline{v}(f), \quad (\lambda\overline{u})(f) = \lambda \cdot \overline{u}(f).$$

Then it can be shown immediately that $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is a vector space under these two operations+and·. Let

$$\mathscr{X}(\widetilde{M}(n_1, n_2, \cdots, n_m)) = \bigcup_{p \in \widetilde{M}} T_p\widetilde{M}(n_1, n_2, \cdots, n_m).$$

A *vector field* on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is a mapping $X : \widetilde{M} \to \mathscr{X}(\widetilde{M}(n_1, n_2, \cdots, n_m))$, i.e., chosen a vector at each point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. Then the dimension and basis of the tangent space $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ are determined in the next result.

Theorem 3.2 *For any point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is*

$$\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$$

with a basis matrix

$$[\frac{\partial}{\partial \overline{x}}]_{s(p) \times n_{s(p)}} =$$

$$\left[\begin{array}{cccccccc} \frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots & 0 \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}} \end{array}\right]$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, namely there is a smoothly functional matrix $[v_{ij}]_{s(p) \times n_{s(p)}}$ such that for any tangent vector $\overline{v}$ at a point p of $\widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\overline{v} = \left\langle [v_{ij}]_{s(p) \times n_{s(p)}}, [\frac{\partial}{\partial \overline{x}}]_{s(p) \times n_{s(p)}} \right\rangle,$$

where $\langle [a_{ij}]_{k \times l}, [b_{ts}]_{k \times l} \rangle = \sum_{i=1}^{k} \sum_{j=1}^{l} a_{ij} b_{ij}$, the inner product on matrixes.

For $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$, the dual space $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called a *co-tangent vector space* at p. Let $f \in \mathscr{X}_p, d \in T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\overline{v} \in T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$. Then the action of d on f, called a *differential operator* $d : \mathscr{X}_p \to \mathbf{R}$, is defined by

$$df = \overline{v}(f).$$

We know the following result.

Theorem 3.3 *For $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ is $\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ with a basis matrix $[d\overline{x}]_{s(p)\times n_{s(p)}} =$*

$$\left[\begin{array}{cccccccc} \frac{dx^{11}}{s(p)} & \cdots & \frac{dx^{1\widehat{s}(p)}}{s(p)} & dx^{1(\widehat{s}(p)+1)} & \cdots & dx^{1n_1} & \cdots & 0 \\ \frac{dx^{21}}{s(p)} & \cdots & \frac{dx^{2\widehat{s}(p)}}{s(p)} & dx^{2(\widehat{s}(p)+1)} & \cdots & dx^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{dx^{s(p)1}}{s(p)} & \cdots & \frac{dx^{s(p)\widehat{s}(p)}}{s(p)} & dx^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & dx^{s(p)n_{s(p)}-1} & dx^{s(p)n_{s(p)}} \end{array}\right]$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, namely for any co-tangent vector d at a point p of $\widetilde{M}(n_1, n_2, \cdots, n_m)$, there is a smoothly functional matrix $[u_{ij}]_{s(p)\times s(p)}$ such that,

$$d = \left\langle [u_{ij}]_{s(p)\times n_{s(p)}}, [d\overline{x}]_{s(p)\times n_{s(p)}} \right\rangle.$$

3.3 Tensor Field

Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A *tensor of type* (r, s) at the point p on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is an $(r+s)$-multilinear function τ,

$$\tau : \underbrace{T_p^*\widetilde{M} \times \cdots \times T_p^*\widetilde{M}}_{r} \times \underbrace{T_p\widetilde{M} \times \cdots \times T_p\widetilde{M}}_{s} \rightarrow \mathbf{R},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$. Denoted by $T_s^r(p, \widetilde{M})$ all tensors of type (r, s) at a point p of $\widetilde{M}(n_1, n_2, \cdots, n_m)$. We know its structure as follows.

Theorem 3.4 *Let $\widetilde{M}(n_1, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, \cdots, n_m)$. Then*

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, \cdots, n_m)$, particularly,

$$\dim T_s^r(p, \widetilde{M}) = (\widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p)))^{r+s}.$$

3.4 Curvature Tensor

A *connection on tensors* of a smoothly combinatorial manifold $\widetilde{M}$ is a mapping $\widetilde{D} : \mathscr{X}(\widetilde{M}) \times T_s^r\widetilde{M} \rightarrow T_s^r\widetilde{M}$ with $\widetilde{D}_X\tau = \widetilde{D}(X, \tau)$ such that for $\forall X, Y \in \mathscr{X}\widetilde{M}$, $\tau, \pi \in T_s^r(\widetilde{M})$,$\lambda \in \mathbf{R}$ and $f \in C^\infty(\widetilde{M})$,

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau + \lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;
(2) $\widetilde{D}_X(\tau \otimes \pi) = \widetilde{D}_X\tau \otimes \pi + \sigma \otimes \widetilde{D}_X\pi$;
(3) for any contraction C on $T_s^r(\widetilde{M})$,

$$\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau).$$

A *combinatorial connection space* is a 2-tuple $(\widetilde{M}, \widetilde{D})$ consisting of a smoothly combinatorial manifold $\widetilde{M}$ with a connection $\widetilde{D}$ on its tensors. Let $(\widetilde{M}, \widetilde{D})$ be a combinatorial connection space. For $\forall X, Y \in \mathscr{X}(\widetilde{M})$, a *combinatorial curvature operator* $\widetilde{\mathcal{R}}(X,Y) : \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ is defined by

$$\widetilde{\mathcal{R}}(X,Y)Z = \widetilde{D}_X \widetilde{D}_Y Z - \widetilde{D}_Y \widetilde{D}_X Z - \widetilde{D}_{[X,Y]} Z$$

for $\forall Z \in \mathscr{X}(\widetilde{M})$.

Let $\widetilde{M}$ be a smoothly combinatorial manifold and $g \in A^2(\widetilde{M}) = \bigcup\limits_{p \in \widetilde{M}} T_2^0(p, \widetilde{M})$. If g is symmetrical and positive, then $\widetilde{M}$ is called a *combinatorial Riemannian manifold*, denoted by $(\widetilde{M}, g)$. In this case, if there is a connection $\widetilde{D}$ on $(\widetilde{M}, g)$ with equality following hold

$$Z(g(X,Y)) = g(\widetilde{D}_Z, Y) + g(X, \widetilde{D}_Z Y)$$

then $\widetilde{M}$ is called a *combinatorial Riemannian geometry*, denoted by $(\widetilde{M}, g, \widetilde{D})$. In this case, calculation shows that ([14])

$$\widetilde{R} = \widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} dx^{\sigma\varsigma} \otimes dx^{\eta\theta} \otimes dx^{\mu\nu} \otimes dx^{\kappa\lambda}$$

with

$$\begin{aligned} \widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \frac{1}{2}\Big(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}} + \frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu\nu}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}}\Big) \\ &+ \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)} \Gamma^{\xi o}_{(\kappa\lambda)(\eta\theta)} g_{(\xi o)(\vartheta\iota)} - \Gamma^{\xi o}_{(\mu\nu)(\eta\theta)} \Gamma_{(\kappa\lambda)(\sigma\varsigma)^{\vartheta\iota}} g_{(\xi o)(\vartheta\iota)}, \end{aligned}$$

where $g_{(\mu\nu)(\kappa\lambda)} = g(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})$.

§4. Principal Fiber Bundles

In classical differential geometry, a principal fiber bundle ([3]) is an application of covering space to smoothly manifolds. Topologically, a covering space ([18]) S' of S consisting of a space S' with a continuous mapping $\pi : S' \to S$ such that each point $x \in S$ has an arcwise connected neighborhood U_x and each arcwise connected component of $\pi^{-1}(U_x)$ is mapped homeomorphically onto U_x by π, such as those shown in Fig.4.1.

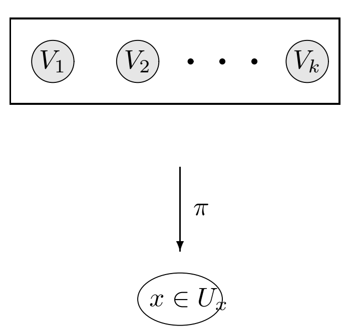


Fig.4.1

where $V_i = \pi^{-1}(U_x)$ for integers $1 \leq i \leq k$.

A *principal fiber bundle* ([3]) consists of a manifold P action by a Lie group $\mathscr{G}$, which is a manifold with group operation $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ given by $(g,h) \to g \circ h$ being C^∞ mapping, a projection $\pi : P \to M$, a base pseudo-manifold M, denoted by $(P, M, \mathscr{G})$, seeing Fig.4.2 such that conditions (1), (2) and (3) following hold.

(1) there is a right freely action of $\mathscr{G}$ on P,, i.e., for $\forall g \in \mathscr{G}$, there is a diffeomorphism $R_g : P \to P$ with $R_g(p) = pg$ for $\forall p \in P$ such that $p(g_1 g_2) = (pg_1)g_2$ for $\forall p \in P$, $\forall g_1, g_2 \in \mathscr{G}$ and $pe = p$ for some $p \in P$, $e \in \mathscr{G}$ if and only if e is the identity element of $\mathscr{G}$.

(2) the map $\pi : P \to M$ is onto with $\pi^{-1}(\pi(p)) = \{pg | g \in \mathscr{G}\}$.

(3) for $\forall x \in M$ there is an open set U with $x \in U$ and a diffeomorphism $T_U : \pi^{-1}(U) \to U \times \mathscr{G}$ of the form $T_U(p) = (\pi(p), s_U(p))$, where $s_U : \pi^{-1}(U) \to \mathscr{G}$ has the property $s_U(pg) = s_U(p)g$ for $\forall g \in \mathscr{G}, p \in \pi^{-1}(U)$.

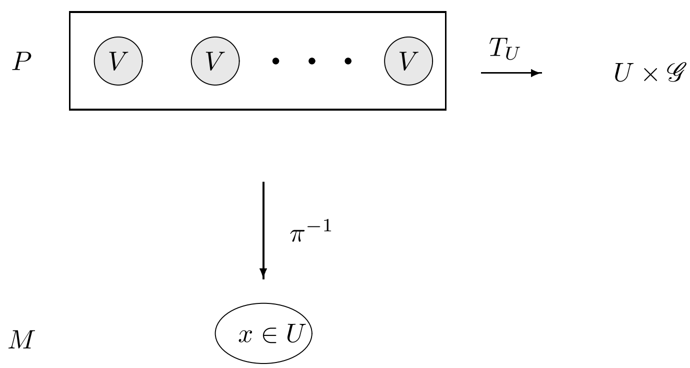


Fig.4.2

where $V = \pi^{-1}(U)$. Now *can we establish principal fiber bundles on smoothly combinatorial manifolds*? This question can be formally presented as follows:

Question *For a family of k principal fiber bundles $P_1(M_1, \mathscr{G}_1)$, $P_2(M_2, \mathscr{G}_2)$,$\cdots$, $P_k(M_k, \mathscr{G}_k)$ over manifolds M_1, M_2, $\cdots$, M_k, how can we construct principal fiber bundles on a smoothly combinatorial manifold consisting of $M_1, M_2, \cdots, M_k$ underlying a connected graph G?*

The answer is YES! For this object, we need some techniques in combinatorics.

4.1 Voltage Graph with Its Lifting

Let G be a connected graph and $(\Gamma; \circ)$ a group. For each edge $e \in E(G), e = uv$, an *orientation* on e is an orientation on e from u to v, denoted by $e = (u, v)$, called *plus orientation* and its *minus orientation*, from v to u, denoted by $e^{-1} = (v, u)$. For a given graph G with plus and minus orientation on its edges, a *voltage assignment* on G is a mapping α from the plus-edges of G into a group Γ satisfying $\alpha(e^{-1}) = \alpha^{-1}(e), e \in E(G)$. These elements $\alpha(e), e \in E(G)$ are called voltages, and (G, α) a *voltage graph* over the group $(\Gamma; \circ)$.

For a voltage graph (G, α), its lifting (See [6], [9] for details) $G^\alpha = (V(G^\alpha), E(G^\alpha); I(G^\alpha))$ is defined by

$$V(G^\alpha) = V(G) \times \Gamma, \ (u, a) \in V(G) \times \Gamma \text{ abbreviated to } u_a;$$

$$E(G^\alpha) = \{(u_a, v_{a\circ b}) | e^+ = (u, v) \in E(G), \alpha(e^+) = b\}.$$

For example, let $G = K_3$ and $\Gamma = Z_2$. Then the voltage graph (K_3, α) with $\alpha : K_3 \to Z_2$ and its lifting are shown in Fig.4.3.

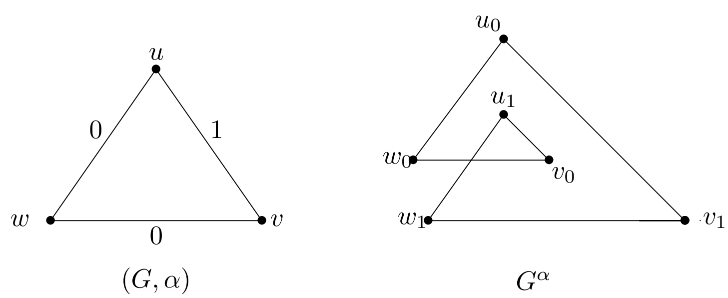


Fig.4.3

Similarly, let G^L be a connected vertex-edge labeled graph with $\theta_L : V(G) \cup E(G) \to L$ of a label set and Γ a finite group. A *voltage labeled graph* on a vertex-edge labeled graph G^L is a 2-tuple $(G^L; \alpha)$ with a voltage assignments $\alpha : E(G^L) \to \Gamma$ such that

$$\alpha(u, v) = \alpha^{-1}(v, u), \quad \forall (u, v) \in E(G^L).$$

Similar to voltage graphs, the importance of voltage labeled graphs lies in their *labeled lifting* G^{L_α} defined by

$$V(G^{L_\alpha}) = V(G^L) \times \Gamma, \ (u, g) \in V(G^L) \times \Gamma \text{ abbreviated to } u_g;$$

$$E(G^L_\alpha) = \{ \ (u_g, v_{g\circ h}) \mid \text{for } \ \forall (u, v) \in E(G^L) \text{ with } \alpha(u, v) = h \ \}$$

with labels $\Theta_L : G^{L_\alpha} \to L$ following:

$$\Theta_L(u_g) = \theta_L(u), \ \text{ and } \ \Theta_L(u_g, v_{g\circ h}) = \theta_L(u, v)$$

for $u, v \in V(G^L)$, $(u, v) \in E(G^L)$ with $\alpha(u, v) = h$ and $g, h \in \Gamma$.

For a voltage labeled graph (G^L, α) with its lifting G^L_α, a *natural projection* $\pi : G^{L_\alpha} \to G^L$ is defined by $\pi(u_g) = u$ and $\pi(u_g, v_{g\circ h}) = (u, v)$ for $\forall u, v \in V(G^L)$ and $(u, v) \in E(G^L)$ with $\alpha(u, v) = h$. Whence, (G^{L_α}, π) is a covering space of the labeled graph G^L. In this covering, we can find

$$\pi^{-1}(u) = \{ \ u_g \mid \forall g \in \Gamma\}$$

for a vertex $u \in V(G^L)$ and

$$\pi^{-1}(u, v) = \{ \ (u_g, v_{g\circ h}) \mid \forall g \in \Gamma \ \}$$

for an edge $(u,v) \in E(G^L)$ with $\alpha(u,v) = h$. Such sets $\pi^{-1}(u)$, π^{-1}(u,v) are called *fibres* over the vertex $u \in V(G^L)$ or edge $(u,v) \in E(G^L)$, denoted by fib_u or $\mathrm{fib}_{(u,v)}$, respectively. A voltage labeled graph with its labeled lifting are shown in Fig.4.4, in where, $G^L = C_3^L$ and $\Gamma = Z_2$.

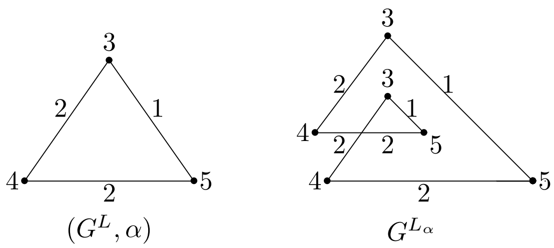


Fig.4.4

A mapping $g : G^L \to G^L$ is *acting on a labeled graph* G^L with a labeling $\theta_L : G^L \to L$ if $g\theta_L(x) = \theta_L g(x)$ for $\forall x \in V(G^L) \cup E(G^L)$, and a group Γ is acting on a labeled graph G^L if each $g \in \Gamma$ is acting on G^L. Clearly, if Γ is acting on a labeled graph G^L, then $\Gamma \leq \mathrm{Aut}G$.

Now let A be a group of automorphisms of G^L. A voltage labeled graph (G^L, α) is called *locally A-invariant* at a vertex $u \in V(G^L)$ if for $\forall f \in A$ and $W \in \pi_1(G^L, u)$, we have

$$\alpha(W) = identity \ \Rightarrow \ \alpha(f(W)) = identity$$

and *locally f-invariant* for an automorphism $f \in \mathrm{Aut}G^L$ if it is locally invariant with respect to the group $\langle f \rangle$ in $\mathrm{Aut}G^L$. Then we know a criterion for lifting automorphisms of voltage labeled graphs.

Theorem 4.1 *Let (G^L, α) be a voltage labeled graph with $\alpha : E(G^L) \to \Gamma$ and $f \in \mathrm{Aut}G^L$. Then f lifts to an automorphism of G^{L_α} if and only if (G^L, α) is locally f-invariant.*

4.2 Combinatorial Principal Fiber Bundles

For construction principal fiber bundles on smoothly combinatorial manifolds, we need to introduce the conception of Lie multi-group. A *Lie multi-group* $\mathscr{L}_G$ is a smoothly combinatorial manifold $\widetilde{M}$ endowed with a multi-group $(\widetilde{\mathscr{A}}(\mathscr{L}_G); \mathscr{O}(\mathscr{L}_G))$, where $\widetilde{\mathscr{A}}(\mathscr{L}_G) = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$ and $\mathscr{O}(\mathscr{L}_G) = \bigcup\limits_{i=1}^{m} \{\circ_i\}$ such that

(i) $(\mathscr{H}_i; \circ_i)$ is a group for each integer i, $1 \leq i \leq m$;

(ii) $G^L[\widetilde{M}] = G$;

(iii) the mapping $(a,b) \to a \circ_i b^{-1}$ is C^∞-differentiable for any integer i, $1 \leq i \leq m$ and $\forall a, b \in \mathscr{H}_i$.

Notice that if $m = 1$, then a Lie multi-group $\mathscr{L}_G$ is nothing but just the Lie group ([24]) in classical differential geometry.

Now let $\widetilde{P}$, $\widetilde{M}$ be a differentiably combinatorial manifolds and $\mathscr{L}_G$ a Lie multi-group $(\widetilde{\mathscr{A}}(\mathscr{L}_G); \mathscr{O}(\mathscr{L}_G))$ with

$$\widetilde{P}=\bigcup_{i=1}^{m}P_i,\ \widetilde{M}=\bigcup_{i=1}^{s}M_i,\ \widetilde{\mathscr{A}}(\mathscr{L}_G)=\bigcup_{i=1}^{m}\mathscr{H}_{\circ_i},\mathscr{O}(\mathscr{L}_G)=\bigcup_{i=1}^{m}\{\circ_i\}.$$

Then a *differentiable principal fiber bundle over* $\widetilde{M}$ *with group* $\mathscr{L}_G$ consists of a differentiably combinatorial manifold $\widetilde{P}$, an action of $\mathscr{L}_G$ on $\widetilde{P}$, denoted by $\widetilde{P}(\widetilde{M},\mathscr{L}_G)$ satisfying following conditions PFB1-PFB3:

PFB1. For any integer $i,\ 1\le i\le m$, $\mathscr{H}_{\circ_i}$ acts differentiably on P_i to the right without fixed point, i.e.,

$$(x,g)\in P_i\times\mathscr{H}_{\circ_i}\to x\circ_i g\in P_i \text{ and } x\circ_i g=x \text{ implies that } g=1_{\circ_i};$$

PFB2. For any integer $i,\ 1\le i\le m$, $M_{\circ_i}$ is the quotient space of a covering manifold $P\in\Pi^{-1}(M_{\circ_i})$ by the equivalence relation R induced by $\mathscr{H}_{\circ_i}$:

$$R_i=\{(x,y)\in P_{\circ_i}\times P_{\circ_i}|\exists g\in\mathscr{H}_{\circ_i}\Rightarrow x\circ_i g=y\},$$

written by $M_{\circ_i}=P_{\circ_i}/\mathscr{H}_{\circ_i}$, i.e., an orbit space of $P_{\circ_i}$ under the action of $\mathscr{H}_{\circ_i}$. These is a canonical projection $\Pi:\widetilde{P}\to\widetilde{M}$ such that $\Pi_i=\Pi|_{P_{\circ_i}}:P_{\circ_i}\to M_{\circ_i}$ is differentiable and each fiber $\Pi_i^{-1}(x)=\{p\circ_i g|g\in\mathscr{H}_{\circ_i},\Pi_i(p)=x\}$ is a closed submanifold of $P_{\circ_i}$ and coincides with an equivalence class of R_i;

PFB3. For any integer $i,\ 1\le i\le m$, $P\in\Pi^{-1}(M_{\circ_i})$ is locally trivial over $M_{\circ_i}$, i.e., any $x\in M_{\circ_i}$ has a neighborhood U_x and a diffeomorphism $T:\Pi^{-1}(U_x)\to U_x\times\mathscr{L}_G$ with

$$T|_{\Pi_i^{-1}(U_x)}=T_i^x:\Pi_i^{-1}(U_x)\to U_x\times\mathscr{H}_{\circ_i};\ x\to\ T_i^x(x)=(\Pi_i(x),\epsilon(x)),$$

called a local trivialization (abbreviated to LT) such that $\epsilon(x\circ_i g)=\epsilon(x)\circ_i g$ for $\forall g\in\mathscr{H}_{\circ_i}$, $\epsilon(x)\in\mathscr{H}_{\circ_i}$.

Certainly, if $m=1$, then $\widetilde{P}(\widetilde{M},\mathscr{L}_G)=P(M,\mathscr{H})$ is just the common principal fiber bundle over a manifold M.

4.3 Construction by Voltage Assignment

Now we show how to construct principal fiber bundles over a combinatorial manifold $\widetilde{M}$.

Construction 4.1 *For a family of principal fiber bundles over manifolds* $M_1,M_2,\cdots,M_l$*, such as those shown in Fig.*4.5*,*

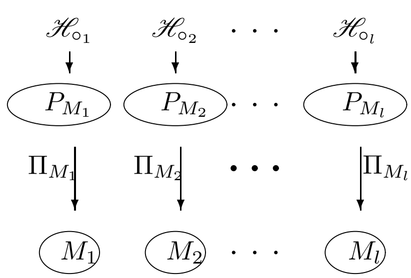


Fig.4.5

where $\mathscr{H}_{\circ_i}$ is a Lie group acting on P_{M_i} for $1 \leq i \leq l$ satisfying conditions PFB1-PFB3, let $\widetilde{M}$ be a differentiably combinatorial manifold consisting of M_i, $1 \leq i \leq l$ and $(G^L[\widetilde{M}], \alpha)$ a voltage graph with a voltage assignment $\alpha : G^L[\widetilde{M}] \to \mathfrak{G}$ over a finite group $\mathfrak{G}$, which naturally induced a projection $\pi : G^L[\widetilde{P}] \to G^L[\widetilde{M}]$. For $\forall M \in V(G^L[\widetilde{M}])$, if $\pi(P_M) = M$, place P_M on each lifting vertex M^{L_α} in the fiber $\pi^{-1}(M)$ of $G^{L_\alpha}[\widetilde{M}]$, such as those shown in Fig.4.6.

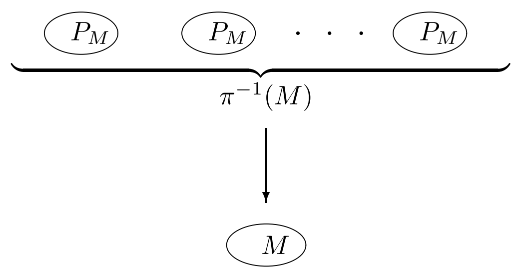


Fig.4.6

Let $\Pi = \pi\Pi_M\pi^{-1}$ for $\forall M \in V(G^L[\widetilde{M}])$. Then $\widetilde{P} = \bigcup\limits_{M \in V(G^L[\widetilde{M}])} P_M$ is a smoothly combinatorial manifold and $\mathscr{L}_G = \bigcup\limits_{M \in V(G^L[\widetilde{M}])} \mathscr{H}_M$ a Lie multi-group by definition. Such a constructed combinatorial fiber bundle is denoted by $\widetilde{P}^{L_\alpha}(\widetilde{M}, \mathscr{L}_G)$.

For example, let $\mathfrak{G} = Z_2$ and $G^L[\widetilde{M}] = C_3$. A voltage assignment $\alpha : G^L[\widetilde{M}] \to Z_2$ and its induced combinatorial fiber bundle are shown in Fig.4.7.

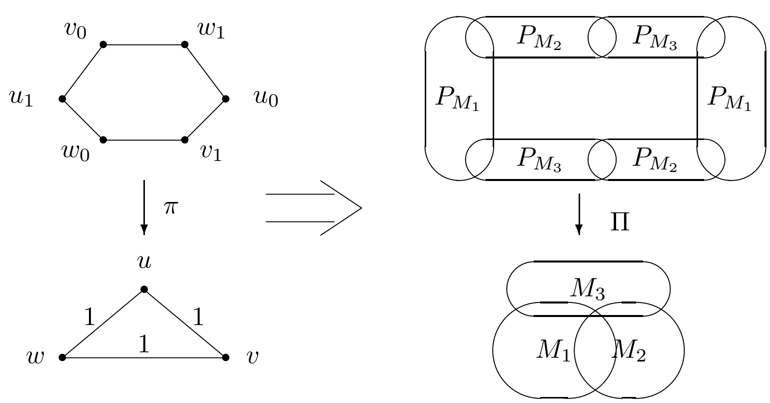


Fig.4.7

Then we know the existence result following.

Theorem 4.2 *A combinatorial fiber bundle $\widetilde{P}^\alpha(\widetilde{M}, \mathscr{L}_G)$ is a principal fiber bundle if and only if for $\forall(M', M'') \in E(G^L[\widetilde{M}])$ and $(P_{M'}, P_{M''}) = (M', M'')^{L_\alpha} \in E(G^L[\widetilde{P}])$, $\Pi_{M'}|_{P_{M'} \cap P_{M''}} = \Pi_{M''}|_{P_{M'} \cap P_{M''}}$.*

We assume $\widetilde{P}^\alpha(\widetilde{M}, \mathscr{L}_G)$ satisfying conditions in Theorem 4.2, i.e., it is indeed a principal fiber bundle over $\widetilde{M}$. An *automorphism of* $\widetilde{P}^\alpha(\widetilde{M}, \mathscr{L}_G)$ is a diffeomorphism $\omega : \widetilde{P} \to \widetilde{P}$ such that $\omega(p\circ_i g)=\omega(p)\circ_i g$ for $g\in\mathscr{H}_{\circ_i}$ and

$$p\in\bigcup_{P\in\pi^{-1}(M_i)}P,\quad \text{where } 1\leq i\leq l.$$

Theorem 4.3 *Let $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ be a principal fiber bundle. Then*

$$\mathrm{Aut}\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)\geq\langle\mathfrak{L}\rangle\,,$$

where $\mathfrak{L}=\{\ \widehat{h}\omega_i\mid\widehat{h}:P_{M_i}\to P_{M_i}$ is $1_{P_{M_i}}$ determined by $h((M_i)_g)=(M_i)_{g\circ_i h}$ for $h\in\mathfrak{G}$ and $g_i\in\mathrm{Aut}P_{M_i}(M_i,\mathscr{H}_{\circ_i}),\ 1\leq i\leq l\}$.

A principal fiber bundle $\widetilde{P}(\widetilde{M},\mathscr{L}_G)$ is called to be *normal* if for $\forall u,v\in\widetilde{P}$, there exists an $\omega\in\mathrm{Aut}\widetilde{P}(\widetilde{M},\mathscr{L}_G)$ such that $\omega(u)=v$. We get the necessary and sufficient conditions of normally principal fiber bundles $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ following.

Theorem 4.4 *$\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ is normal if and only if $P_{M_i}(M_i,\mathscr{H}_{\circ_i})$ is normal, $(\mathscr{H}_{\circ_i};\circ_i)=(\mathscr{H};\circ)$ for $1\leq i\leq l$ and $G^{L_\alpha}[\widetilde{M}]$ is transitive by diffeomorphic automorphisms in $\mathrm{Aut}G^{L_\alpha}[\widetilde{M}]$.*

4.4 Connection on Principal Fiber Bundles over Combinatorial Manifolds

A *local connection* on a principal fiber bundle $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ is a linear mapping ${}^i\Gamma_u:T_x(\widetilde{M})\to T_u(\widetilde{P})$ for an integer $i,\ 1\leq i\leq l$ and $u\in\Pi_i^{-1}(x)={}^iF_x,\ x\in M_i$, enjoys with properties following:

- (i) $(d\Pi_i)^i\Gamma_u=$ identity mapping on $T_x(\widetilde{M})$;
- (ii) ${}^i\Gamma_{{}^iR_g\circ_i u}=d\,{}^iR_g\circ_i{}^i\Gamma_u$, where ${}^iR_g$ is the right translation on P_{M_i};
- (iii) the mapping $u\to{}^i\Gamma_u$ is C^{∞}.

Similarly, a *global connection* on a principal fiber bundle $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ is a linear mapping $\Gamma_u:T_x(\widetilde{M})\to T_u(\widetilde{P})$ for a $u\in\Pi^{-1}(x)=F_x$, $x\in\widetilde{M}$ with conditions following hold:

- (i) $(d\Pi)\Gamma_u=$ identity mapping on $T_x(\widetilde{M})$;
- (ii) $\Gamma_{R_g\circ u}=dR_g\circ\Gamma_u$ for $\forall g\in\mathscr{L}_G$, $\forall\circ\in\mathscr{O}(\mathscr{L}_G)$, where R_g is the right translation on $\widetilde{P}$;
- (iii) the mapping $u\to\Gamma_u$ is C^{∞}.

Local or global connections on combinatorial principal fiber bundles are characterized by results following.

Theorem 4.5 *For an integer $i,\ 1\leq i\leq l$, a local connection ${}^i\Gamma$ in $\widetilde{P}$ is an assignment ${}^iH:u\to{}^iH_u\subset T_u(\widetilde{P})$, of a subspace ${}^iH_u$ of $T_u(\widetilde{P})$ to each $u\in{}^iF_x$ with*

- (i) *$T_u(\widetilde{P})=\ {}^iH_u\oplus^i V_u$, $u\in{}^iF_x$;*
- (ii) *$(d\ {}^iR_g)\ {}^iH_u=\ {}^iH_{u\circ_i g}$ for $\forall u\in{}^iF_x$ and $\forall g\in\mathscr{H}_{\circ_i}$;*
- (iii) *${}^iH$ is a C^{∞}-distribution on $\widetilde{P}$.*

Theorem 4.6 *A global connection Γ in $\widetilde{P}$ is an assignment $H:u\to H_u\subset T_u(\widetilde{P})$, of a subspace H_u of $T_u(\widetilde{P})$ to each $u\in F_x$ with*

- (i) *$T_u(\widetilde{P})=\ H_u\oplus V_u$, $u\in F_x$;*
- (ii) *$(dR_g)H_u=H_{u\circ g}$ for $\forall u\in F_x$, $\forall g\in\mathscr{L}_G$ and $\circ\in\mathscr{O}(\mathscr{L}_G)$;*
- (iii) *H is a C^{∞}-distribution on $\widetilde{P}$.*

Theorem 4.7 *Let $^i\Gamma$ be a local connections on $\widetilde{P}^\alpha(\widetilde{M},\mathscr{L}_G)$ for $1\leq i\leq l$. Then a global connection on $\widetilde{P}^\alpha(\widetilde{M},\mathscr{L}_G)$ exists if and only if $(\mathscr{H}_{\circ_i};\circ_i)=(\mathscr{H};\circ)$, i.e., $\mathscr{L}_G$ is a group and $^i\Gamma|_{M_i\cap M_j}=\ ^j\Gamma|_{M_i\cap M_j}$ for $(M_i,M_j)\in E(G^L[\widetilde{M}])$, $1\leq i,j\leq l$.*

A *curvature form* of a local or global connection is a $\mathfrak{Y}(\mathscr{H}_{\circ_i},\circ_i)$ or $\mathfrak{Y}(\mathscr{L}_G)$-valued 2-form

$$^i\Omega=(d\ ^i\omega)h,\quad \text{or}\quad \Omega=(d\omega)h,$$

where $(d\ ^i\omega)h(X,Y)=d\ ^i\omega(hX,hY)$, $\quad (d\omega)h(X,Y)=d\omega(hX,hY)$ for $X,Y\in\mathscr{X}(P_{M_i})$ or $X,Y\in\mathscr{X}(\widetilde{P})$. Notice that a 1-form $\omega h(X_1,X_2)=0$ if and only if $^ih(X_1)=0$ or $^ih(X_12)=0$. We generalize classical structural equations and Bianchi's identity on principal fiber bundles following.

Theorem 4.8(E.Cartan) *Let $^i\omega$, $1\leq i\leq l$ and ω be local or global connection forms on a principal fiber bundle $\widetilde{P}^\alpha(\widetilde{M},\mathscr{L}_G)$. Then*

$$(d\ ^i\omega)(X,Y)=-[\ ^i\omega(X),^i\omega(Y)]+\ ^i\Omega(X,Y)$$

and

$$d\omega(X,Y)=-[\omega(X),\omega(Y)]+\Omega(X,Y)$$

for vector fields $X,Y\in\mathscr{X}(P_{M_i})$ or $\mathscr{X}(\widetilde{P})$.

Theorem 4.9(Bianchi) *Let $^i\omega$, $1\leq i\leq l$ and ω be local or global connection forms on a principal fiber bundle $\widetilde{P}^\alpha(\widetilde{M},\mathscr{L}_G)$. Then*

$$(d\ ^i\Omega)h=0,\quad \textit{and}\quad (d\Omega)h=0.$$

§5. Applications

A *gauge field* is such a mathematical model with local or global symmetries under a group, a finite-dimensional Lie group in most cases action on its gauge basis at an individual point in space and time, together with a set of techniques for making physical predictions consistent with the symmetries of the model, which is a generalization of Einstein's principle of covariance to that of internal field characterized by the following ([3],[23],[24]).

Gauge Invariant Principle *A gauge field equation, particularly, the Lagrange density of a gauge field is invariant under gauge transformations on this field.*

We wish to find gauge fields on combinatorial manifolds, and then to characterize WORLD by combinatorics. A *globally* or *locally combinatorial gauge field* is a combinatorial field $\widetilde{M}$ under a gauge transformation $\tau_{\widetilde{M}}:\widetilde{M}\to\widetilde{M}$ independent or dependent on the field variable $\overline{x}$. If a combinatorial gauge field $\widetilde{M}$ is consisting of gauge fields $M_1,M_2,\cdots,M_m$, we can easily find that $\widetilde{M}$ is a globally combinatorial gauge field only if each gauge field is global.

Let M_i, $1 \leq i \leq m$ be gauge fields with a basis B_{M_i} and $\tau_i : B_{M_i} \to B_{M_i}$ a gauge transformation, i.e., $\mathscr{L}_{M_i}(B_{M_i}^{\tau_i}) = \mathscr{L}_{M_i}(B_{M_i})$. A gauge transformation $\tau_{\widetilde{M}} : \bigcup\limits_{i=1}^{m} B_{M_i} \to \bigcup\limits_{i=1}^{m} B_{M_i}$ is such a transformation on the gauge multi-basis $\bigcup\limits_{i=1}^{m} B_{M_i}$ and Lagrange density $\mathscr{L}_{\widetilde{M}}$ with $\tau_{\widetilde{M}}|_{M_i} = \tau_i, \mathscr{L}_{\widetilde{M}}|_{M_i} = \mathscr{L}_{M_i}$ for integers $1 \leq i \leq m$ such that

$$\mathscr{L}_{\widetilde{M}}(\bigcup_{i=1}^{m} B_{M_i})^{\tau_{\widetilde{M}}} = \mathscr{L}_{\widetilde{M}}(\bigcup_{i=1}^{m} B_{M_i}).$$

A multi-basis $\bigcup\limits_{i=1}^{m} B_{M_i}$ is a *combinatorial gauge basis* if for any automorphism $g \in \mathrm{Aut}G^L[\widetilde{M}]$,

$$\mathscr{L}_{\widetilde{M}}(\bigcup_{i=1}^{m} B_{M_i})^{\tau_{\widetilde{M}} \circ g} = \mathscr{L}_{\widetilde{M}}(\bigcup_{i=1}^{m} B_{M_i}),$$

where $\tau_{\widetilde{M}} \circ g$ means $\tau_{\widetilde{M}}$ composting with an automorphism g, a bijection on gauge multi-basis $\bigcup\limits_{i=1}^{m} B_{M_i}$. Whence, a combinatorial field consisting of gauge fields $M_1, M_2, \cdots, M_m$ is a combinatorial gauge field if $M_1^{\alpha} = M_2^{\alpha}$ for $\forall M_1^{\alpha}, M_2^{\alpha} \in \Omega_{\alpha}$, where Ω_{α}, $1 \leq \alpha \leq s$ are orbits of $M_1, M_2, \cdots, M_m$ under the action of $\mathrm{Aut}G^L[\widetilde{M}]$. Therefore, combining existent gauge fields underlying a connected graph G in space enables us to find more combinatorial gauge fields. For example, combinatorial gravitational fields $\widetilde{M}(t)$ determined by tensor equations

$$R_{(\mu\nu)(\sigma\tau)} - \frac{1}{2} g_{(\mu\nu)(\sigma\tau)} R = -8\pi G \mathscr{E}_{(\mu\nu)(\sigma\tau)}$$

in a combinatorial Riemannian manifold $(\widetilde{M}, g, \widetilde{D})$ with $\widetilde{M} = \widetilde{M}(n_1, n_2, \cdots, n_m)$.

Now let $\overset{1}{\omega}$ be the local connection 1-form, $\overset{2}{\Omega} = \widetilde{d}\,\overset{1}{\omega}$ the curvature 2-form of a local connection on $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$ and $\Lambda : \widetilde{M} \to \widetilde{P}$, $\Pi \circ \Lambda = \mathrm{id}_{\widetilde{M}}$ be a local cross section of $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$. Consider

$$\widetilde{A} = \Lambda^* \overset{1}{\omega} = \sum_{\mu\nu} A_{\mu\nu} dx^{\mu\nu},$$

$$\widetilde{F} = \Lambda^* \overset{2}{\Omega} = \sum F_{(\mu\nu)(\kappa\lambda)} dx^{\mu\nu} \wedge dx^{\kappa\lambda}, \qquad \widetilde{d}\,\widetilde{F} = 0,$$

called the *combinatorial gauge potential* and *combinatorial field strength*, respectively. Let $\gamma : \widetilde{M} \to \mathbf{R}$ and $\Lambda' : \widetilde{M} \to \widetilde{P}$, $\Lambda'(\overline{x}) = e^{i\gamma(\overline{x})}\Lambda(\overline{x})$. If $\widetilde{A}' = \Lambda'^* \overset{1}{\omega}$, then we have

$$\overset{1}{\omega}{}' (X) = g^{-1} \overset{1}{\omega} (X') g + g^{-1} dg, \ g \in \mathscr{L}_G,$$

for $dg \in T_g(\mathscr{L}_G)$, $X = \widetilde{d} R_g X'$ by properties of local connections on combinatorial principal fiber bundles discussed in Section 4.4, which finally yields equations following

$$\widetilde{A}' = \widetilde{A} + \widetilde{d}\,\widetilde{A}, \qquad \widetilde{d}\,\widetilde{F}' = \widetilde{d}\,\widetilde{F},$$

i.e., the gauge transformation law on field. This equation enables one to obtain the local form of $\widetilde{F}$ as they contributions to Maxwell or Yang-Mills fields in classical gauge fields theory.

Certainly, combinatorial fields can be applied to any many-body system in natural or social science, such as those in mechanics, cosmology, physical structure, economics,$\cdots$, etc..

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