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A Spacetime Geodesics of the Schwarzschild Space and Its Deformation Retract

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Abstract: A Smarandache multi-space is such a union space $\bigcup\limits_{i=1}^{n} S_i$ of $S_1, S_2, \cdots, S_n$ for an integer $n \geq 1$. In this paper, we deduce the spacetime geodesic of Schwarzschild space, i.e., a Smarandachely Schwarzschild space with $n = 1$ by using Lagrangian equations. The deformation retract of this space will be presented. The relation between folding and deformation retract of this space will be achieved.

Keywords: Schwarzschild space, spacetime, folding, Smarandache multi-space, deformation retract.

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§1. Introduction

The Dirichlet problem for boundaries which are S^1-bundles over some compact manifolds. In general relativity such boundaries often arise in gravitational thermodynamics. The classic example is that of the trivial bundle $\sum \equiv S^1 \times S^2$. Manifolds with complete Ricci-flat metrics [1] admitting such boundaries are known; they are the Euclidean's Schwarzschild metric and the flat metric with periodic identification. The Schwarzschild metric result by taking the limit k $\to$ 0 and L $\to$ 0 while keeping r+ fixed:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad (1)$$

Here $t \in [0,\infty)$ and replaces the ψ coordinate in the previous two examples. The metric has a bolt singularity at r = 2m which can be made regular by identifying the coordinate t with a period of 8πm. The radial coordinate r has the range [2m,∞) and constant r slices of the regular metric have the trivial product topology of $S^1 \times S^2$. The four-metric therefore has the topology of $R^1 \times S^2$. For a boundary $\sum \equiv S^1 \times S^2$, the pair ($\alpha$, β) constitutes the canonical boundary data with the interpretation that α represents the radius of a spherical cavity immersed in a thermal bath with temperature T = $\frac{1}{2\pi\beta}$. It is known that for such canonical boundary data, apart from the obvious infilling flat-space solution with proper identification, there are in general two black hole solutions distinguished by their masses which become degenerate at a certain

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value of the squashing, i.e., the ratio of the two radii $\frac{\beta}{\alpha}$. This can be seen in the following way. First rewrite the Schwarzschild metric (1) in the following form:

$$ds^2 = \left(1 - \frac{2m}{r}\right) 64\,\pi^2 m^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad (2)$$

where t $= 8\pi\tau$ such that τ has unit period. With this definition one can simply read off the proper length – alternatively the radius – of the S^1fibre and that of the S^2base. They are

$$\alpha^2 = r^2 \qquad (3)$$

and

$$\beta^2 = 16m^2\left(1 - \frac{2m}{r}\right) \qquad (4)$$

It is easy to see that for a given (α, β), r is uniquely determined whereas m is given by the positive solutions of the following equation:

$$m^3 - \frac{1}{2}\alpha m^2 + \frac{1}{32}\alpha\beta^2 = 0 \qquad (5)$$

By solving this equation for m, the two Schwarzschild infilling geometries are found3. There are in general two positive roots of Eq.(5) provided $\frac{\beta^2}{\alpha^2} \leq \frac{16}{27}$. When the equality holds the two solutions become degenerate and beyond this value of squashing they turn complex. The remaining root of Eq.(5) is always negative. Therefore the two infilling solutions appear and disappear in pairs as the boundary data is varied [1,12].

Next let us recall the concept of a metric in four-dimensions. Considering only flat space, we have

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 \qquad (6)$$

Now we see that $ds^2 > 0, ds^2 = 0$ and $ds^2 < 0$ correspond to space-like, null and time-like geodesics, we note that massless particles, such as the photon, move on null geodesics. That can be interpreted as saying that in 4-dimensional space, the photon does not move and that for a photon, and time does not pass. Particularly intriguing is the mathematical possibility of a negative metric. Now it is extremely interesting that there is a geometry in which two separated points may still have a zero distance analogous to the corresponding to a null geodesic [7].

§2. **Definitions and Background**

(i) Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \rightarrow N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : J \to M$, the induced path $f \circ \gamma : J \to N$ is a piecewise geodesic and of the same length as γ [13]. If f does not preserve the lengths, it is called topological folding. Many types of foldings are discussed in [3,4,5,6,8,9]. Some applications are discussed in [2,10].

(ii) A subset A of a topological space X is called a retract of X, if there exists a continuous map $r : X \to A$ such that([11])

(a) X is open

(b) $r(a) = a$, $\forall\ a \in A$.

(iii) A subset A of a topological space X is said to be a deformation retract if there exists a retraction $r : X \to A$, and a homotopy $f : X \times I \to X$ such that([11])

$f(x, 0) = x, \forall\ x \in X,$

$f(x, 1) = r(x), \forall\ x \in X,$

$f(a, t) = a, \forall\ a \in A, t \in [0, 1]$.

§3. Main Results

In this paper we discuss the deformation retract of the Schwarzschild space with metric:

$$ds^2 = \left(1 - \frac{2m}{r}\right) 64\,\pi^2 m^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

Then, the coordinate of Schwarzschild space are given by:

$x_1 = \pm\sqrt{c_1 + (1 - \frac{2m}{r})64\pi^2 m^2 t^2}$

$x_2 = \pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}$

$x_3 = \pm\sqrt{c_3 + r^2\theta^2}$

$x_4 = \pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}$

where c_1, c_2, c_3 and c_4 are the constant of integration. Now, by using the Lagrangian equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial x_i'}\right) - \frac{\partial T}{\partial x_i} = 0, \quad i = 1, 2, 3, 4,$$

find a geodesic which is a subspace of Schwarzschild space.

Since

$$T = \frac{1}{2}\overline{ds}^2$$

$$T = \frac{1}{2}\left\{\left(1 - \frac{2m}{r}\right) 64\,\pi^2 m^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

Then, the Lagrangian equations are

$$\frac{d}{ds}\left(\left(1 - \frac{2m}{r}\right) 64\,\pi^2 m^2 t'\right) = 0 \qquad (7)$$

$$
\begin{aligned}
&\frac{d}{ds}\left(\left(1-\frac{2m}{r}\right)^{-1} r'\right)- \\
&\left(128\,\pi^2 m^3 dt^2+\frac{2m}{(2m-r)^2}dr^2+r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right)\right)=0 && (8)\\
&\frac{d}{ds}\left(r^2\theta'\right)-r^2\sin 2\theta\, d\phi^2=0 && (9)\\
&\frac{d}{ds}\left(r^2\sin^2\theta\ \phi'\right)=0 && (10)
\end{aligned}
$$

From equation (7), we obtain

$$\left(1-\frac{2m}{r}\right)64\,\pi^2 m^2 t'=\delta=\text{constant},$$

if $\delta=0$, we have two cases:

(i) $t'=0$, or $t=cons\tan t=\beta$, if $\beta=0$, we obtain the following coordinates:

$x_1=\pm\sqrt{c_1}$

$x_2=\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}$

$x_3=\pm\sqrt{c_3+r^2\theta^2}$

$x_4=\pm\sqrt{c_4+r^2\sin^2\theta\phi^2}$

This is the geodesic hyper spacetime S_1 of the Schwarzchild space$S$, i.e. $dS^2\succ 0$. This is a retraction.

(ii) If $m=0$, we obtain the following coordinates:

$x_1=\pm\sqrt{c_1}$

$x_2=\pm\sqrt{c_2+r^2+8\pi^2\ln(r)}$

$x_3=\pm\sqrt{c_3+r^2\theta^2}$

$x_4=\pm\sqrt{c_4+r^2\sin^2\theta\phi^2}$

This is the geodesic hyper spacetime S_2 of the Schwarzchild space$S$, i.e. $dS^2\succ 0$. This is a retraction.

From equation (8), we obtain

$$r^2\sin^2\theta\ \phi'=\alpha=\text{constant},$$

if $\alpha=0$, we have two cases:

(a) If $\phi'=0$, or $\phi=cons\tan t=\zeta$, if $\zeta=0$, we obtain the following coordinates:

$x_1=\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}$

$x_2=\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}$

$x_3 = \pm\sqrt{c_3 + r^2\theta^2}$

$x_4 = \pm\sqrt{c_4}$

This is the geodesic hyper spacetime S_3 of the Schwarzchild space$S$, i.e. $dS^2 \succ 0$. This is a retraction.

(b) If $\theta = 0$, , we obtain the following coordinates:

$x_1 = \pm\sqrt{c_1 + (1 - \frac{2m}{r})64\pi^2 m^2 t^2}$

$x_2 = \pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}$

$x_3 = \pm\sqrt{c_3}$

$x_4 = \pm\sqrt{c_4}$

If $c_3 = c_4 = 0$, then $x_1^2 + x_2^2 + x_3^2 - x_4^2 \succ 0$, which is the great circle S_4in the Schwarzchild spacetime geodesic. These geodesic is a retraction in Schwarzchild space.

Now, we are in a postion to formulate the following theorem.

Theorem 1 *The retraction of Schwarzchild space are spacetime geodesic.*

The deformation retract of the Schwarzchild space is defined by:$\varphi : S \times I \to S$,where S is the Schwarzchild space and I is the closed interval $[0, 1]$. The retraction of Schwarzchild space$S$ is given by:$R : S \to S_1, S_2, S_3, S_4$.

Then, the deformation retracts of the Schwarzchild space$S$ into a hyper spacetime geodesic $S_1 \subset S$ is given by:

$$\begin{aligned}
\varphi(m,t) &= (1-t)\{\pm\sqrt{c_1 + (1 - \frac{2m}{r})64\pi^2 m^2 t^2}, \\
&\pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}, \pm\sqrt{c_3 + r^2\theta^2}, \\
&\pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}\} + t\ \{\pm\sqrt{c_1}, \pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}, \\
&\pm\sqrt{c_3 + r^2\theta^2}, \pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}\},
\end{aligned}$$

where $\varphi(m, 0) = S$ and $\varphi(m, 1) = S_1$.

The deformation retracts of the Schwarzchild space S into a hyper spacetime geodesic $S_2 \subset S$ is given by:

$$\begin{aligned}
\varphi(m,t) &= (1-t)\{\pm\sqrt{c_1 + (1 - \frac{2m}{r})64\pi^2 m^2 t^2}, \\
&\pm\sqrt{c_2 + r^2 + 4mr + 8\pi^2 \ln(r - 2m)}, \pm\sqrt{c_3 + r^2\theta^2}\ \pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}\} \\
&+t\ \{\pm\sqrt{c_1}, \pm\sqrt{c_2 + r^2 + 8\pi^2 \ln(r)}, \pm\sqrt{c_3 + r^2\theta^2}\ , \pm\sqrt{c_4 + r^2 \sin^2\theta\ \phi^2}\},
\end{aligned}$$

The deformation retracts of the Schwarzchild space S into a hyper spacetime geodesic $S_3 \subset S$ is given by

$$\begin{aligned}\varphi(m,t) &= (1-t)\{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2},\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}\,,\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\\ &+t\ \{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2},\ \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4}\}\end{aligned}$$

The deformation retracts of the Schwarzchild space S into a spacetime geodesic $S_4 \subset S$ is given by

$$\begin{aligned}\varphi(m,t) &= (1-t)\{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2},\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\\ &+t\ \{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\ ,\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3}\,,\pm\sqrt{c_4}\}\end{aligned}$$

Now, we are going to discuss the folding f of the Schwarzchild space$S$. Let $f : S \to S$, where $f(x_1, x_2, x_3, x_4) = (|x_1|\,, x_2, x_3, x_4)$.An isometric folding of the Schwarzchild space$S$ into itself may be defined by

$$\begin{gathered}f:\{\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2},\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ \pm\sqrt{c_3+r^2\theta^2}\,,\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\to\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{gathered}$$

The deformation retracts of the folded Schwarzchild space$S$ into the folded hyper spacetime geodesic $S_1 \subset S$ is

$$\begin{gathered}\varphi_f:\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ \pm\sqrt{c_3+r^2\theta^2}\,,\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\times I\to\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ \sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{gathered}$$

with

$$\begin{aligned}\varphi_f(m,t) &= (1-t)\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\\ &+t\ \{|\pm\sqrt{c_1}|\ , \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2},\\ &\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

The deformation retracts of the folded Schwarzchild space S into the folded hyper spacetime geodesic $S_2 \subset S$ is

$$\begin{aligned}&\varphi_f : \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\} \times I \to \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

with

$$\begin{aligned}\varphi_f(m,t) &= (1-t)\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \pm\\ &\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\\ &+t\ \{|\pm\sqrt{c_1}|\ , \pm\sqrt{c_2+r^2+8\pi^2\ln(r)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

The deformation retracts of the folded Schwarzchild space S into the folded hyper spacetime geodesic $S_3 \subset S$ is

$$\begin{aligned}&\varphi_f : \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|, \pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\} \times I \to \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)}, \pm\sqrt{c_3+r^2\theta^2}, \pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

with

$$\begin{aligned}\varphi_f(m,t) &= (1-t)\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\ \pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\\ &+t\ \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4}\}\end{aligned}$$

The deformation retracts of the folded Schwarzchild space S into the folded hyper spacetime geodesic $S_4 \subset S$ is

$$\begin{aligned}&\varphi_f:\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\\ &\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\times I\to\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ &c_2+r^2+4mr+8\pi^2\ln(r-2m),\pm\sqrt{c_3+r^2\theta^2},\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}\end{aligned}$$

with

$$\begin{aligned}\varphi_f(m,t) &= (1-t)\{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|,\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\ \pm\sqrt{c_3+r^2\theta^2},\\ &\pm\sqrt{c_4+r^2\sin^2\theta\ \phi^2}\}+t\ \{\left|\pm\sqrt{c_1+(1-\frac{2m}{r})64\pi^2m^2t^2}\right|\\ &\pm\sqrt{c_2+r^2+4mr+8\pi^2\ln(r-2m)},\pm\sqrt{c_3},\pm\sqrt{c_4}\}\end{aligned}$$

Then the following theorem has been proved.

Theorem 2 *Under the defined folding, the deformation retract of the folded Schwarzchild space into the folded hyper spacetime geodesic is different from the deformation retract of Schwarzchild space into hyper spacetime geodesic.*

References

[1] M.M.Akbar and G.W.Gibbons: *Ricci-flat Metrics with U(1) Action and Dirichle Boundary value. Problem in Riemannian Quantum Gravity and Isoperimetric Inequalities*, Arxiv: hep-th/ 0301026 v1.

[2] P. DI-Francesco, Folding and coloring problem in Mathematics and Physics, *Bull. Amer. Math. Soc.* 37, (3), (2002), 251-307.

[3] A. E. El-Ahmady, and H. Rafat, A calculation of geodesics in chaotic flat space and its folding, *Chaos, Solitons and fractals*, 30 (2006), 836-844.

[4] M. El-Ghoul, A. E. El-Ahmady and H. Rafat, Folding-Retraction of chaotic dynamical manifold and the VAK of vacuum fluctuation. *Chaos, Solutions and Fractals*, 20 , (2004), 209-217.

[5] M. El-Ghoul, A. E. El-Ahmady, H. Rafat and M.Abu-Saleem, The fundamental group of the connected sum of manifolds and their foldings, *Journal of the Chungcheong Mathematical Society*, Vol. 18, No. 2(2005), 161-173.

[6] M. El-Ghoul, A. E. El-Ahmady , H. Rafat and M.Abu-Saleem, Folding and Retraction of manifolds and their fundamental group, *International Journal of Pure and Applied Mathematics*, Vol. 29, No.3(2006), 385-392.

[7] El Naschie MS, Einstein's dream and fractal geometry, *Chaos, Solitons and Fractals* 24, (2005), 1-5.

[8] H. Rafat,Tiling of topological spaces and their Cartesian product, *International Journal of Pure and Applied Mathematics*, Vol. 27, No. 3, (2006), 517-522.

[9] H. Rafat, On Tiling for some types of manifolds and their foldings,*Journal of the Chungcheong Mathematical Society* , (Accepted).

[10] J.Nesetril and P.O. de Mendez, Folding, *Journal of Combinatorial Theory*, Series B, 96, (2006), 730-739.

[11] W. S. Massey, *Algebric topology, An introduction*, Harcourt Brace and World, New York (1967).

[12] A. Z. Petrov, *Einstein Spaces*, Pergaman Oxford, London, New York (1969).

[13] S. A. Robertson, Isometric folding of Riemannian manifolds, *Proc. Roy. Soc. Edinburgh*, 77, (1977) , 275-284.