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Achromatic Coloring on Double Star Graph Families

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Abstract: The purpose of this article is to find the achromatic number, i.e., Smarandachely achromatic 1-coloring for the central graph, middle graph, total graph and line graph of double star graph $K_{1,n,n}$ denoted by $C(K_{1,n,n})$, $M(K_{1,n,n})$, $T(K_{1,n,n})$ and $L(K_{1,n,n})$ respectively.

Keywords: Smarandachely achromatic k-coloring, Smarandachely achromatic number, central graph, middle graph, total graph, line graph and achromatic coloring.

AMS(2000): 05C15

§1. Preliminaries

For a given graph $G = (V, E)$ we do an operation on G, by subdividing each edge exactly once and joining all the non adjacent vertices of G. The graph obtained by this process is called central graph [10] of G denoted by $C(G)$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [4] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [1,5] of G, denoted by $T(G)$ is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices

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x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The line graph [1,5] of G denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $2n+1$ vertices and $2n$ edges.

For given graph G, an integer $k \geq 1$, a *Smarandachely achromatic k-coloring* of G is a proper vertex coloring of G in which every pair of colors appears on at least k pairs of adjacent vertices. The *Smarandachely achromatic number* of G denoted $\chi_c^S(G)$, is the greatest number of colors in a Smarandachely achromatic k-coloring of G. Certainly, $\chi_c^S(G) \geq k$. Now if $k = 1$, i.e., a Smarandachely achromatic 1-coloring and $\chi_c^S(G)$ are usually abbreviated to *achromatic coloring* [2,3,6,7,8,9,11] and $\chi_c(G)$.

The achromatic number was introduced by Harary, Hedetniemi and Prins [6]. They considered homomorphisms from a graph G onto a complete graph K_n. A homomorphism from a graph G to a graph G' is a function $\phi : V(G) \to V(G')$ such that whenever u and v are adjacent in G, $u\phi$ and $v\phi$ are adjacent in G'. They show that, for every (complete) n-coloring τ of a graph G there exists a (complete) homomorphism ϕ of G onto K_n and conversely. They noted that the smallest n for which such a complete homomorphism exists is just the chromatic number $\chi = \chi(G)$ of G. They considered the largest n for which such a homomorphism exists. This was later named as the achromatic number $\psi(G)$ by Harary and Hedetniemi [6]. In the first paper [6] it is shown that there is a complete homomorphism from G onto K_n if and if only $\chi(G) \leq n \leq \psi(G)$.

§2. Achromatic Coloring on central graph of double star graph

Algorithm 2.1

Input: The number n of $K_{1,n,n}$.
Output: Assigning achromatic coloring for the vertices in $C(K_{1,n,n})$.

```
begin
for i = 1 to n
{
V_1 = {u_i};
C(u_i) = i;
V_2 = {s_i};
C(s_i) = n + 1;
V_3 = {e_i};
C(e_i) = i;
V_4 = {v_i};
C(v_i) = n + 1 + i;
}
```

$V_4 = \{v\}$;
$C(v) = n + 1$;
$V = V_1 \cup V_2 \cup V_3 \cup V_4$;
end

Theorem 2.1 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[C(K_{1,n,n})] = 2n + 1.$$

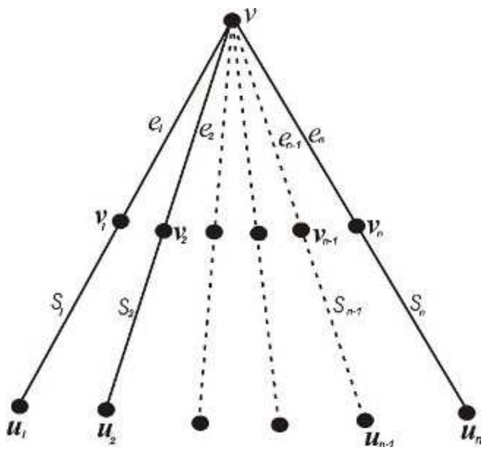

Fig.1

Double star graph $K_{1,n,n}$

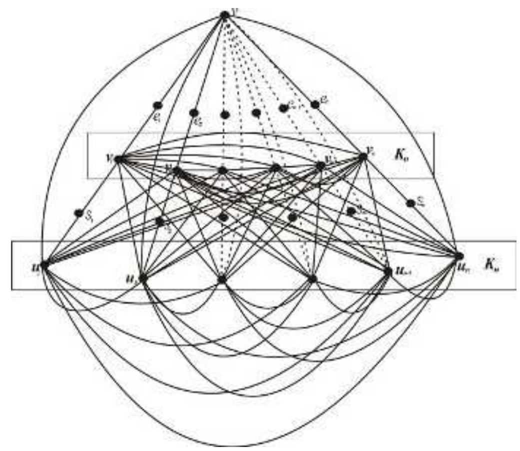

Fig.2

Central graph of double star graph $C(K_{1,n,n})$

Proof Let v,v_1,v_2,$\cdots$,v_n and u_1 ,u_2,..,u_n be the vertices in $K_{1,n,n}$, the vertex v be adjacent to $v_i (1 \le i \le n)$. The vertices $v_i (1 \le i \le n)$ be adjacent to $u_i (1 \le i \le n)$. Let the edge vv_i and uu_i $(1 \le i \le n)$ be subdivided by the vertices $e_i (1 \le i \le n)$ and $s_i (1 \le i \le n)$ in $C(K_{1,n,n})$. Clearly $V[C(K_{1,n,n})] = \{v\} \cup \{v_i/1 \le i \le n\} \cup \{u_i/1 \le i \le n\} \cup \{e_i/1 \le i \le n\} \cup \{s_i/1 \le i \le n\}$. The vertices $v_i (1 \le i \le n)$ induce a clique of order n (say K_n) and the vertices $v, u_i (1 \le i \le n)$ induce a clique of order $n + 1$ (say K_{n+1}) in $C(K_{1,n,n})$ respectively. Now consider the vertex set $V[C(K_{1,n,n})]$ and the color classes $C_1 = \{c_1, c_2, c_3, .., c_n\}$ and $C_2 = \{c_1, c_2, c_3, ..., c_n, c_{n+1}\}$, assign a proper coloring to $C(K_{1,n,n})$ by Algorithm 2.1.

To prove the above said coloring is achromatic, we consider any pair (c_i, c_j).

Step 1

If $i = 1$, $j = 2, 3..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will accommodate the color pair (c_i, c_j).

Step 2

If $i = 2$, $j = 1, 2, 3, ..., n$. The edges joining the vertices (u_i, u_j), (u_i, v), (u_i, v_j), (e_i, v_i), (u_i, s_i) and (v_i, s_i), will accommodate the color pair (c_i, c_j).

Step 3

If $i = 3$, $j = 1, 2, ..n$. The edges joining the vertices (u_i, u_j), (u_i, v), (u_i, v_j), (e_i, v_i), (u_i, s_i) and (v_i, s_i), will accommodate the color pair (c_i, c_j). Similarly if $i = n$, $j = 1, 2, ..n-1$, then the edges joining the vertex pair (u_i, u_j), (u_i, v), (u_i, v_j), (e_i, v_i), (u_i, s_i) and (v_i, s_i), will stand for the color pair (c_i, c_j). Now this coloring will accommodate all the pairs of the color class. Thus we have $\chi_c[C(K_{1,n,n})] \geq 2n + 1$. The number of edges of

$$C[K_{1,n,n}] = \left\{5n + 2\frac{n(n-1)}{2} + n(n-1)\right\} = 4n + 2n^2 < \binom{2n+1}{2}.$$

Therefore, $\chi_c[C(K_{1,n,n})] \leq 2n + 1$. Hence $\chi_c[C(K_{1,n,n})] = 2n + 1$. □

Example 2.3

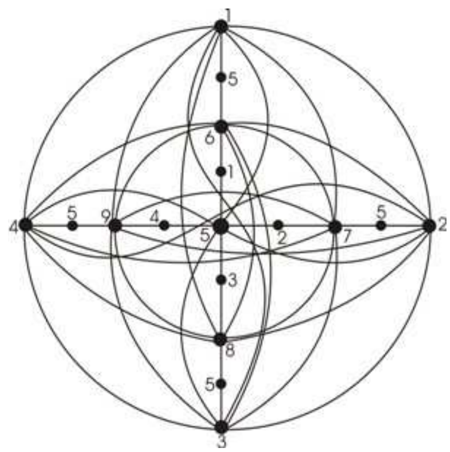

Fig.3
Central graph of $C(K_{1,4,4})$
$\chi_c[C(K_{1,4,4})] = 9$

§3. Achromatic coloring on middle graph of double star graph

Algorithm 3.1

Input: The number n of $K_{1,n,n}$.
Output: Assigning achromatic coloring for vertices in $M(K_{1,n,n})$.
begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
}
$V_2 = \{v\}$;
$C(v) = n + 1$;
for $i = 1$ to n

{
$V_3 = \{v_i\}$;
$C(v_i) = n + 2$;
}
for $i = 2$ to n
{
$V_4 = \{s_i\}$;
$C(s_i) = n + 3$;
}
$C(s_1) = n + 1$;
for $i = 1$ to $n - 1$
{
$V_5 = \{u_i\}$;
$C(u_i) = 1$;
}
$C(u_n) = C(v)$;
$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$;
end

Theorem 3.1 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[M(K_{1,n,n})] = n + 3.$$

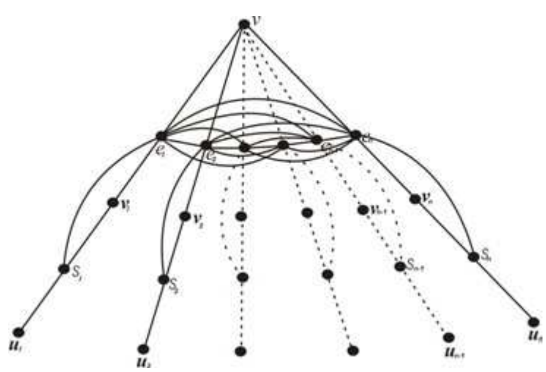

Fig.4

Middle graph of double star graph $M(K_{1,n,n})$

Proof Let $V(K_{1,n,n}) = \{v\} \cup \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\}$. By definition of middle graph, each edge vv_i and $v_iu_i (1 \leq i \leq n)$ in $K_{1,n,n}$ are subdivided by the vertices u_i and s_i in $M(K_{1,n,n})$ and the vertices v,e_1,e_2,...,e_n induce a clique of order $n+1$(say K_{n+1})

in $M(K_{1,n,n})$. i.e.,$V[M(K_{1,n,n})] = \{v\} \cup \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\} \cup \{e_i/1 \leq i \leq n\} \cup \{s_i/1 \leq i \leq n\}$. Now consider the vertex set $V[M(K_{1,n,n})]$ and colour class $C = \{c_1, c_2, c_3, .., c_n, c_{n+1}, c_{n+2}, c_{n+3}\}$, assign a proper coloring to $M(K_{1,n,n})$ by Algorithm 3.1.

To prove the above said coloring is achromatic, we consider any pair (c_i, c_j)

Step 1

If $i = 1$, $j = 2, 3, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will accommodate the color pair (c_i, c_j).

Step 2

If $i = 2$, $j = 1, 2, 3, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will accommodate the color pair (c_i, c_j).

Step 3

If $i = 3$, $j = 1, 2, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will accommodate the color pair (c_i, c_j). Similarly if $i = n$, $j = 1, 2, ..., n-1$, then the edges joining the vertex pair (e_i, e_j), (e_i, v), (e_i, s_j), (e_i, v_i), (e_i, s_i), (s_j, u_j) and (u_n, s_n), will stand for the color pair (c_i, c_j). Now this coloring will accommodate all the pairs of the color class.

Thus we have $\chi_c[M(K_{1,n,n})] \geq n + 3$. The number of edges in $M[K_{1,n,n}] = \dfrac{n^2 + 9n}{2} < \dbinom{n+4}{2}$. Therefore, $\chi_c[M(K_{1,n,n})] \leq n + 3$. Hence $\chi_c[M(K_{1,n,n})] = n + 3$. □

Example 3.3

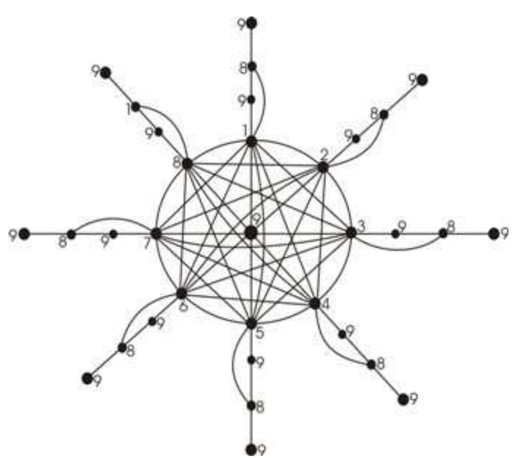

Fig.5

Middle graph of $M(K_{1,8,8})$

$\chi_c[M(K_{1,8,8})] = 11$

§4. Achromatic Coloring on Total Graph of Double Star Graph

Algorithm 4.1

Input: The number "n" of $K_{1,n,n}$.
Output: Assigning achromatic coloring for vertices in $T(K_{1,n,n})$.
begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
$V_2 = \{v_i\}$;
$C(v_i) = n + 2$;
$V_3 = \{s_i\}$;
$C(s_i) = n + 3$;
$V_4 = \{u_i\}$;
$C(u_i) = n + 1$;
}
$V_5 = \{v\}$;
$C(v) = n + 1$;
$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$;
end

Theorem 4.1 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[T(K_{1,n,n})] = n + 3.$$

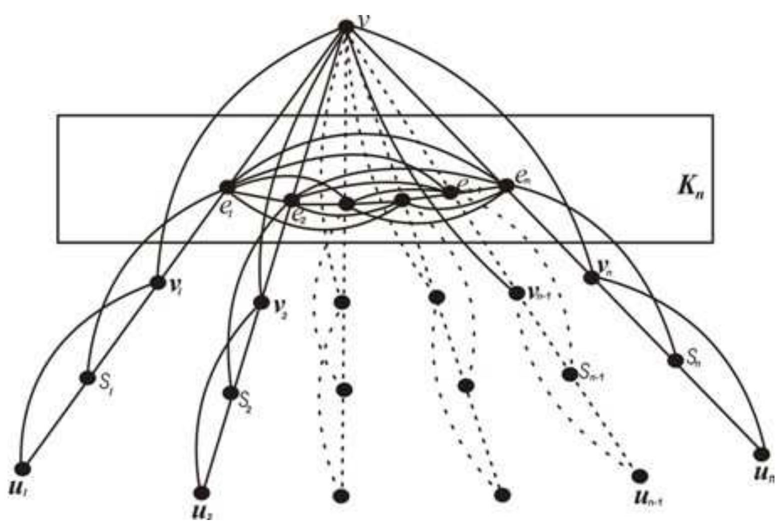

Fig.6

Total graph of double star graph $T(K_{1,n,n})$

Proof Let $V(K_{1,n,n}) = \{v, v_1, v_2, .., v_n\} \cup \{u_1, u_2, .., u_n\}$ and $E(K_{1,n,n}) = \{e_1, e_2, ..., e_n\} \cup \{s_1, s_2, s_3, ..., s_n\}$. By the definition of total graph, we have $V[T(K_{1,n,n})] = \{v\} \cup \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\} \cup \{e_i/1 \leq i \leq n\} \cup \{s_i/1 \leq i \leq n\}$, in which the vertices v, e_1, e_2,...,e_n induce a clique of order $n+1$ (say K_{n+1}). Now consider the vertex set $V[T(K_{1,n,n})]$ and colour class $C = \{c_1, c_2, c_3, ...c_n, c_{n+1}, c_{n+2}, c_{n+3}\}$, assign a proper coloring to $T(K_{1,n,n})$ by Algorithm 4.1.

To prove the above said coloring is achromatic, we consider any pair (c_i, c_j).

Step 1

If $i = 1$, $j = 1, 2, 3, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), and (e_i, v_i) will accommodate the color pair (c_i, c_j).

Step 2

If $i = 2$, $j = 1, 2, ...n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), and (e_i, v_i) will accommodate the color pair (c_i, c_j).

Step 3

If $i = 3$, $j = 1, 2, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, v), (e_i, s_j), and (e_i, v_i) will accommodate the color pair (c_i, c_j). Similarly if $i = n$, $j = 1, 2, ..n - 1$, then the edges joining the vertex pair (e_i, e_j), (e_i, v), (e_i, s_j), and (e_i, v_i) will stand for the color pair (c_i, c_j).

Thus any pair in the color class is adjacent by at least one edge. Thus we have $\chi_c[T(K_{1,n,n})] \geq n + 3$. The number of edges of $T[K_{1,n,n}] = \dfrac{n^2 + 13n}{2} < \dbinom{n+4}{2}$. Therefore, $\chi_c[(K_{1,n,n})] \leq n + 3$. Hence $\chi_c[T(K_{1,n,n})] = n + 3$. □

Example 4.3

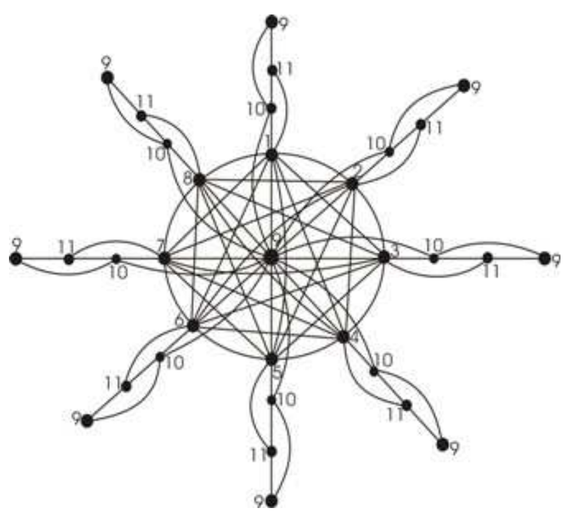

Fig.7

Total graph of $T(K_{1,8,8})$

$\chi_c[T(K_{1,8,8})] = 11$

§5. Achromatic Coloring on Line Graph of Double Star Graphs

Algorithm 5.1

Input: The number n of $K_{1,n,n}$.
Output: Assigning achromatic coloring for vertices in $L(K_{1,n,n})$.
begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
$V_2 = \{s_i\}$;
$C(s_i) = n + 1$;
}
$V = V_1 \cup V_2$;
end

Theorem 5.1 *For any double star graph $K_{1,n,n}$,the achromatic number,*

$$\chi_c[L(K_{1,n,n})] = n + 1.$$

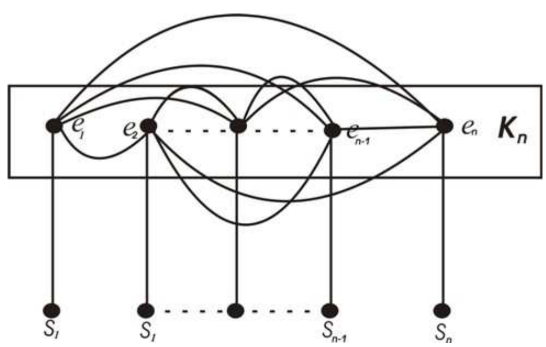

Fig.8

Line graph of double star graph $L(K_{1,n,n})$

Proof Let $V(K_{1,n,n}) = \{v\} \cup \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\}$ and $E(K_{1,n,n}) = \{e_1, e_2, ...e_n\} \cup \{s_1, s_2, s_3, ..., s_n\}$ By the definition of Line graph, each edge of $K_{1,n,n}$ taken to be as vertex in $L(K_{1,n,n})$. The vertices e_1, e_2, ..., e_n induce a clique of order n in $L(K_{1,n,n})$.i.e., $V[L(K_{1,n,n})] = \{e_i/1 \leq i \leq n\} \cup \{s_i/1 \leq i \leq n\}$. Now consider the vertex set $V[L(K_{1,n,n})]$ and colour class $C = \{c_1, c_2, c_3, ...c_n, c_{n+1}\}$, assigned a proper coloring to $L(K_{1,n,n})$ by Algorithm 5.1.

To prove the above said coloring is achromatic, we consider any pair (c_i, c_j).

Step 1

If $i = 1$, $j = 1, 2, 3, ..., n$. The edges joining the vertices (e_i, e_j), and (e_i, s_i) will accommodate the color pair (c_i, c_j).

Step 2

If $i = 2$, $j = 1, 2, ...n$. The edges joining the vertices (e_i, e_j), and (e_i, s_i), will accommodate the color pair (c_i, c_j).

Step 3

If $i = 3$, $j = 1, 2, ..., n$. The edges joining the vertices (e_i, e_j), (e_i, s_i) will accommodate the color pair(c_i, c_j). Similarly if $i = n$,$j = 1, 2, ..n - 1$, then the edges joining the vertex pair (e_i, e_j), (e_i, s_i) will stand for the color pair (c_i, c_j).

Thus any pair in the color class is adjacent by at least one edge we have $\chi_c[L[K_{1,n,n})] \geq n + 1$. The number of edges of edges of $L[K_{1,n,n}] = \dfrac{n^2+n}{2} < \dbinom{n+2}{2}$. Therefore, $\chi_c[L(K_{1,n,n})] \leq n + 1$. Hence $\chi_c[L(K_{1,n,n})] = n + 1$. □

Example 5.3

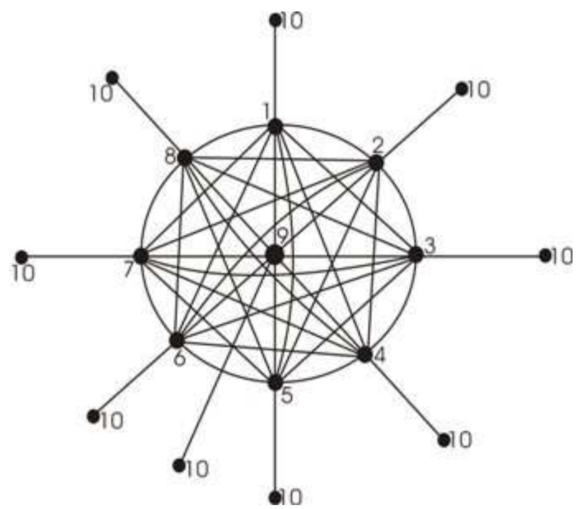

Fig.9

Line graph of $L(K_{1,9,9})$
$\chi_c[L(K_{1,9,9})] = 10$

§6. Main Theorems

Theorem 6.1 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[L(K_{1,n,n})] = \chi[M(K_{1,n,n})] = \chi[T[K_{1,n,n})] = n + 1.$$

Theorem 6.2 *For any double star graph $K_{1,n,n}$, the achromatic number,*

$$\chi_c[M(K_{1,n,n})] = \chi_c[T(K_{1,n,n})] = n + 3.$$

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