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Simple Path Covers in Graphs

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Abstract: A simple path cover of a graph G is a collection ψ of paths in G such that every edge of G is in exactly one path in ψ and any two paths in ψ have at most one vertex in common. More generally, for any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common. Thus if $k = 1$ and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G. The minimum cardinality of a simple path cover of G is called the simple path covering number of G and is denoted by $\pi_s(G)$. In this paper we initiate a study of this parameter.

Key Words: Smarandache path k-cover, simple path cover, simple path covering number.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [5]. All graphs in this paper are assumed to be connected and non-trivial.

If $P = (v_0, v_1, v_2, \ldots, v_n)$ is a path or a cycle in a graph G, then $v_1, v_2, \ldots, v_{n-1}$ are called internal vertices of P and v_0, v_n are called external vertices of P. If $P = (v_0, v_1, v_2, \ldots, v_n)$ and $Q = (v_n = w_0, w_1, w_2, \ldots, w_m)$ are two paths in G, then the walk obtained by concatenating P and Q at v_n is denoted by $P \circ Q$ and the path $(v_n, v_{n-1}, \ldots, v_2, v_1, v_0)$ is denoted by P^{-1}. For a unicyclic graph G with cycle C, if w is a vertex of degree greater than 2 on C with $deg\ w = k$, let $e_1, e_2, \ldots, e_{k-2}$ be the edges of $E(G) - E(C)$ incident with w. Let $T_i, 1 \leq i \leq k-2$, be the maximal subtree of G such that T_i contains the edge e_i and w is a pendant vertex of T_i. Then $T_1, T_2, \ldots, T_{k-2}$ are called the *branches* of G at w. Also the maximal subtree T of G such that $V(T) \cap V(C) = \{w\}$ is called the *subtree* rooted at w.

The concept of path cover and path covering number of a graph was introduced by Harary [6]. Preliminary results on this parameter were obtained by Harary and Schwenk [7], Peroche [9] and Stanton et al. [10], [11].

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Definition 1.1([6]) *A path cover of a graph G is a collection ψ of paths in G such that every edge of G is in exactly one path in ψ. The minimum cardinality of a path cover of G is called the path covering number of G and is denoted by $\pi(G)$ or simply π.*

Theorem 1.2([10]) *For any tree T with k vertices of odd degree, $\pi(T) = \frac{k}{2}$.*

Theorem 1.3([7]) *The path covering number of the complete graph K_p is given by $\pi(K_p) = \lceil \frac{p}{2} \rceil$. (For any real number x, $\lceil x \rceil$ denotes the least positive integer $\geq x$.)*

Theorem 1.4([4]) *Let G be a unicyclic graph with unique cycle C. Let m denote the number of vertices of degree greater than 2 on C. Let k denote the number of vertices of odd degree. Then*

$$\pi(G) = \begin{cases} 2 & \text{if } m = 0 \\ \frac{k}{2} + 1 & \text{if } m = 1 \\ \frac{k}{2} & \text{otherwise} \end{cases}$$

Theorem 1.5([4]) *For any graph G, $\pi(G) \geq \lceil \frac{\Delta}{2} \rceil$.*

The concepts of graphoidal cover and acyclic graphoidal cover were introduced by Acharya et al. [1] and Arumugam et al. [4].

Definition 1.6([1]) *A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions.*

(*i*) *Every path in ψ has at least two vertices.*
(*ii*) *Every vertex of G is an internal vertex of at most one path in ψ.*
(*iii*) *Every edge of G is in exactly one path in ψ.*

If further no member of ψ is a cycle in G, then ψ is called an acyclic graphoidal cover of G. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$. Similarly we define the acyclic graphoidal covering number $\eta_a(G)$.

An elaborate review of results in graphoidal covers with several interesting applications and a large collection of unsolved problems is given in Arumugam et al.[2].

For any graph $G = (V, E)$, $\psi = E$ is trivially an acyclic graphoidal cover and has the interesting property that any two paths in ψ have at most one vertex in common. Motivated by this observation we introduced the concept of simple acyclic graphoidal covers in graphs [3].

Definition 1.7([3]) *A simple acyclic graphoidal cover of a graph G is an acyclic graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of G and is denoted by $\eta_{as}(G)$ or simply η_{as}.*

Definition 1.8 *Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be an interior vertex of ψ if it is an internal vertex of some path in ψ, otherwise it is said to* *be an exterior vertex of ψ.*

Theorem 1.9([3]) *For any simple acyclic graphoidal cover ψ of a graph G, let t_ψ denote the number of exterior vertices of ψ. Let $t = min\ t_\psi$, where the minimum is taken over all simple acyclic graphoidal covers ψ of G. Then $\eta_{as}(G) = q - p + t$.*

Theorem 1.10([3]) *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let m denote the number of vertices of degree greater than 2 on C. Then*

$$\eta_{as}(G) = \begin{cases} 3 & \text{if } m = 0 \\ n+2 & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n & \text{if } m \geq 3 \end{cases}$$

Theorem 1.11([3]) *Let m and n be integers with $n \geq m \geq 4$. Then*

$$\eta_{as}(K_{m,n}) = \begin{cases} mn - m - n & \text{if } n \leq \binom{m}{2} \\ & \\ mn - m - n + r & \text{if } n = \binom{m}{2} + r, r > 0. \end{cases}$$

In this paper we introduce the concept of simple path cover and simple path covering number π_s of a graph G and initiate a study of this parameter. We observe that the concept of simple path cover is a special case of Smarandache path k-cover [8]. For any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common. Thus if $k = 1$ and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G.

§2. Main results

Definition 2.1 *A simple path cover of a graph G is a path cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple path cover of G is called the simple path covering number of G and is denoted by $\pi_s(G)$. Any simple path cover ψ of G for which $|\psi| = \pi_s(G)$ is called a minimum simple path cover of G.*

Example 2.2 Consider the graph G given in Fig.2.1.

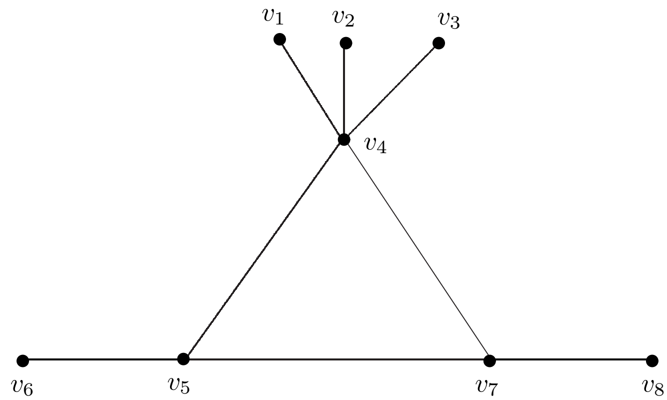

Fig. 2.1

Then $\psi = \{(v_1, v_4, v_7, v_8), (v_3, v_4, v_5, v_6), (v_2, v_4), (v_7, v_5)\}$ is a minimum simple path cover of G so that $\pi_s(G) = 4$.

Remark 2.3 Every path in a simple path cover of a graph G is an induced path.

Theorem 2.4 *For any simple path cover ψ of a graph G, let $t_\psi = \sum_{P\in\psi} t(P)$, where $t(P)$ denotes the number of internal vertices of P and let $t = max\ t_\psi$, where the maximum is taken over all simple path covers ψ of G. Then $\pi_s(G) = q - t$.*

Proof Let ψ be any simple path cover of G. Then

$$\begin{aligned} q &= \sum_{P\in\psi} |E(P)| \\ &= \sum_{P\in\psi} (t(P) + 1) \\ &= |\psi| + \sum_{P\in\psi} t(P) \\ &= |\psi| + t_\psi \end{aligned}$$

Hence $|\psi| = q - t_\psi$ so that $\pi_s(G) = q - t$. □

Corollary 2.5 *For any graph G with k vertices of odd degree $\pi_s(G) = \frac{k}{2} + \sum_{v\in V(G)} \left\lfloor \frac{deg\ v}{2} \right\rfloor - t$.*

Proof Since $q = \frac{k}{2} + \sum_{v\in V(G)} \left\lfloor \frac{deg\ v}{2} \right\rfloor$ the result follows. □

Corollary 2.6 *For any graph G, $\pi_s(G) \geq \frac{k}{2}$ where k is the number of vertices of odd degree in G. Further, the following are equivalent.*

(i) $\pi_s(G) = \frac{k}{2}$.

(ii) There exists a simple path cover ψ of G such that every vertex v in G is an internal vertex of $\left\lfloor \frac{deg\ v}{2} \right\rfloor$ paths in ψ.

(ii) There exists a simple path cover ψ of G such that every vertex of odd degree is an

external vertex of exactly one path in ψ and no vertex of even degree is an external vertex of any path in ψ.

Remark 2.7 For any (p, q)-graph G, $\pi_s(G) \leq q$. Further, equality holds if and only if G is complete. Hence it follows from Theorem 1.3 that $\pi_s(K_n) = \pi(K_n)$ if and only if $n = 2$.

Remark 2.8 Since any path cover of a tree T is a simple path cover of T, it follows from Theorem 1.2 that $\pi_s(T) = \pi(T) = \frac{k}{2}$, where k is the number of vertices of odd degree in T.

We now proceed to determine the value of π_s for unicyclic graphs and wheels.

Theorem 2.9 *Let G be a unicyclic graph with cycle C. Let m denote the number of vertices of degree greater than 2 on C. Let k be the number of vertices of odd degree. Then*

$$\pi_s(G) = \begin{cases} 3 & \textit{if } m = 0 \\ \frac{k}{2} + 2 & \textit{if } m = 1 \\ \frac{k}{2} + 1 & \textit{if } m = 2 \\ \frac{k}{2} & \textit{if } m \geq 3 \end{cases}$$

Proof Let $C = (v_1, v_2, \ldots, v_r, v_1)$.

Case 1. $m = 0$.

Then $G = C$ so that $\pi_s(G) = 3$.

Case 2. $m = 1$.

Let v_1 be the unique vertex of degree greater than 2 on C. Let G_1 be the tree rooted at v_1. Then G_1 has k vertices of odd degree and hence $\pi_s(G_1) = \frac{k}{2}$. Let ψ_1 be a minimum simple path cover of G_1.

If $deg\ v_1$ is odd, then $deg_{G_1}\ v_1$ is odd. Let P be the path in ψ_1 having v_1 as a terminal vertex. Now, let

$P_1 = P \circ (v_1, v_2)$

$P_2 = (v_2, v_3, \ldots, v_r)$ and

$P_3 = (v_r, v_1)$.

If $deg\ v_1$ is even, then $deg_{G_1}\ v_1$ is even. Let $P = (x_1, x_2, \ldots, x_r, v_1, x_{r+1}, \ldots, x_s)$ be a path in ψ_1 having v_1 as an internal vertex. Now, let

$P_1 = (x_1, x_2, \ldots, x_r, v_1, v_2)$

$P_2 = (x_s, x_{s-1}, \ldots, x_{r+1}, v_1, v_r)$ and

$P_3 = (v_2, v_3, \ldots, v_r)$.

Then $\psi = \{\psi_1 - \{P\}\} \cup \{P_1, P_2, P_3\}$ is a simple path cover of G and hence $\pi_s(G) \leq |\psi_1| + 2 = \frac{k}{2} + 2$. Further, for any simple path cover ψ of G, all the k vertices of odd degree and at least two vertices on C are terminal vertices of paths in ψ. Hence $t \leq q - \frac{k}{2} - 2$, so that $\pi_s(G) = q - t \geq \frac{k}{2} + 2$. Thus $\pi_s(G) = \frac{k}{2} + 2$.

Case 3. $m = 2$.

Let v_1 and v_i, where $2 \leq i \leq r$, be the vertices of degree greater than 2 on C. Let P and Q denote respectively the (v_1, v_i)-section and (v_i, v_1)-section of C. Let v_j be an internal vertex of P(say). Let R_1 and R_2 be the (v_1, v_j)-section of P and (v_j, v_i)-section P respectively. Let G_1 be the graph obtained by deleting all the internal vertices of P.

Subcase 3.1 Both $deg\ v_1$ and $deg\ v_i$ are odd.

Then both $deg_{G_1}\ v_1$ and $deg_{G_1}\ v_i$ are even. Hence G_1 is a tree with $k-2$ odd vertices so that $\pi_s(G_1) = \frac{k}{2} - 1$. Let ψ_1 be a minimum simple path cover of G_1 . Then $\psi = \psi_1 \cup \{R_1, R_2\}$ is a simple path cover of G and $|\psi| = \frac{k}{2} + 1$.Hence $\pi_s(G) \leq \frac{k}{2} + 1$.

Subcase 3.2 Both $deg\ v_1$ and $deg\ v_i$ are even.

Then $deg_{G_1}\ v_1$ and $deg_{G_1}\ v_i$ are odd. Hence G_1 is a tree with $k+2$ vertices of odd degree so that $\pi_s(G_1) = \frac{k}{2} + 1$. Let ψ_1 be a minimum simple path cover of G_1.

Suppose v_1 and v_i are terminal vertices of two different paths in ψ_1, say P_1 and P_2 respectively. Now, let

$Q_1 = P_1 \circ R_1$

$Q_2 = P_2 \circ R_2^{-1}$ and

$\psi\ = \{\psi_1 - \{P_1, P_2\}\} \cup \{Q_1, Q_2\}$.

Suppose there exists a path P_1 in ψ_1 having both v_1 and v_i as its end vertices. Then let $P_1 = Q$. Let P_2 be an u_1-w_1 path in ψ_1 having v_1 as an internal vertex and P_3 be an u_2-w_2 path in ψ_1 having v_i as an internal vertex. Let S_1 and S_2 be the (u_1, v_1)-section of P_2 and (w_1, v_1)-section of P_2 respectively. Let S_3 and S_4 be the (u_2, v_i)-section of P_3 and (w_2, v_i)-section of P_3 respectively. Now, let

$Q_1 = S_1 \circ P_1 \circ S_3^{-1}$

$Q_2 = S_2 \circ R_1$

$Q_3 = S_4 \circ R_2^{-1}$ and

$\psi\ = \{\psi_1 - \{P_1, P_2, P_3\}\} \cup \{Q_1, Q_2, Q_3\}$.

Then ψ is a simple path cover of G and $|\psi| = |\psi_1| = \frac{k}{2} + 1$ and hence $\pi_s(G) \leq \frac{k}{2} + 1$.

Subcase 3.3 $deg\ v_1$ is odd and $deg\ v_i$ is even.

Then $deg_{G_1}\ v_1$ is even and $deg_{G_1}\ v_i$ is odd. Hence G_1 is a tree with k vertices of odd degree so that $\pi_s(G_1) = \frac{k}{2}$. Let ψ_1 be a minimum simple path cover of G_1. Let P_1 be the path in ψ_1 having v_i as a terminal vertex.

If $E(P_1) \cap E(Q) = \phi$, let

$Q_1 = P_1 \circ R_2^{-1}$

$Q_2 = R_1$ and

$\psi\ = \{\psi_1 - \{P_1\}\} \cup \{Q_1, Q_2\}$.

Suppose $E(P_1) \cap E(Q) \neq \phi$. Since $deg_{G_1}\ v_i \geq 3$, there exists an u_1-w_1 path in ψ_1, say P_2, having v_i as an internal vertex. Let S_1 and S_2 be the (w_1, v_i)-section of P_2 and (u_1, v_i)-section of P_2 respectively. Now, let

$Q_1 = P_1 \circ S_1^{-1}$

$Q_2 = S_2 \circ R_2^{-1}$

$Q_3 = R_1$ and

$\psi = \{\psi_1 - \{P_1, P_2\}\} \cup \{Q_1, Q_2, Q_3\}$.

Then ψ is a simple path cover of G and $|\psi| = |\psi_1| + 1 = \frac{k}{2} + 1$. Hence $\pi_s(G) \le \frac{k}{2} + 1$.

Thus in either of the above subcases, we have $\pi_s(G) \le \frac{k}{2} + 1$. Also, for any simple path cover ψ of G all the k vertices of odd degree and at least one vertex on C are terminal vertices of paths in ψ. Hence $t \le q - \frac{k}{2} - 1$, so that $\pi_s(G) = q - t \ge \frac{k}{2} + 1$.

Hence $\pi_s(G) = \frac{k}{2} + 1$.

Case 4. $m \ge 3$.

Let $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$, where $1 \le i_1 < i_2 < \cdots < i_s \le r$ and $s \ge 3$, be the vertices of degree greater than 2 on C. Let ψ_{i_j}, $1 \le j \le s$, be a minimum simple path cover of the tree rooted at v_{i_j}. Consider the vertices v_{i_1}, v_{i_2} and v_{i_3}. For each j, where $1 \le j \le 3$, let P_j be the path in ψ_{i_j} in which v_{i_j} is a terminal vertex if $deg\ v_{i_j}$ is odd, otherwise let P_j be an u_j-w_j path in ψ_{i_j} in which v_{i_j} is an internal vertex and R_j and S_j be the (u_j, v_{i_j}) and (w_j, v_{i_j})-sections of P_j respectively. Further, let $P = (v_{i_1}, v_{i_1+1}, \ldots, v_{i_2})$, $Q = (v_{i_2}, v_{i_2+1}, \ldots, v_{i_3})$ and $R = (v_{i_3}, v_{i_3+1}, \ldots, v_{i_1})$.

If $deg\ v_{i_1}, deg\ v_{i_2}$ and $deg\ v_{i_3}$ are even, let $Q_1 = R_1 \circ P \circ R_2^{-1}$, $Q_2 = S_2 \circ Q \circ R_3^{-1}$ and $Q_3 = S_3 \circ R \circ S_1^{-1}$.

If $deg\ v_{i_1}, deg\ v_{i_2}$ and $deg\ v_{i_3}$ are odd, let $Q_1 = P_1 \circ P$, $Q_2 = P_2 \circ Q$ and $Q_3 = P_3 \circ R$.

If $deg\ v_{i_1}, deg\ v_{i_2}$ are odd and $deg\ v_{i_3}$ is even, let $Q_1 = P_1 \circ P \circ P_2^{-1}$, $Q_2 = R_3 \circ Q^{-1}$ and $Q_3 = S_3 \circ R$.

If $deg\ v_{i_1}, deg\ v_{i_2}$ are even and $deg\ v_{i_3}$ is odd, let $Q_1 = R_1 \circ P \circ R_2^{-1}$, $Q_2 = S_2 \circ Q \circ P_3^{-1}$ and $Q_3 = R \circ S_1^{-1}$.

Then $\psi = (\bigcup_{j=1}^{s} \psi_{i_j} - \{P_1, P_2, P_3\}) \cup \{Q_1, Q_2, Q_3\}$ is a simple path cover of G such that every vertex of odd degree is an external vertex of exactly one path in ψ and no vertex of even degree is an external vertex of any path in ψ. Hence $\pi_s(G) = \frac{k}{2}$. □

Corollary 2.10 *Let G be as in Theorem 2.9. Then $\pi_s(G) = \pi(G)$ if and only if $m \ge 3$.*

Proof The proof follows from Theorem 2.9 and Theorem 1.4. □

We observe that there are infinite families of graphs such as trees and unicyclic graphs having at least three vertices of degree greater than 2 on C for which $\pi_s = \pi$ and so the following problem arises naturally.

Problem 2.11 *Characterize graphs for which $\pi_s = \pi$.*

Theorem 2.12 *For a wheel $W_n = K_1 + C_{n-1}$, we have*

$$\pi_s(W_n) = \begin{cases} 6 & \text{if } n = 4 \\ \left\lfloor \frac{n}{2} \right\rfloor + 3 & \text{if } n \ge 5 \end{cases}$$

Proof Let $V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0 v_i : 1 \le i \le n-1\} \cup \{v_i v_{i+1} : 1 \le i \le n-2\} \cup \{v_1 v_{n-1}\}$.

If $n = 4$, then $W_n = K_4$ and hence $\pi_s(W_n) = 6$.

Now, suppose $n \geq 5$. Let $r = \left\lfloor \frac{n}{2} \right\rfloor$

If n is odd, let

$P_i = (v_i, v_0, v_{r+i}),\ i = 1, 2, \ldots, r.$

$P_{r+1} = (v_1, v_2, \ldots, v_r),$

$P_{r+2} = (v_1, v_{2r}, v_{2r-1}, \ldots, v_{r+2})$ and

$P_{r+3} = (v_r, v_{r+1}, v_{r+2}).$

If n is even, let

$P_i = (v_i, v_0, v_{r-1+i}),\ i = 1, 2, \ldots, r-1$.

$P_r = (v_0, v_{2r-1}),$

$P_{r+1} = (v_1, v_2, \ldots, v_{r-1}),$

$P_{r+2} = (v_1, v_{2r-1}, \ldots, v_{r+1})$ and

$P_{r+3} = (v_{r-1}, v_r, v_{r+1}).$

Then $\psi = \{P_1, P_2, \ldots, P_{r+3}\}$ is a simple path cover of W_n. Hence $\pi_s(W_n) \leq r+3 = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Further, for any simple path cover ψ of W_n at least three vertices on $C = (v_1, v_2, \ldots, v_{n-1})$ are terminal vertices of paths in ψ. Hence $t \leq q - \frac{k}{2} - 3$, so that $\pi_s(W_n) = q - t \geq \frac{k}{2} + 3 = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Thus $\pi_s(W_n) = \left\lfloor \frac{n}{2} \right\rfloor + 3$. □

Remark 2.13 Since every simple acyclic graphoidal cover of a graph G is a simple path cover of G and every simple path cover of G is a path cover of G, we have $\eta_{as} \geq \pi_s \geq \pi$. These parameters may be either equal or all distinct as shown below. For the graph G_1 given in Figure 2, $\eta_{as}(G_1) = 7, \pi_s(G_1) = 6, \pi(G_1) = 5$ and for the graph G_2 given in Fig.2.2, we have $\eta_{as}(G_2) = \pi_s(G_2) = \pi(G_2) = 3$.

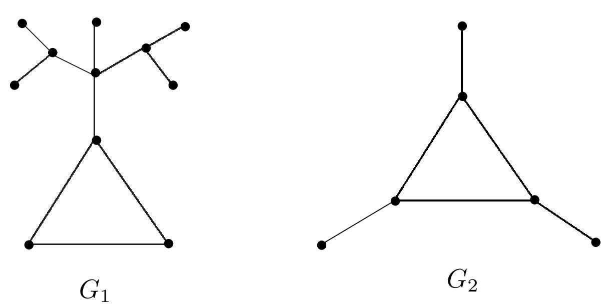

Fig.2.2

Problem 2.14 *Characterize graphs for which $\eta_{as} = \pi_s = \pi$.*

We now proceed to obtain some bounds for π_s.

Theorem 2.15 *For any graph G, $\pi_s(G) \geq \left\lceil \frac{\Delta}{2} \right\rceil$. Further, the following are equivalent.*

(i) $\pi_s(G) = \left\lceil \frac{\Delta}{2} \right\rceil$.

(ii) $\eta_{as}(G) = \Delta - 1$.

(iii) G is homeomorphic to a star.

Proof Since $\pi_s \geq \pi$, the inequality follows from Theorem 1.5.

Suppose $\pi_s(G) = \lceil \frac{\Delta}{2} \rceil$. Let $\psi = \{P_1, P_2, \ldots, P_r\}$, where $r = \lceil \frac{\Delta}{2} \rceil$ be a minimum simple path cover of G. Let v be a vertex of G with $deg\ v = \Delta$. Then v lies on each P_i and v is an internal vertex of all the paths in ψ except possibly for at most one path. Hence $V(P_i) \cap V(P_j) = \{v\}$, for all $i \neq j$, so that G is homeomorphic to a star. Obviously, if G is homeomorphic to a star, then $\pi_s(G) = \lceil \frac{\Delta}{2} \rceil$. Thus (i) and (iii) are equivalent. Similarly the equivalence of (ii) and (iii) can be proved. □

Theorem 2.16 *For any graph G, $\pi_s(G) \geq \binom{\omega}{2}$, where ω is the clique number of G.*

Proof Let H be a maximum clique in G so that $|E(H)| = \binom{\omega}{2}$. Let ψ be a simple path cover of G. Since any path in ψ covers at most one edge of H, it follows that $\pi_s(G) \geq \binom{\omega}{2}$. □

In the following theorem we characterize cubic graphs for which $\pi_s = \binom{\omega}{2}$.

Theorem 2.17 *Let G be a cubic graph. Then $\pi_s(G) = \binom{\omega}{2}$ if and only if $G = K_4$.*

Proof Let G be a cubic graph with $\pi_s(G) = \binom{\omega}{2}$. Clearly $\omega = 3$ or 4. Suppose $\omega = 3$. Then it follows from Corollary 2.6 that $\pi_s(G) \geq \frac{p}{2}$ so that $p = 6$. Hence G is isomorphic to the cartesian product $K_3 \times K_2$ and it can be shown that $\pi_s(K_3 \times K_2) = 6 \neq \binom{\omega}{2}$. Thus $\omega = 4$ and consequently $G = K_4$. □

Problem 2.18 *Characterize graphs for which $\pi_s(G) = \binom{\omega}{2}$.*

If $\Delta \leq 3$, then every simple path cover of G is a simple acyclic graphoidal cover of G and hence $\eta_{as}(G) = \pi_s(G)$. However, the converse is not true. For the complete graph $K_p (p \geq 5)$, $\pi_s = \eta_{as}$ whereas $\Delta \geq 4$. We now prove that the converse is true for trees and unicyclic graphs.

Theorem 2.19 *Let G be a tree. Then $\eta_{as}(G) = \pi_s(G)$ if and only if $\Delta \leq 3$.*

Proof Let G be a tree with $\eta_{as}(G) = \pi_s(G)$.

Suppose $\Delta \geq 4$. Let v be a vertex of G with $deg\ v \geq 4$.

Let ψ be a minimum simple acyclic graphoidal cover of G. Let P_1 and P_2 be two paths in ψ having v as a terminal vertex. Let $Q = P_1 \circ P_2^{-1}$. Since G is a tree, Q is an induced path and hence $\psi_1 = (\psi - \{P_1, P_2\}) \cup \{Q\}$ is a simple path cover of G with $|\psi_1| = |\psi| - 1 = \eta_{as} - 1$ so that $\pi_s(G) \leq \eta_{as}(G) - 1$, which is a contradiction. Hence $\Delta \leq 3$. □

Theorem 2.20 *Let G be a unicyclic graph. Then $\eta_{as}(G) = \pi_s(G)$ if and only if $\Delta \leq 3$.*

Proof Let G be a unicyclic graph with $\eta_{as}(G) = \pi_s(G)$. Let k denote the number of vertices of odd degree and n be the number of pendant vertices of G.

It follows from Theorem 1.10 and Theorem 2.9 that $k = 2n$. Now, suppose $\Delta > 3$. Then

$$\begin{aligned} 2q &= \sum_{\substack{v \in V(G) \\ deg\ v = 1}} deg\ v + \sum_{\substack{v \in V(G) \\ deg\ v > 1 \\ \text{and is odd}}} deg\ v + \sum_{\substack{v \in V(G) \\ deg\ v > 1 \\ \text{and is even}}} deg\ v \\ &> n + 3(k - n) + 2(p - k) \\ &= 2p, \end{aligned}$$

which is a contradiction. Hence $\Delta \leq 3$. $\square$

The above results lead to the following problem.

Problem 2.21 *Characterize graphs for which $\eta_{as}(G) = \pi_s(G)$.*

In the following theorem we establish a relation connecting the parameters η_{as} and π_s.

Theorem 2.22 *For any (p,q)-graph $G, \eta_{as}(G) \leq \pi_s(G)+q-p+n-\frac{k}{2}$, where n and k respectively denote the number of pendant vertices and the number of odd vertices of G. Further, equality holds if and only if $\pi_s(G) = \frac{k}{2}$.*

Proof Let ψ be a minimum simple path cover of G. Let $i(v)$ denote the number of paths in ψ having $v \in V$ as an internal vertex. If $i(v) \geq 2$, let P_i, where $1 \leq i \leq i(v)$, be the u_i-w_i path in ψ having v as an internal vertex and let Q_i and R_i, where $2 \leq i \leq i(v)$, respectively denote the (v, w_i)-section and (v, u_i)-section of P_i. Let ψ_1 be the collection of paths obtained from ψ by replacing $P_2, P_3, \ldots, P_{i(v)}$ by $Q_2, Q_3, \ldots, Q_{i(v)}$ and $R_2, R_3, \ldots, R_{i(v)}$ for every $v \in V$ with $i(v) \geq 2$. Then ψ_1 is a simple acyclic graphoidal cover of G with $|\psi_1| = \pi_s(G) + \sum\limits_{\substack{v\in V\\ i(v)\geq 2}} (i(v)-1)$.

Since $i(v) \leq \left\lfloor \frac{deg\ v}{2} \right\rfloor$, it follows that

$$\begin{aligned}
\eta_{as}(G) &\leq \pi_s(G) \ + \sum_{\substack{v\in V\\ deg\ v\geq 4}} \left(\left\lfloor \frac{deg\ v}{2}\right\rfloor - 1\right)\\
&= \pi_s(G) \ + \sum_{\substack{v\in V\\ deg\ v\geq 2}} \left(\left\lfloor \frac{deg\ v}{2}\right\rfloor - 1\right)\\
&= \pi_s(G) \ + \sum_{\substack{v\in V\\ deg\ v\geq 2}} \left\lfloor \frac{deg\ v}{2}\right\rfloor - (p-n)\\
&= \pi_s(G) \ + \sum_{\substack{v\in V\\ deg\ v\geq 2\\ \text{and is odd}}} \frac{deg\ v-1}{2} \ + \sum_{\substack{v\in V\\ deg\ v\geq 2\\ \text{and is even}}} \frac{deg\ v}{2} - p + n\\
&= \pi_s(G) \ + \sum_{\substack{v\in V\\ deg\ v\geq 2\\ \text{and is odd}}} \frac{deg\ v}{2} - \frac{k-n}{2} \ + \sum_{\substack{v\in V\\ deg\ v\geq 2\\ \text{and is even}}} \frac{deg\ v}{2} - p + n\\
&= \pi_s(G) \ + \sum_{\substack{v\in V\\ deg\ v\geq 2}} \frac{deg\ v}{2} - \frac{k}{2} + \frac{n}{2} - p + n\\
&= \pi_s(G) \ + \sum_{v\in V} \frac{deg\ v}{2} - \frac{k}{2} - p + n.\\
&= \pi_s(G) + q - p + n - \frac{k}{2}.
\end{aligned}$$

Thus $\eta_{as}(G) \leq \pi_s(G)+q-p+n-\frac{k}{2}$. Further, it is clear that $\eta_{as}(G) = \pi_s(G)+q-p+n-\frac{k}{2}$ if and only if there exist a minimum simple path cover ψ of G such that $i(v) = \left\lfloor \frac{deg\ v}{2}\right\rfloor$ for all $v \in V$. Hence it follows from Corollary 2.6 that $\eta_{as}(G) = \pi_s(G) + q - p + n - \frac{k}{2}$ if and only if $\pi_s = \frac{k}{2}$. $\square$

Corollary 2.23 *If $\pi_s(G) = \frac{k}{2}$, then $\eta_{as}(G) = q - p + n$.*

Proof Suppose $\pi_s(G) = \frac{k}{2}$. By Theorem 2.22, we have $\eta_{as}(G) \leq q-p+n$. Hence it follows from Theorem 1.9 that $\eta_{as}(G) = q-p+n$. □

Remark 2.24 The converse of Corollary 2.23 is not true. For example, $\eta_{as}(K_{4,5}) = q-p = 11$, whereas $\pi_s(K_{4,5}) > 2 = \frac{k}{2}$.

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