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Extending Homomorphism Theorem to Multi-Systems

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Abstract: The multi-laterality of WORLD implies multi-systems to be its best candidate model for ones cognition on nature, which is also included in an ancient book of China, *TAO TEH KING* written by Lao Zi, an ancient philosopher of China. Then *how it works to mathematics, not suspended in thought*? This paper explains this action by mathematical logic on mathematical systems generalized to Smarandache systems, or such systems with combinatorial structures, i.e., combinatorial systems, and shows how to extend the homomorphism theorem in abstract algebra to multi-systems or combinatorial systems. All works in this paper are motivated by a combinatorial speculation of mine which is reformed on combinatorial systems and can be also applied to geometry.

Key Words: Homomorphism theorem, multi-system, combinatorial system.

AMS(2000): 05E15, 08A02, 15A03, 20E07, 51M15.

§1. Introduction

The WORLD is a multi-lateral one. The entirely realization on WORLD is very difficult for the limitation of mankind on the earth. In Fig.1.1, it is shown part of the WORLD by eyes of a man on the earth.

Fig.1.1

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[2]Reported at the *4th International Conference on Number Theory and Smarandache Problems*, March 22-24, 2008, Shanxi Xiangyang, P.R.China.

The multi-laterality of the WORLD implies multi-systems to be its best candidate model for ones cognition on nature. This is also included in a well-known Chinese ancient book *TAO TEH KING* written by *LAO ZI*. In this book, only with nearly 5000 words, we can find many sentences for cognition of our world, such as those of the following (see [5] for details), each of them is explained by a concrete figure.

SENTENCE 1. *All things that we can acknowledge is determined by our eyes, or ears, or nose, or tongue, or body or passions, i.e., these six organs.* Such as those shown in Fig.1.2.

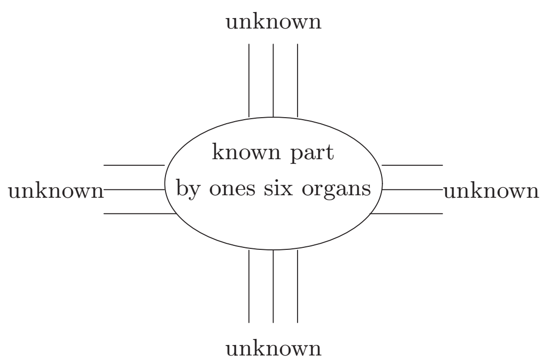

Fig.1.2

SENTENCE 2. *The Tao gives birth to One. One gives birth to Two. Two gives birth to Three. Three gives birth to all things. All things have their backs to the female and stand facing the male. When male and female combine, all things achieve harmony.* Shown in Fig.1.3.

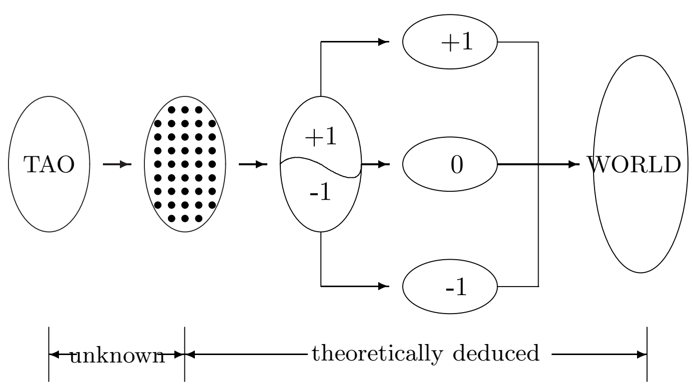

Fig.1.3

SENTENCE 3. *Mankind follows the earth. Earth follows the universe. The universe follows the Tao. The Tao follows only itself.* Such as those shown in Fig.1.4

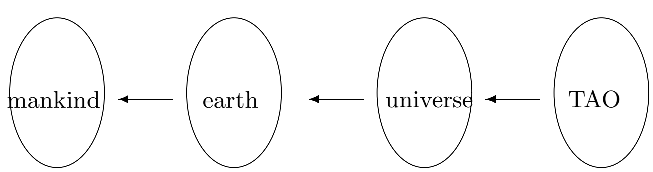

Fig.1.4

SENTENCE 4. Have *and* Not have *exist jointly ahead of the birth of the earth and the sky.* This means that any thing have two sides. One is the positive. Another is the negative. We can not say a thing existing or not just by our six organs because its existence independent on our living.

What can we learn from these words? How can we apply them in mathematics of the 21st century? All these sentences mean that our world is a multi-one. For characterizing its behavior, We should construct a multi-system model for the WORLD, also called parallel universes ([23]), such as those shown in Fig.1.5.

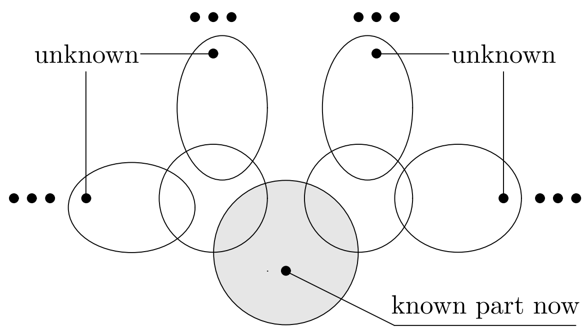

Fig.1.5

Whence, it is a Smarandache multi-system with a combinatorial structure G, i.e., a combinatorial system $\mathscr{C}_G$.

In this paper, we will characterize such systems, and generalize the well-known homomorphism theorem in group theory to multi-systems, particularly, to multi-groups, multi-rings and multi-modules (see [11] – [14] for details). In the remain part of this section, we recall some terminologies in mathematical logic and define what is a mathematical system. These Smarandache systems and combinatorial systems are introduced in Section 2. After that, we show how to generalize the homomorphism theorem of groups to multi-systems in the following sections. Terminologies and notations not defined here follow the reference [1], [18], [24] for algebra, [2], [3] and [7] – [9] for graphs.

A *proposition* p on a set Σ is a declarative sentence on elements in Σ that is either true or false but not both. The statements *it is not the case that* p and *it is the opposite case that* p are still propositions, called the *negation* or *anti-proposition* of p, denoted by non-p or anti-p, respectively. Generally, non $-\ p \neq$ anti $-\ p$. The structure of anti-p is very clear, but non-p is not. An oppositive or negation of a proposition are shown in Fig.1.6(1) and (2).

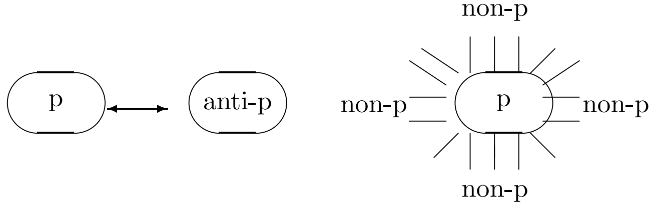

Fig.1.6

A proposition and its non-proposition jointly exist in the world. Its truth or false can be only decided by logic inference, independent on one knowing it or not.

A norm inference is called implication. An *implication* $p \to q$, *i.e., if* p *then* q, is a proposition that is false when p is true but q false and true otherwise. There are three propositions related with $p \to q$, namely, $q \to p$, $\neg q \to \neg p$ and $\neg p \to \neg q$, called the *converse*, *contrapositive* and *inverse* of $p \to q$. Two propositions are called *equivalent* if they have the same truth value. It can be shown immediately that *an implication and its contrapositive are equivalent*. This fact is commonly used in mathematical proofs, i.e., we can either prove the proposition $p \to q$ or $\neg q \to \neg p$ in the proof of $p \to q$, not the both.

A *rule* on a set Σ is a mapping

$$\underbrace{\Sigma \times \Sigma \cdots \times \Sigma}_{n} \to \Sigma$$

for some integers n. A *mathematical system* is a pair $(\Sigma; \mathcal{R})$, where Σ is a set consisting mathematical objects, infinite or finite and $\mathcal{R}$ is a collection of rules on Σ by logic providing all these resultants are still in Σ, i.e., elements in Σ is closed under rules in $\mathcal{R}$.

Two mathematical systems $(\Sigma_1; \mathcal{R}_1)$ and $(\Sigma_2; \mathcal{R}_2)$ are *isomorphic* if there is a $1-1$ mapping $\omega : \Sigma_1 \to \Sigma_2$ such that for elements $a, b, \cdots, c \in \Sigma_1$,

$$\omega(\mathcal{R}_1(a, b, \cdots, c)) = \mathcal{R}_2(\omega(a), \omega(b), \cdots, \omega(c)) \in \Sigma_2.$$

Generally, we do not distinguish isomorphic systems in mathematics. Examples for mathematical systems are shown in the following.

Example 1.1 A *group* $(G; \circ)$ in classical algebra is a mathematical system $(\Sigma_G; \mathcal{R}_G)$, where $\Sigma_G = G$ and

$$\mathcal{R}_G = \{R_1^G; R_2^G, R_3^G\},$$

with

R_1^G: $(x \circ y) \circ z = x \circ (y \circ z)$ for $\forall x, y, z \in G$;

R_2^G: there is an element $1_G \in G$ such that $x \circ 1_G = x$ for $\forall x \in G$;

R_3^G: for $\forall x \in G$, there is an element $y, y \in G$, such that $x \circ y = 1_G$.

Then, the well-known homomorphism theorem of groups is restated in the next.

Homomorphism Theorem *Let* $\sigma : G \to G'$ *be a homomorphism from groups* G *to* G'. *Then*

$$G/\mathrm{Ker}\sigma \cong G'.$$

□

Example 1.2 A ring $(R; +, \circ)$ with two binary closed operations $+$, $\circ$is a mathematical system $(\Sigma; \mathcal{R})$, where $\Sigma = R$ and $\mathcal{R} = \{R_1; R_2, R_3, R_4\}$ with

R_1: $x + y, x \circ y \in R$ for $\forall x, y \in R$;

R_2: $(R; +)$ is a commutative group, i.e., $x + y = y + x$ for $\forall x, y \in R$;

R_3: $(R; \circ)$ is a semigroup;

R_4: $x \circ (y+z) = x \circ y + x \circ z$ and $(x+y) \circ z = x \circ z + y \circ z$ for $\forall x, y, z \in R$.

Example 1.3 An Euclidean geometry on the plane $\mathbf{R}^2$ is a a mathematical system $(\Sigma_E; \mathcal{R}_E)$, where $\Sigma_E = \{\text{points and lines on } \mathbf{R}^2\}$ and $\mathcal{R}_E = \{\text{Hilbert's 21 axioms on Euclidean geometry}\}$.

A mathematical $(\Sigma; \mathcal{R})$ can be constructed dependent on the set Σ or on rules $\mathcal{R}$. The former requires each rule in $\mathcal{R}$ closed in Σ. But the later requires that $\mathcal{R}(a, b, \cdots, c)$ in the final set $\widehat{\Sigma}$, which means that $\widehat{\Sigma}$ maybe an extended of the set Σ. In this case, we say $\widehat{\Sigma}$ is generated by Σ under rules $\mathcal{R}$, denoted by $\langle \Sigma; \mathcal{R} \rangle$.

§2. Combinatorial System

By the view of *LAO ZHI* in Section 1, we should construct such mathematical systems $(\Sigma; \mathcal{R})$ for the WORLD in which a proposition with its non-proposition validated turn up in the set Σ, or invalidated but in multiple ways in Σ, which is a Smarandache system defined in the next.

Definition 2.1 *A rule in a mathematical system* $(\Sigma; \mathcal{R})$ *is said to be Smarandachely denied if it behaves in at least two different ways within the same set* Σ*, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ *is a mathematical system which has at least one Smarandachely denied rule in* $\mathcal{R}$*.*

Definition 2.2 *For an integer* $m \geq 2$*, let* $(\Sigma_1; \mathcal{R}_1)$*,* $(\Sigma_2; \mathcal{R}_2)$*,* $\cdots$*,* $(\Sigma_m; \mathcal{R}_m)$ *be* m *mathematical systems different two by two. A Smarandache multi-space is a pair* $(\widetilde{\Sigma}; \widetilde{\mathcal{R}})$ *with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad \textit{and} \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

Certainly, we can construct Smarandache systems by applying Smarandache multi-spaces, particularly, Smarandache geometries ([4], [7]-[17]).

These Smarandache systems $(\Sigma; \mathcal{R})$ defined in Definition 2.1 consider the behavior of a proposition and its non-proposition in the same set Σ without distinguishing the guises of these non-propositions. In fact, there are many appearing ways for non-propositions of a proposition in Σ. For describing their behavior, the combinatorial systems are needed.

Definition 2.3 *A combinatorial system* $\mathscr{C}_G$ *is a union of mathematical systems* $(\Sigma_1; \mathcal{R}_1)$,$(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ *for an integer* m*, i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i; \bigcup_{i=1}^{m} \mathcal{R}_i)$$

with an underlying connected graph structure G*, where*

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

Unless its combinatorial G structure, these cardinalities $|\Sigma_i \bigcap \Sigma_j|$, called the *coupling constants* in a combinatorial system $\mathscr{C}_G$ also determine its structure if $\Sigma_i \bigcap \Sigma_j \neq \emptyset$ for integers $1 \leq i, j \leq m$. For emphasizing its coupling constants, we denote a combinatorial system $\mathscr{C}_G$ by $\mathscr{C}_G(l_{ij}, 1 \leq i, j \leq m)$ if $l_{ij} = |\Sigma_i \bigcap \Sigma_j| \neq 0$.

Let $\mathscr{C}_G^{(1)}$ and $\mathscr{C}_G^{(2)}$ be two combinatorial systems with

$$\mathscr{C}_G^{(1)} = (\bigcup_{i=1}^{m} \Sigma_i^{(1)};\ \bigcup_{i=1}^{m} \mathcal{R}_i^{(1)}), \quad \mathscr{C}_G^{(2)} = (\bigcup_{i=1}^{n} \Sigma_i^{(2)};\ \bigcup_{i=1}^{n} \mathcal{R}_i^{(2)}).$$

A *homomorphism* $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ is a mapping $\varpi : \bigcup\limits_{i=1}^{m} \Sigma_i^{(1)} \to \bigcup\limits_{i=1}^{n} \Sigma_i^{(2)}$ and $\varpi : \bigcup\limits_{i=1}^{m} \mathcal{R}_i^{(1)}) \to \bigcup\limits_{i=1}^{n} \mathcal{R}_i^{(2)}$ such that

$$\varpi|_{\Sigma_i}(a\mathcal{R}_i^{(1)}b) = \varpi|_{\Sigma_i}(a)\varpi|_{\Sigma_i}(\mathcal{R}_i^{(1)})\varpi|_{\Sigma_i}(b)$$

for $\forall a, b \in \Sigma_i^{(1)}$, $1 \leq i \leq m$, where $\varpi|_{\Sigma_i}$ denotes the constraint mapping of ϖ on the mathematical system $(\Sigma_i, \mathcal{R}_i)$. Further more, if $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ is a $1-1$ mapping, then we say these $\mathscr{C}_G^{(1)}$ and $\mathscr{C}_G^{(2)}$ are *isomorphic* with an isomorphism ϖ between them.

A homomorphism $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ naturally induces a mappings $\varpi|_G$ on the graph G_1 and G_2 by

$$\varpi|_G : V(G_1) \to \varpi(V(G_1)) \subset V(G_2) \text{ and}$$

$$\varpi|_G : (\Sigma_i, \Sigma_j) \in E(G_1) \to (\varpi(\Sigma_i), \varpi(\Sigma_j)) \in E(G_2), 1 \leq i, j \leq m.$$

With these notations, a criterion for isomorphic combinatorial systems is presented in the following.

Theorem 2.1 *Two combinatorial systems $\mathscr{C}_G^{(1)}$ and $\mathscr{C}_G^{(2)}$ are isomorphic if and only if there is a $1-1$ mapping $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ such that*

(i) $\varpi|_{\Sigma_i^{(1)}}$ is an isomorphism and $\varpi|_{\Sigma_i^{(1)}}(x) = \varpi|_{\Sigma_j^{(1)}}(x)$ for $\forall x \in \Sigma_i^{(1)} \cap \Sigma_j^{(1)}$, $1 \leq i, j \leq m$;

(ii) $\varpi|_G : G_1 \to G_2$ is an isomorphism.

Proof If $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ is an isomorphism, considering the constraint mappings of ϖ on the mathematical system $(\Sigma_i, \mathcal{R}_i)$ for an integer $i, 1 \leq i \leq m$ and the graph $G_1^{(1)}$, then we find isomorphisms $\varpi|_{\Sigma_i^{(1)}}$ and $\varpi|_G$.

Conversely, if these isomorphism $\varpi|_{\Sigma_i^{(1)}}, 1 \leq i \leq m$ and $\varpi|_G$ exist, we can construct a mapping $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ by

$$\varpi(a) = \varpi|_{\Sigma_1}(a) \text{ if } a \in \Sigma_i \text{ and } \varpi(\circ) = \varpi|_{\Sigma_1}(\circ) \text{ if } \circ \in \mathcal{R}_i, 1 \leq i \leq m.$$

Then we know that

$$\varpi|_{\Sigma_i}(a\mathcal{R}_i^{(1)}b) = \varpi|_{\Sigma_i}(a)\varpi|_{\Sigma_i}(\mathcal{R}_i^{(1)})\varpi|_{\Sigma_i}(b)$$

for $\forall a, b \in \Sigma_i^{(1)}$, $1 \leq i \leq m$ by definition. Whence, $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ is a homomorphism. Similarly, we can know that $\varpi^{-1} : \mathscr{C}_G^{(2)} \to \mathscr{C}_G^{(1)}$ is also an homomorphism. Therefore, ϖ is an isomorphism between $\mathscr{C}_G^{(1)}$ and $\mathscr{C}_G^{(2)}$. □

For understanding well the multiple behavior of the WORLD, namely, its overlap and hybrid, a combinatorial system should be constructed. Then *what is its relation with classical mathematical sciences*? *What is its developing way for mathematical sciences*? I have presented an idea of combinatorial notion in Chapter 5 of [7], then stated formally as the *combinatorial conjecture for mathematics* in [10] and [16], the last was reported at *the 2nd Conference on Combinatorics and Graph Theory of China* in 2006.

Combinatorial Conjecture *Any mathematical system* $(\Sigma; \mathcal{R})$ *is a combinatorial system* $\mathscr{C}_G(l_{ij}, 1 \leq i, j \leq m)$.

This conjecture is not just like an open problem, but more like a deeply thought, which opens a entirely way for advancing the modern mathematics. Here, we need further clarification for this conjecture. In fact, it indeed means combinatorial notions following for researchers.

(1) There is a combinatorial structure and finite rules for a classical mathematical system, which means one can make combinatorialization for all classical mathematical subjects.

(2) One can generalizes a classical mathematical system by this combinatorial notion such that it is a particular case in this generalization.

(3) One can make one combination of different branches in mathematics and find new results after then.

Whence, a mathematical system can not be ended if it has not been combinatorialization and all mathematical systems can not be ended if its combinatorialization has not completed yet. The reader can applies this combinatorial notion to all of his research work, and then finds his combinatorial fields.

§3. Algebraic Systems

Let $(\mathscr{A}; \circ)$ be an algebraic system and $\mathscr{B} \subset \mathscr{A}$, if $(\mathscr{B}; \circ)$ is still an algebraic system, then we call it an *algebraic sub-system of* $(\mathscr{A}; \circ)$, denoted by $\mathscr{B} \prec \mathscr{A}$. Similarly, an algebraic sub-system is called a *subgroup* if it is group itself.

Let $(\mathscr{A}; \circ)$ be an algebraic system and $\mathscr{B} \prec \mathscr{A}$. For $\forall a \in \mathscr{A}$, define a coset $a \circ \mathscr{B}$ of $\mathscr{B}$ in $\mathscr{A}$ by

$$a \circ \mathscr{B} = \{a \circ b | \forall b \in \mathscr{B}\}.$$

Define a *quotient set* $\mathfrak{S} = \mathscr{A}/\mathscr{B}$ consists of all cosets of $\mathscr{B}$ in $\mathscr{A}$ and let R be a minimal set with $\mathfrak{S} = \{r \circ \mathscr{B} | r \in R\}$, called a *representation of* $\mathfrak{S}$. Then

Theorem 3.1 *If* $(\mathscr{B}; \circ)$ *is a subgroup of an associative system* $(\mathscr{A}; \circ)$, *then*

(*i*) *for* $\forall a, b \in \mathscr{A}$, $(a \circ \mathscr{B}) \cap (b \circ \mathscr{B}) = \emptyset$ *or* $a \circ \mathscr{B} = b \circ \mathscr{B}$, *i.e.,* $\mathfrak{S}$ *is a partition of* $\mathscr{A}$;

(*ii*) *define an operation* $\bullet$ *on* $\mathfrak{S}$ *by*

$$(a\circ\mathscr{B})\bullet(b\circ\mathscr{B})=(a\circ b)\circ\mathscr{B},$$

then $(\mathfrak{S};\bullet)$ *is an associative algebraic system, called a quotient system of* $\mathscr{A}$ *to* $\mathscr{B}$*. Particularly, if there is a representation* R *whose each element has an inverse in* $(\mathscr{A};\circ)$ *with unit* $1_{\mathscr{A}}$*, then* $(\mathfrak{S};\bullet)$ *is a group, called a quotient group of* $\mathscr{A}$ *to* $\mathscr{B}$*.*

Proof For (i), notice that if

$$(a\circ\mathscr{B})\cap(b\circ\mathscr{B})\neq\emptyset$$

for $a,b\in\mathscr{A}$, then there are elements $c_1,c_2\in\mathscr{B}$ such that $a\circ c_1=b\circ c_2$. By assumption, $(\mathscr{B};\circ)$ is a subgroup of $(\mathscr{A};\circ)$, we know that there exists an inverse element $c_1^{-1}\in\mathscr{B}$, i.e., $a=b\circ c_2\circ c_1^{-1}$. Therefore, we get that

$$\begin{aligned}a\circ\mathscr{B} &= (b\circ c_2\circ c_1^{-1})\circ\mathscr{B}\\ &= \{(b\circ c_2\circ c_1^{-1})\circ c|\forall c\in\mathscr{B}\}\\ &= \{b\circ c|\forall c\in\mathscr{B}\}\\ &= b\circ\mathscr{B}\end{aligned}$$

by the associative law and $(\mathscr{B};\circ)$ is a group gain, i.e., $(a\circ\mathscr{B})\cap(b\circ\mathscr{B})=\emptyset$ or $a\circ\mathscr{B}=b\circ\mathscr{B}$.

By definition of $\bullet$ on $\mathfrak{S}$ and (i), we know that $(\mathfrak{S};\bullet)$ is an algebraic system. For $\forall a,b,c\in\mathscr{A}$, by the associative laws in $(\mathscr{A};\circ)$, we find that

$$\begin{aligned}((a\circ\mathscr{B})\bullet(b\circ\mathscr{B}))\bullet(c\circ\mathscr{B}) &= ((a\circ b)\circ\mathscr{B})\bullet(c\circ\mathscr{B})\\ &= ((a\circ b)\circ c)\circ\mathscr{B}=(a\circ(b\circ c))\circ\mathscr{B}\\ &= (a\circ\mathscr{B})\circ((b\circ c)\circ\mathscr{B})\\ &= (a\circ\mathscr{B})\bullet((b\circ\mathscr{B})\bullet(c\circ\mathscr{B})).\end{aligned}$$

Now if there is a representation R whose each element has an inverse in $(\mathscr{A};\circ)$ with unit $1_{\mathscr{A}}$, then it is easy to know that $1_{\mathscr{A}}\circ\mathscr{B}$ is the unit and $a^{-1}\circ\mathscr{B}$ the inverse element of $a\circ\mathscr{B}$ in $\mathfrak{S}$. Whence, $(\mathfrak{S};\bullet)$ is a group. □

Corollary 3.1 *For a subgroup* $(\mathscr{B};\circ)$ *of a group* $(\mathscr{A};\circ)$*,* $(\mathfrak{S};\bullet)$ *is a group.*

Corollary 3.2(Lagrange theorem) *For a subgroup* $(\mathscr{B};\circ)$ *of a group* $(\mathscr{A};\circ)$*,*

$$|\mathscr{B}|\mid|\mathscr{A}|.$$

Proof Since $a\circ c_1=a\circ c_2$ implies that $c_1=c_2$ in this case, we know that

$$|a\circ\mathscr{B}|=|\mathscr{B}|$$

for $\forall a\in\mathscr{A}$. Applying Theorem 2.2.4(i), we find that

$$|\mathscr{A}| = \sum_{r\in R} |r \circ \mathscr{B}| = |R||\mathscr{B}|,$$

for a representation R, i.e., $|\mathscr{B}| \mid |\mathscr{A}|$. □

Although the operation $\bullet$ in $\mathfrak{S}$ is introduced by the operation $\circ$ in $\mathscr{A}$, it may be $\bullet \neq \circ$. Now if $\bullet = \circ$, i.e.,

$$(a \circ \mathscr{B}) \circ (b \circ \mathscr{B}) = (a \circ b) \circ \mathscr{B}, \qquad (3-1)$$

the subgroup $(\mathscr{B}; \circ)$ is called a *normal subgroup of* $(\mathscr{B}; \circ)$, denoted by $\mathscr{B} \trianglelefteq \mathscr{A}$. In this case, if there exist inverses of a, b, we know that

$$\mathscr{B} \circ b \circ \mathscr{B} = b \circ \mathscr{B}$$

by product a^{-1} from the left on both side of $(3-1)$. Now since $(\mathscr{B}; \circ)$ is a subgroup, we get that

$$b^{-1} \circ \mathscr{B} \circ b = \mathscr{B},$$

which is the usually definition for a normal subgroup of a group. Certainly, we can also get

$$b \circ \mathscr{B} = \mathscr{B} \circ b$$

by this equality, which can be used to define a *normal subgroup of a algebraic system* with or without inverse element for an element in this system.

Now let $\varpi : \mathscr{A}_1 \to \mathscr{A}_2$ be a homomorphism from an algebraic system $(\mathscr{A}_1; \circ_1)$ with unit $1_{\mathscr{A}_1}$ to $(\mathscr{A}_2; \circ_2)$ with unit $1_{\mathscr{A}_2}$. Define the *inverse set* $\varpi^{-1}(a_2)$ *for an element* $a_2 \in \mathscr{A}_2$ by

$$\varpi^{-1}(a_2) = \{a_1 \in \mathscr{A}_1 | \varpi(a_1) = a_2\}.$$

Particularly, if $a_2 = 1_{\mathscr{A}_2}$, the inverse set $\varpi^{-1}(1_{\mathscr{A}_2})$ is important in algebra and called the *kernel of* ϖ and denoted by $\mathrm{Ker}(\varpi)$, which is a normal subgroup of $(\mathscr{A}_1; \circ_1)$ if it is associative and each element in $\mathrm{Ker}(\varpi)$ has inverse element in $(\mathscr{A}_1; \circ_1)$. In fact, by definition, for $\forall a, b, c \in \mathscr{A}_1$, we know that

(1) $(a \circ b) \circ c = a \circ (b \circ c) \in \mathrm{Ker}(\varpi)$ for $\varpi((a \circ b) \circ c) = \varpi(a \circ (b \circ c)) = 1_{\mathscr{A}_2}$;

(2) $1_{\mathscr{A}_2} \in \mathrm{Ker}(\varpi)$ for $\varpi(1_{\mathscr{A}_1}) = 1_{\mathscr{A}_2}$;

(3) $a^{-1} \in \mathrm{Ker}(\varpi)$ for $\forall a \in \mathrm{Ker}(\varpi)$ if a^{-1} exists in $(\mathscr{A}_1; \circ_1)$ since $\varpi(a^{-1}) = \varpi^{-1}(a) = 1_{\mathscr{A}_2}$;

(4) $a \circ \mathrm{Ker}(\varpi) = \mathrm{Ker}(\varpi) \circ a$ for

$$\varpi(a \circ \mathrm{Ker}(\varpi)) = \varpi(\mathrm{Ker}(\varpi) \circ a) = \varpi^{-1}(\varpi(a))$$

by definition. Whence, $\mathrm{Ker}(\varpi)$ is a normal subgroup of $(\mathscr{A}_1; \circ_1)$.

Theorem 3.2 *Let* $\varpi : \mathscr{A}_1 \to \mathscr{A}_2$ *be an onto homomorphism from associative systems* $(\mathscr{A}_1; \circ_1)$ *to* $(\mathscr{A}_2; \circ_2)$ *with units* $1_{\mathscr{A}_1}$, $1_{\mathscr{A}_2}$. *Then*

$$\mathscr{A}_1/\mathrm{Ker}(\varpi) \ \cong \ (\mathscr{A}_2;\circ_2)$$

if each element of $\mathrm{Ker}(\varpi)$ *has an inverse in* $(\mathscr{A}_1;\circ_1)$.

Proof We have known that $\mathrm{Ker}(\varpi)$ is a subgroup of $(\mathscr{A}_1;\circ_1)$. Whence $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ is a quotient system. Define a mapping $\varsigma : \mathscr{A}_1/\mathrm{Ker}(\varpi) \to \mathscr{A}_2$ by

$$\varsigma(a \circ_1 \mathrm{Ker}(\varpi)) = \varpi(a).$$

We prove this mapping is an isomorphism. Notice that ς is onto by that ϖ is an onto homomorphism. Now if $a \circ_1 \mathrm{Ker}(\varpi) \neq b \circ_1 \mathrm{Ker}(\varpi)$, then $\varpi(a) \neq \varpi(b)$. Otherwise, we find that $a \circ_1 \mathrm{Ker}(\varpi) = b \circ_1 \mathrm{Ker}(\varpi)$, a contradiction. Whence, $\varsigma(a \circ_1 \mathrm{Ker}(\varpi)) \neq \varsigma(b \circ_1 \mathrm{Ker}(\varpi))$, i.e., ς is a bijection from $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ to $\mathscr{A}_2$.

Since ϖ is a homomorphism, we get that

$$\begin{aligned}
&\varsigma((a \circ_1 \mathrm{Ker}(\varpi)) \circ_1 (b \circ_1 \mathrm{Ker}(\varpi))) \\
&= \varsigma(a \circ_1 \mathrm{Ker}(\varpi)) \circ_2 \varsigma(b \circ_1 \mathrm{Ker}(\varpi)) \\
&= \varpi(a) \circ_2 \varpi(b),
\end{aligned}$$

i.e., ς is an isomorphism from $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ to $(\mathscr{A}_2;\circ_2)$. □

Corollary 3.3 *Let* $\varpi : \mathscr{A}_1 \to \mathscr{A}_2$ *be an onto homomorphism from groups* $(\mathscr{A}_1;\circ_1)$ *to* $(\mathscr{A}_2;\circ_2)$. *Then*

$$\mathscr{A}_1/\mathrm{Ker}(\varpi) \ \cong \ (\mathscr{A}_2;\circ_2).$$

§4. Multi-Operation Systems

A *multi-operation system* is a pair $(\mathscr{H};\widetilde{O})$ with a set $\mathscr{H}$ and an operation set

$$\widetilde{O} = \{\circ_i \mid 1 \leq i \leq l\}$$

on $\mathscr{H}$ such that each pair $(\mathscr{H};\circ_i)$ is an algebraic system. A multi-operation system $(\mathscr{H};\widetilde{O})$ is *associative* if for $\forall a,b,c \in \mathscr{H}$, $\forall \circ_1, \circ_2 \in \widetilde{O}$, there is

$$(a \circ_1 b) \circ_2 c = a \circ_1 (b \circ_2 c).$$

Such a system is called an *associative multi-operation system*. A multi-operation system $(\mathscr{H};\widetilde{O})$ is *distributive* if $\widetilde{O} = \mathcal{O}_1 \cup \mathcal{O}_1$ with $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ such that

$$a \circ_1 (b \circ_2 c) = (a \circ_1 b) \circ_2 (a \circ_1 c) \text{ and } (b \circ_2 c) \circ_1 a = (b \circ_1 a) \circ_2 (c \circ_1 a)$$

for $\forall a,b,c \in \mathscr{H}$ and $\forall \circ_1 \in \mathcal{O}_1$, $\circ_2 \in \mathcal{O}_2$. Denoted such a system by $(\mathscr{H};\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$.

Let $(\mathscr{H},\widetilde{O})$ be a multi-operation system and $\mathscr{G} \subset \mathscr{H}$, $\widetilde{Q} \subset \widetilde{O}$. If $(\mathscr{G};\widetilde{Q})$ is itself a multi-operation system, we call $(\mathscr{G};\widetilde{Q})$ a *multi-operation subsystem of* $(\mathscr{H},\widetilde{O}))$, denoted by $(\mathscr{G};\widetilde{Q}) \prec (\mathscr{H},\widetilde{O})$. In those of subsystems, the $(\mathscr{G};\widetilde{O})$ is taking over an important position in the following.

Assume $(\mathscr{G};\widetilde{O}) \prec (\mathscr{H},\widetilde{O})$. For $\forall a \in \mathscr{H}$ and $\circ_i \in \widetilde{O}$, where $1 \leq i \leq l$, define a coset $a \circ_i \mathscr{G}$ by

$$a \circ_i \mathscr{G} = \{a \circ_i b | \text{ for } \forall b \in \mathscr{G}\},$$

and let

$$\mathscr{H} = \bigcup_{a\in R, \circ\in \widetilde{P}\subset\widetilde{O}} a \circ \mathscr{G}.$$

Then the set

$$\mathscr{Q} = \{a \circ \mathscr{G} | a \in R, \circ \in \widetilde{P} \subset \widetilde{O}\}$$

is called a *quotient set of* $\mathscr{G}$ *in* $\mathscr{H}$ *with a representation pair* $(R,\widetilde{P})$, denoted by $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$.

Two multi-operation systems $(\mathscr{H}_1;\widetilde{O}_1)$ and $(\mathscr{H}_2;\widetilde{O}_2)$ are called *homomorphic* if there is a mapping $\omega : \mathscr{H}_1 \to \mathscr{H}_2$ with $\omega : \widetilde{O}_1 \to \widetilde{O}_2$ such that for $a_1, b_1 \in \mathscr{H}_1$ and $\circ_1 \in \widetilde{O}_1$, there exists an operation $\circ_2 = \omega(\circ_1) \in \widetilde{O}_2$ enables that

$$\omega(a_1 \circ_1 b_1) = \omega(a_1) \circ_2 \omega(b_1).$$

Similarly, if ω is a bijection, $(\mathscr{H}_1;\widetilde{O}_1)$ and $(\mathscr{H}_2;\widetilde{O}_2)$ are called *isomorphic*, and if $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$, ω is called an *automorphism on* $\mathscr{H}$.

Theorem 4.1 *Let* $(\mathscr{H},\widetilde{O})$ *be an associative multi-operation system with a unit* $1_\circ$ *for* $\forall \circ \in \widetilde{O}$ *and* $\mathscr{G} \subset \mathscr{H}$.

(i) If $\mathscr{G}$ *is closed for operations in* $\widetilde{O}$ *and for* $\forall a \in \mathscr{G}, \circ \in \widetilde{O}$, *there exists an inverse element* $a_\circ^{-1}$ *in* $(\mathscr{G};\circ)$, *then there is a representation pair* $(R,\widetilde{P})$ *such that the quotient set* $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ *is a partition of* $\mathscr{H}$, *i.e., for* $a, b \in \mathscr{H}, \forall \circ_1, \circ_2 \in \widetilde{O}$, $(a \circ_1 \mathscr{G}) \cap (b \circ_2 \mathscr{G}) = \emptyset$ *or* $a \circ_1 \mathscr{G} = b \circ_2 \mathscr{G}$.

(ii) For $\forall \circ \in \widetilde{O}$, *define an operation* $\circ$ *on* $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ *by*

$$(a \circ_1 \mathscr{G}) \circ (b \circ_2 \mathscr{G}) = (a \circ b) \circ_1 \mathscr{G}.$$

Then $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})};\widetilde{O})$ *is an associative multi-operation system. Particularly, if there is a representation pair* $(R,\widetilde{P})$ *such that for* $\circ' \in \widetilde{P}$, *any element in* R *has an inverse in* $(\mathscr{H};\circ')$, *then* $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})},\circ')$ *is a group.*

Proof For $a, b \in \mathscr{H}$, if there are operations $\circ_1, \circ_2 \in \widetilde{O}$ with $(a \circ_1 \mathscr{G}) \cap (b \circ_2 \mathscr{G}) \neq \emptyset$, then there must exist $g_1, g_2 \in \mathscr{G}$ such that $a \circ_1 g_1 = b \circ_2 g_2$. By assumption, there is an inverse element c_1^{-1} in the system $(\mathscr{G};\circ_1)$. We find that

$$\begin{aligned} a \circ_1 \mathscr{G} &= (b \circ_2 g_2 \circ_1 c_1^{-1}) \circ_1 \mathscr{G} \\ &= b \circ_2 (g_2 \circ_1 c_1^{-1} \circ_1 \mathscr{G}) = b \circ_2 \mathscr{G} \end{aligned}$$

by the associative law. This implies that $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ is a partition of $\mathscr{H}$.

Notice that $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ is closed under operations in $\widetilde{P}$ by definition. It is a multi-operation system. For $\forall a,b,c\in R$ and operations $\circ_1,\circ_2,\circ_3,\circ^1,\circ^2\in\widetilde{P}$ we know that

$$\begin{aligned}((a\circ_1\mathscr{G})\circ^1(b\circ_2\mathscr{G}))\circ^2(c\circ_3\mathscr{G}) &= ((a\circ^1 b)\circ_1\mathscr{G})\circ^2(c\circ_3\mathscr{G})\\ &= ((a\circ^1 b)\circ^2 c)\circ_1\mathscr{G}\end{aligned}$$

and

$$\begin{aligned}(a\circ_1\mathscr{G})\circ^1((b\circ_2\mathscr{G})\circ^2(c\circ_3\mathscr{G})) &= (a\circ_1\mathscr{G})\circ_1((b\circ^2 c)\circ_2\mathscr{G})\\ &= (a\circ^1(b\circ^2 c))\circ_1\mathscr{G}.\end{aligned}$$

by definition. Since $(\mathscr{H},\widetilde{O})$ is associative, we have $(a\circ^1 b)\circ^2 c=a\circ^1(b\circ^2 c)$. Whence, we get that

$$((a\circ_1\mathscr{G})\circ^1(b\circ_2\mathscr{G}))\circ^2(c\circ_3\mathscr{G})=(a\circ_1\mathscr{G})\circ^1((b\circ_2\mathscr{G})\circ^2(c\circ_3\mathscr{G})),$$

i.e., $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})};\widetilde{O})$ is an associative multi-operation system.

If any element in R has an inverse in $(\mathscr{H};\circ')$, then we know that $\mathscr{G}$ is a unit and $a^{-1}\circ'\mathscr{G}$ is the inverse element of $a\circ'\mathscr{G}$ in the system $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})},\circ')$, namely, it is a group again. □

Let $\mathcal{I}(\widetilde{O})$ be the set of all units $1_\circ,\circ\in\widetilde{O}$ in a multi-operation system $(\mathscr{H};\widetilde{O})$. Define a *multi-kernel* $\widetilde{\mathrm{Ker}}\omega$ of a homomorphism $\omega:(\mathscr{H}_1;\widetilde{O}_1)\to(\mathscr{H}_2;\widetilde{O}_2)$ by

$$\widetilde{\mathrm{Ker}}\omega=\{\ a\in\mathscr{H}_1\mid\omega(a)=1_\circ\in\mathcal{I}(\widetilde{O}_2)\ \}.$$

Then we know the homomorphism theorem for multi-operation systems in the following.

Theorem 4.2 *Let ω be an onto homomorphism from associative systems $(\mathscr{H}_1;\widetilde{O}_1)$ to $(\mathscr{H}_2;\widetilde{O}_2)$ with $(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)$ an algebraic system with unit $1_{\circ^-}$ for $\forall\circ^-\in\widetilde{O}_2$ and inverse x^{-1} for $\forall x\in(\mathcal{I}(\widetilde{O}_2)$ in $((\mathcal{I}(\widetilde{O}_2);\circ^-)$. Then there are representation pairs $(R_1,\widetilde{P}_1)$ and $(R_2,\widetilde{P}_2)$, where $\widetilde{P}_1\subset\widetilde{O},\widetilde{P}_2\subset\widetilde{O}_2$ such that*

$$\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}}\omega;\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}\ \cong\ \frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$$

if each element of $\widetilde{\mathrm{Ker}}\omega$ has an inverse in $(\mathscr{H}_1;\circ)$ for $\circ\in\widetilde{O}_1$.

Proof Notice that $\widetilde{\mathrm{Ker}}\omega$ is an associative subsystem of $(\mathscr{H}_1;\widetilde{O}_1)$. In fact, for $\forall k_1,k_2\in\widetilde{\mathrm{Ker}}\omega$ and $\forall\circ\in\widetilde{O}_1$, there is an operation $\circ^-\in\widetilde{O}_2$ such that

$$\omega(k_1\circ k_2)=\omega(k_1)\circ^-\omega(k_2)\in\mathcal{I}(\widetilde{O}_2)$$

since $\mathcal{I}(\widetilde{O}_2)$ is an algebraic system. Whence, $\widetilde{\mathrm{Ker}}\omega$ is an associative subsystem of $(\mathscr{H}_1;\widetilde{O}_1)$. By assumption, for any operation $\circ\in\widetilde{O}_1$ each element $a\in\widetilde{\mathrm{Ker}}\omega$ has an inverse a^{-1} in $(\mathscr{H}_1;\circ)$. Let $\omega:(\mathscr{H}_1;\circ)\to(\mathscr{H}_2;\circ^-)$. We know that

$$\omega(a \circ a^{-1}) = \omega(a) \circ^{-} \omega(a^{-1}) = 1_{\circ^{-}},$$

i.e., $\omega(a^{-1}) = \omega(a)^{-1}$ in $(\mathscr{H}_2; \circ^{-})$. Because $\mathcal{I}(\widetilde{O}_2)$ is an algebraic system with an inverse x^{-1} for $\forall x \in \mathcal{I}(\widetilde{O}_2)$ in $((\mathcal{I}(\widetilde{O}_2); \circ^{-})$, we find that $\omega(a^{-1}) \in \mathcal{I}(\widetilde{O}_2)$, namely, $a^{-1} \in \widetilde{\mathrm{Ker}\omega}$.

Define a mapping $\sigma : \frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)} \to \frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$ by

$$\sigma(a \circ \mathrm{Ker}\omega) = \sigma(a) \circ^{-} \mathcal{I}(\widetilde{O}_2)$$

for $\forall a \in R_1, \circ \in \widetilde{P}_1$, where $\omega : (\mathscr{H}_1; \circ) \to (\mathscr{H}_2; \circ^{-})$. We prove σ is an isomorphism. Notice that σ is onto by that ω is an onto homomorphism. Now if $a \circ_1 \widetilde{\mathrm{Ker}\omega} \neq b \circ_2 \mathrm{Ker}(\varpi)$ for $a, b \in R_1$ and $\circ_1, \circ_2 \in \widetilde{P}_1$, then $\omega(a) \circ_1^{-} \mathcal{I}(\widetilde{O}_2) \neq \omega(b) \circ_2^{-} \mathcal{I}(\widetilde{O}_2)$. Otherwise, we find that $a \circ_1 \widetilde{\mathrm{Ker}\omega} = b \circ_2 \widetilde{\mathrm{Ker}\omega}$, a contradiction. Whence, $\sigma(a \circ_1 \widetilde{\mathrm{Ker}\omega}) \neq \sigma(b \circ_2 \widetilde{\mathrm{Ker}\omega})$, i.e., σ is a bijection from $\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}$ to $\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$.

Since ω is a homomorphism, we get that

$$\begin{aligned}
\sigma((a \circ_1 \widetilde{\mathrm{Ker}\omega}) \circ (b \circ_2 \widetilde{\mathrm{Ker}\omega})) &= \sigma(a \circ_1 \widetilde{\mathrm{Ker}\omega}) \circ^{-} \sigma(b \circ_2 \widetilde{\mathrm{Ker}\omega}) \\
&= (\omega(a) \circ_1^{-} \mathcal{I}(\widetilde{O}_2)) \circ^{-} (\omega(b) \circ_2^{-} \mathcal{I}(\widetilde{O}_2)) \\
&= \sigma((a \circ_1 \widetilde{\mathrm{Ker}\omega}) \circ^{-} \sigma(b \circ_2 \widetilde{\mathrm{Ker}\omega}),
\end{aligned}$$

i.e., σ is an isomorphism from $\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}$ to $\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$. □

Corollary 4.1 *Let $(\mathscr{H}_1; \widetilde{O}_1)$, $(\mathscr{H}_2; \widetilde{O}_2)$ be multi-operation systems with groups $(\mathscr{H}_2; \circ_1)$, $(\mathscr{H}_2; \circ_2)$ for $\forall \circ_1 \in \widetilde{O}_1$, $\forall \circ_2 \in \widetilde{O}_2$ and $\omega : (\mathscr{H}_1; \widetilde{O}_1) \to (\mathscr{H}_2; \widetilde{O}_2)$ a homomorphism. Then there are representation pairs $(R_1, \widetilde{P}_1)$ and $(R_2, \widetilde{P}_2)$, where $\widetilde{P}_1 \subset \widetilde{O}_1, \widetilde{P}_2 \subset \widetilde{O}_2$ such that*

$$\frac{(\mathscr{H}_1; \widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega}; \widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)} \;\cong\; \frac{(\mathscr{H}_2; \widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2); \widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}.$$

Particularly, if $(\mathscr{H}_2; \widetilde{O}_2)$ is a group, we get an interesting result following.

Corollary 4.2 *Let $(\mathscr{H}; \widetilde{O})$ be a multi-operation system and $\omega : (\mathscr{H}; \widetilde{O}) \to (\mathscr{A}; \circ)$ a onto homomorphism from $(\mathscr{H}; \widetilde{O})$ to a group $(\mathscr{A}; \circ)$. Then there are representation pairs $(R, \widetilde{P})$, $\widetilde{P} \subset \widetilde{O}$ such that*

$$\frac{(\mathscr{H}; \widetilde{O})}{(\widetilde{\mathrm{Ker}\omega}; \widetilde{O})}|_{(R,\widetilde{P})} \;\cong\; (\mathscr{A}; \circ).$$

§5. Multi-Rings

An associative multi-operation system $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ is said to be a *multi-group* if $(\mathscr{H}; \circ)$ is a group for $\forall \circ \in \mathcal{O}_1 \cup \mathcal{O}_2$, a *multi-ring* (or *multi-field*) if $\mathcal{O}_1 = \{\cdot_i | 1 \leq i \leq l\}$, $\mathcal{O}_2 = \{+_i | 1 \leq i \leq l\}$ with rings (or multi-field) $(\mathscr{H}; +_i, \cdot_i)$ for $1 \leq i \leq l$. We call them *l-group*, *l-ring* or *l-field* for abbreviation. It is obvious that a multi-group is a group if $|\mathcal{O}_1 \cup \mathcal{O}_2| = 1$ and a ring or field if $|\mathcal{O}_1| = |\mathcal{O}_2| = 1$ in classical algebra. Likewise, We also denote these units of a l-ring $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ by $1_{\cdot_i}$ and 0_{+_i} in the ring $(\mathscr{H}; +_i, \cdot_i)$. Notice that for $\forall a \in \mathscr{H}$, by these distribute laws we find that

$$a \cdot_i b = a \cdot_i (b +_i 0_{+_i}) = a \cdot_i b +_i a \cdot_i 0_{+_i},$$

$$b \cdot_i a = (b +_i 0_{+_i}) \cdot_i a = b \cdot_i a +_i 0_{+_i} \cdot_i a$$

for $\forall b \in \mathscr{H}$. Whence,

$$a \cdot_i 0_{+_i} = 0_{+_i} \text{ and } 0_{+_i} \cdot_i a = 0_{+_i}.$$

Similarly, a multi-operation subsystem of $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ is said a *multi-subgroup*, *multi-subring* or *multi-subfield* if it is a *multi-group*, *multi-ring* or *multi-field* itself.

Now let $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ be an associative multi-operation system. We find these criterions for multi-subgroups and multi-subrings of $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ in the following.

Theorem 5.1 *Let $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2$ be a multi-group, $\mathcal{H} \subset \mathscr{H}$. Then $(\mathcal{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ is a*

(i) multi-subgroup if and only if for $\forall a, b \in \mathcal{H},\ \circ \in \mathcal{O}_1 \cup \mathcal{O}_2,\ a \circ b_\circ^{-1} \in \mathcal{H}$;

(ii) multi-subring if and only if for $\forall a, b \in \mathcal{H},\ \cdot_i \in \mathcal{O}_1$ and $\forall +_i \in \mathcal{O}_2)$, $a \cdot_i b,\ a +_i b_{+_i}^{-1} \in \mathcal{H}$, particularly, a multi-field if $a \cdot_i b_{\cdot_i}^{-1},\ a +_i b_{+_i}^{-1} \in \mathcal{H}$, where, $\mathcal{O}_1 = \{\cdot_i | 1 \leq i \leq l\}$, $\mathcal{O}_2 = \{+_i | 1 \leq i \leq l\}$.

Proof The necessity of conditions (i) and (ii) is obvious. Now we consider their sufficiency.

For (i), we only need to prove that $(\mathcal{H}; \circ)$ is a group for $\forall \circ \in \mathcal{O}_1 \cup \mathcal{O}_2$. In fact, it is associative by the definition of multi-groups. For $\forall a \in \mathcal{H}$, we get that $1_\circ = a \circ a_\circ^{-1} \in \mathcal{H}$ and $1_\circ \circ a_\circ^{-1} \in \mathcal{H}$. Whence, $(\mathcal{H}; \circ)$ is a group.

Similarly for (ii), the conditions $a \cdot_i b,\ a +_i b_{+_i}^{-1} \in \mathcal{H}$ imply that $(\mathcal{H}; +_i)$ is a group and closed in operation $\cdot_i \in \mathcal{O}_1$. These associative or distributive laws are hold by $(\mathscr{H}; +_i, \cdot_i)$ being a ring for any integer $i, 1 \leq i \leq l$. Particularly, $a \cdot_i b_{\cdot_i}^{-1} \in \mathcal{H}$ imply that $(\mathcal{H}; \cdot_i)$ is also a group. Whence, $(\mathscr{H}; +_i, \cdot_i)$ is a field for any integer $i, 1 \leq i \leq l$ in this case. □

A multi-ring $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ with $\mathcal{O}_1 = \{\cdot_i | 1 \leq i \leq l\}$, $\mathcal{O}_2 = \{+_i | 1 \leq i \leq l\}$ is *integral* if for $\forall a, b \in \mathscr{H}$ and an integer $i,\ 1 \leq i \leq l$, $a \circ_i b = b \circ_i a$, $1_{\circ_i} \neq 0_{+i}$ and $a \circ_i b = 0_{+_i}$ implies that $a = 0_{+_i}$ or $b = 0_{+_i}$. If $l = 1$, an integral l-ring is the integral ring by definition. For the case of multi-rings with finite elements, an integral multi-ring is nothing but a multi-field. See the next result.

Theorem 5.2 *A finitely integral multi-ring is a multi-field.*

Proof Let $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ be a finitely integral multi-ring with $\mathscr{H} = \{a_1, a_2 \cdots, a_n\}$, where $\mathcal{O}_1 = \{\cdot_i | 1 \leq i \leq l\}$, $\mathcal{O}_2 = \{+_i | 1 \leq i \leq l\}$. For any integer $i, 1 \leq i \leq l$, choose an element $a \in \mathscr{H}$ and $a \neq 0_{+_i}$. Then

$$a \circ_i a_1,\ a \circ_i a_2, \cdots,\ a \circ_i a_n$$

are n elements. If $a \circ_i a_s = a \circ_i a_t$, i.e., $a \circ_i (a_s +_i a_t^{-1}) = 0_{+_i}$. By definition, we know that

$a_s +_i a_t^{-1} = 0+_i$, namely, $a_s = a_t$. That is, these $a \circ_i a_1,\ a \circ_i a_2, \cdots,\ a \circ_i a_n$ are different two by two. Whence,

$$\mathscr{H} = \{\ a \circ_i a_1,\ a \circ_i a_2, \cdots,\ a \circ_i a_n\ \}.$$

Now assume $a \circ_i a_s = 1_{\cdot_i}$, then $a^{-1} = a_s$, i.e., each element of $\mathscr{H}$ has an inverse in $(\mathscr{H}; \cdot_i)$, which implies it is a commutative group. Therefore, $(\mathscr{H}; +_i, \cdot_i)$ is a field for any integer $i, 1 \leq i \leq l$. □

Corollary 5.1 *Any finitely integral ring is a field.*

Let $(\mathscr{H}; \mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1)$, $(\mathscr{H}; \mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)$ be multi-rings with $\mathcal{O}_1^k = \{\cdot_i^k | 1 \leq i \leq l_k\}$, $\mathcal{O}_2^k = \{+_i^k | 1 \leq i \leq l_k\}$ for $k = 1, 2$ and $\varrho : (\mathscr{H}; \mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1) \to (\mathscr{H}; \mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)$ a homomorphism. Define a *zero kernel* $\widetilde{\mathrm{Ker}}\varrho$ of ϱ by

$$\widetilde{\mathrm{Ker}_0}\varrho = \{a \in \mathscr{H} | \varrho(a) = 0_{+_i^2}, 1 \leq i \leq l_2\}.$$

Then, for $\forall h \in \mathscr{H}$ and $a \in \widetilde{\mathrm{Ker}_0}\varrho$, $\varrho(a \cdot_i^1 h) = 0_{+_i}\varrho(\cdot_i)h = 0_{+_i}$, i.e., $a \cdot_i h \in \widetilde{\mathrm{Ker}_0}\varrho$. Similarly, $h \cdot_i a \in \widetilde{\mathrm{Ker}_0}\varrho$. These properties imply the conception of multi-ideals of a multi-ring introduced following.

Choose a subset $\mathcal{I} \subset \mathscr{H}$. For $\forall h \in \mathscr{H},\ a \in \mathcal{I}$, if there are

$$h \circ_i a \in \mathcal{I} \ \text{ and } \ a \circ_i h \in \mathcal{H},$$

then $\mathcal{I}$ is said a *multi-ideal*. Previous discussion shows that the zero kernel $\widetilde{\mathrm{Ker}_0}\varrho$ of a homomorphism ϱ on a multi-ring is a multi-ideal. Now let $\mathcal{I}$ be a multi-ideal of $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$. According to Corollary 4.1, we know that there is a representation pair (R_2, P_2) such that

$$\widetilde{\mathcal{I}} = \{a +_i \mathcal{I} \mid a \in R_2,\ +_i \in P_2\}$$

is a commutative multi-group. By the distributive laws, we find that

$$\begin{aligned}(a +_i \mathcal{I}) \cdot_j (b +_k \mathcal{I}) &= a \cdot_j b +_k a \cdot_j \mathcal{I} +_i \mathcal{I}b +_k \mathcal{I} \cdot_j \mathcal{I} \\ &= a \cdot_j b +_k \mathcal{I}.\end{aligned}$$

Similarly, we also know these associative and distributive laws follow in $(\widetilde{\mathcal{I}}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$. Whence, $(\widetilde{\mathcal{I}}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ is also a multi-ring, called the *quotient multi-ring of* $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$, denoted by $(\mathscr{H} : \mathcal{I})$.

Define a mapping $\varrho : (\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2) \to (\mathscr{H} : \mathcal{I})$ by $\varrho(a) = a +_i \mathcal{I}$ for $\forall a \in \mathscr{H}$ if $a \in a +_i \mathcal{I}$. Then it can be checked immediately that it is a homomorphism with

$$\widetilde{\mathrm{Ker}_0}\varrho \ = \ \mathcal{I}.$$

Therefore, we conclude that *any multi-ideal is a zero kernel of a homomorphism on a multi-ring*. The following result is a special case of Theorem 4.2.

Theorem 5.3 *Let $(\mathscr{H}_1;\mathcal{O}_1^1\hookrightarrow\mathcal{O}_2^1)$ and $(\mathscr{H}_2;\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2)$ be multi-rings and $\omega:(\mathscr{H}_1;\mathcal{O}_2^1)\rightarrow(\mathscr{H}_2;\mathcal{O}_2^2)$ be an onto homomorphism with $(\mathcal{I}(\mathcal{O}_2^2);\mathcal{O}_2^2)$ be a multi-operation system, where $\mathcal{I}(\mathcal{O}_2^2)$ denotes all units in $(\mathscr{H}_2;\mathcal{O}_2^2)$. Then there exist representation pairs $(R_1,\widetilde{P}_1),(R_2,\widetilde{P}_2)$ such that*

$$(\mathscr{H}:\mathcal{I})|_{(R_1,\widetilde{P}_1)}\ \cong\ \frac{(\mathscr{H}_2;\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2)}{(\mathcal{I}(\mathcal{O}_2^2);\mathcal{O}_2^2)}|_{(R_2,\widetilde{P}_2)}.$$

Particularly, if $(\mathscr{H}_2;\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2)$ is a ring, we get an interesting result following.

Corollary 5.2 *Let $(\mathscr{H};\mathcal{O}_1\hookrightarrow\mathcal{O}_2)$ be a multi-ring, $(R;+,\cdot)$ a ring and $\omega:(\mathscr{H};\mathcal{O}_2)\rightarrow(R;+)$ be an onto homomorphism. Then there exists a representation pair $(R,\widetilde{P})$ such that*

$$(\mathscr{H}:\mathcal{I})|_{(R,\widetilde{P})}\ \cong\ (R;+,\cdot).$$

§6. Finite Dimensional Multi-Modules

Let $\mathcal{O}=\{\ +_i\ |\ 1\leq i\leq m\}$, $\mathcal{O}_1=\{\cdot_i|1\leq i\leq m\}$ and $\mathcal{O}_2=\{\dot{+}_i|1\leq i\leq m\}$ be operation sets, $(\mathscr{M};\mathcal{O})$ a commutative m-group with units 0_{+_i} and $(\mathscr{R};\mathcal{O}_1\hookrightarrow\mathcal{O}_2)$ a multi-ring with a unit $1.$ for $\forall\cdot\in\mathcal{O}_1$. For any integer i, $1\leq i\leq m$, define a binary operation $\times_i:\mathscr{R}\times\mathscr{M}\rightarrow\mathscr{M}$ by $a\times_i x$ for $a\in\mathscr{R}$, $x\in\mathscr{M}$ such that for $\forall a,b\in\mathscr{R}$, $\forall x,y\in\mathscr{M}$, conditions following hold:

(*i*) $a\times_i(x+_i y)=a\times_i x+_i a\times_i y$;

(*ii*) $(a\dot{+}_i b)\times_i x=a\times_i x+_i b\times_i x$;

(*iii*) $(a\cdot_i b)\times_i x=a\times_i(b\times_i x)$;

(*iv*) $1_{\cdot_i}\times_i x\ =\ x$.

Then $(\mathscr{M};\mathcal{O})$ is said an *algebraic multi-module over* $(\mathscr{R};\mathcal{O}_1\hookrightarrow\mathcal{O}_2)$ abbreviated to an *m-module* and denoted by $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$. In the case of $m=1$, It is obvious that $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ is a *module*, particularly, if $(\mathscr{R};\mathcal{O}_1\hookrightarrow\mathcal{O}_2)$ is a field, then $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ is a *linear space* in classical algebra.

For any integer k, $a_i\in\mathscr{R}$ and $x_i\in\mathscr{M}$, where $1\leq i,\ k\leq s$, equalities following are hold by induction on the definition of m-modules.

$$a\times_k(x_1+_k x_2+_k\cdots+_k x_s)=a\times_k x_1+_k a\times_k x_2+_k\cdots+_k a_s\times_k x,$$

$$(a_1\dot{+}_k a_2\dot{+}_k\cdots\dot{+}_k a_s)\times_k x=a_1\times_k x+_k a_2\times_k x+_k\cdots+_k a_s\times_k x,$$

$$(a_1\cdot_k a_2\cdot_k\cdots\cdot_k a_s)\times_k x=a_1\times_k(a_2\times_k\cdots\times(a_s\times_k x)\cdots)$$

and

$$1_{\cdot_{i_1}}\times_{i_1}(1_{\cdot_{i_2}}\times_{i_2}\cdots\times_{i_{s-1}}(1_{\cdot_{i_s}}\times_{i_s}x)\cdots)\ =\ x$$

for integers $i_1,i_2,\cdots,i_s\in\{1,2,\cdots,m\}$.

Notice that for $\forall a,x\in\mathscr{M}$, $1\leq i\leq m$,

$$a\times_i x=a\times_i(x+_i 0_{+_i})=a\times_i x+_i a\times_i 0_{+_i},$$

we find that $a\times_i 0_{+_i}=0_{+_i}$. Similarly, $0_{\dot{+}_i}\times_i a=0_{+_i}$. Applying this fact, we know that

$$a \times_i x +_i a^-_{\dot{+}_i} \times_i x = (a \dot{+}_i a^-_{\dot{+}_i}) \times_i x = 0_{\dot{+}_i} \times_i x = 0_{+_i}$$

and

$$a \times_i x +_i a \times_i x^-_{+_i} = a \times_i (x +_i x^-_{+_i}) = a \times_i 0_{+_i} = 0_{+_i}.$$

We know that

$$(a \times_i x)^-_{+_i} = a^-_{\dot{+}_i} \times_i x = a \times_i x^-_{+_i}.$$

Notice that $a \times_i x = 0_{+_i}$ does not always mean $a = 0_{\dot{+}_i}$ or $x = 0_{+_i}$ in an m-module $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ unless $a^-_{\dot{+}_i}$ is existing in $(\mathscr{R}; \dot{+}_i, \cdot_i)$ if $x \neq 0_{+_i}$.

Now choose $\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1) : \mathscr{R}_1(\mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1))$ an m-module with operation sets $\mathcal{O}_1 = \{ +'_i \mid 1 \leq i \leq m\}$, $\mathcal{O}_1^1 = \{\cdot_i^1 | 1 \leq i \leq m\}$, $\mathcal{O}_2^1 = \{\dot{+}_i^1 | 1 \leq i \leq m\}$ and $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2))$ an n-module with operation sets $\mathcal{O}_2 = \{ +''_i \mid 1 \leq i \leq n\}$, $\mathcal{O}_1^2 = \{\cdot_i^2 | 1 \leq i \leq n\}$, $\mathcal{O}_2^2 = \{\dot{+}_i^2 | 1 \leq i \leq n\}$. They are said *homomorphic* if there is a mapping $\iota : \mathscr{M}_1 \to \mathscr{M}_2$ such that for any integer $i, 1 \leq i \leq m$,

(*i*) $\iota(x +'_i y) = \iota(x) +'' \iota(y)$ for $\forall x, y \in \mathscr{M}_1$, where $\iota(+'_i) = +'' \in \mathcal{O}_2$;

(*ii*) $\iota(a \times_i x) = a \times_i \iota(x)$ for $\forall x \in \mathscr{M}_1$.

If ι is a bijection, these modules $\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1) : \mathscr{R}_1(\mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1))$ and $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2))$ are said to be *isomorphic*, denoted by

$$\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1) : \mathscr{R}_1(\mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1)) \cong \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)).$$

Let $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ be an m-module. For a multi-subgroup $(\mathscr{N}; \mathcal{O})$ of $(\mathscr{M}; \mathcal{O})$, if for any integer i, $1 \leq i \leq m$, $a \times_i x \in \mathscr{N}$ for $\forall a \in \mathscr{R}$ and $x \in \mathscr{N}$, then by definition it is itself an m-module, called a multi-submodule of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$.

Now if $\mathbf{Mod}(\mathscr{N}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ is a multi-submodule of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$, by Theorem 4.2, we can get a quotient multi-group $\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})}$ with a representation pair $(R, \widetilde{P})$ under operations

$$(a +_i \mathscr{N}) + (b +_j \mathscr{N}) = (a + b) +_i \mathscr{N}$$

for $\forall a, b \in R, + \in \mathcal{O}$. For convenience, we denote elements $x +_i \mathscr{N}$ in $\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})}$ by $\overline{x^{(i)}}$. For an integer $i, 1 \leq i \leq m$ and $\forall a \in \mathscr{R}$, define

$$a \times_i \overline{x^{(i)}} = \overline{(a \times_i x)^{(i)}}.$$

Then it can be shown immediately that

(*i*) $a \times_i (\overline{x^{(i)}} +_i \overline{y^{(i)}}) = a \times_i \overline{x^{(i)}} +_i a \times_i \overline{y^{(i)}}$;

(*ii*) $(a \dot{+}_i b) \times_i \overline{x^{(i)}} = a \times_i \overline{x^{(i)}} +_i b \times_i \overline{x^{(i)}}$;

(*iii*) $(a \cdot_i b) \times_i \overline{x^{(i)}} = a \times_i (b \times_i \overline{x^{(i)}})$;

(*iv*) $1_{\cdot_i} \times_i \overline{x^{(i)}} = \overline{x^{(i)}}$, i.e.,$(\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})}:\mathscr{R})$ is also an m-module, called a quotient module of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ to $\mathbf{Mod}(\mathscr{N}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$. Denoted by $\mathbf{Mod}(\mathscr{M}/\mathscr{N})$.

The result on homomorphisms of m-modules following is an immediately consequence of Theorem 4.2.

Theorem 6.1 *Let $\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1):\mathscr{R}_1(\mathcal{O}_1^1\hookrightarrow\mathcal{O}_2^1))$, $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2):\mathscr{R}_2(\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2))$ be multi-modules with $\mathcal{O}_1=\{\ +_i'\ |\ 1\leq i\leq m\}$, $\mathcal{O}_2=\{\ +_i''\ |\ 1\leq i\leq n\}$, $\mathcal{O}_1^1=\{\cdot_i^1|1\leq i\leq m\}$, $\mathcal{O}_2^1=\{\dot{+}_i^1|1\leq i\leq m\}$, $\mathcal{O}_1^2=\{\cdot_i^2|1\leq i\leq n\}$, $\mathcal{O}_2^2=\{\dot{+}_i^2|1\leq i\leq n\}$ and $\iota:\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1):\mathscr{R}_1(\mathcal{O}_1^1\hookrightarrow\mathcal{O}_2^1))\rightarrow\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2):\mathscr{R}_2(\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2))$ be a onto homomorphism with $(\mathcal{I}(\mathcal{O}_2);\mathcal{O}_2)$ a multi-group, where $\mathcal{I}(\mathcal{O}_2^2)$ denotes all units in the commutative multi-group $(\mathscr{M}_2;\mathcal{O}_2)$. Then there exist representation pairs $(R_1,\widetilde{P}_1),(R_2,\widetilde{P}_2)$ such that*

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)}\ \cong\ \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))|_{(R_2,\widetilde{P}_2)},$$

where $\mathscr{N}=\mathrm{Ker}\iota$ is the kernel of ι. Particularly, if $(\mathcal{I}(\mathcal{O}_2);\mathcal{O}_2)$ is trivial, i.e., $|\mathcal{I}(\mathcal{O}_2)|=1$, then

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)}\ \cong\ \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2):\mathscr{R}_2(\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2))|_{(R_2,\widetilde{P}_2)}.$$

Proof Notice that $(\mathcal{I}(\mathcal{O}_2);\mathcal{O}_2)$ is a commutative multi-group. We can certainly construct a quotient module $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))$. Applying Theorem 2.3.6, we find that

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)}\ \cong\ \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))|_{(R_2,\widetilde{P}_2)}.$$

Notice that $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))=\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2):\mathscr{R}_2(\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2))$ in the case of $|\mathcal{I}(\mathcal{O}_2)|=1$. We get the isomorphism as desired. □

Corollary 6.1 *Let $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ be an m-module with $\mathcal{O}=\{\ +_i\ |\ 1\leq i\leq m\}$, $\mathcal{O}_1=\{\cdot_i|1\leq i\leq m\}$, $\mathcal{O}_2=\{\dot{+}_i|1\leq i\leq m\}$, M a module on a ring $(R;+,\cdot)$ and $\iota:\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1):\mathscr{R}_1(\mathcal{O}_1^1\hookrightarrow\mathcal{O}_2^1))\rightarrow M$ a onto homomorphism with $\mathrm{Ker}\iota=\mathscr{N}$. Then there exists a representation pair $(R',\widetilde{P})$ such that*

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R',\widetilde{P})}\ \cong\ M,$$

particularly, if $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ is a module $\mathscr{M}$, then

$$\mathscr{M}/\mathscr{N}\ \cong\ M.$$

For constructing multi-submodules of an m-module $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ with $\mathcal{O}=\{\ +_i\ |\ 1\leq i\leq m\}$, $\mathcal{O}_1=\{\cdot_i|1\leq i\leq m\}$, $\mathcal{O}_2=\{\dot{+}_i|1\leq i\leq m\}$, a general way is described in the following.

Let $\widehat{S}\subset\mathscr{M}$ with $|\widehat{S}|=n$. Define its *linearly spanning set* $\left\langle\widehat{S}|\mathscr{R}\right\rangle$ in $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ to be

$$\left\langle\widehat{S}|\mathscr{R}\right\rangle=\{\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\alpha_{ij}\times_i x_{ij}|\ \alpha_{ij}\in\mathscr{R},x_{ij}\in\widehat{S}\ \},$$

where

$$\begin{aligned}\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} a_{ij}\times_{ij} x_i \quad = \quad & a_{11}\times_1 x_{11} +_1 \cdots +_1 a_{1n}\times_1 x_{1n} \\ & +^{(1)} a_{21}\times_2 x_{21} +_2 \cdots +_2 a_{2n}\times_2 x_{2n} \\ & +^{(2)} \cdots\cdots\cdots\cdots\cdots\cdots +^{(3)} \\ & a_{m1}\times_m x_{m1} +_m \cdots +_m a_{mn}\times_m x_{mn}\end{aligned}$$

with $+^{(1)}, +^{(2)}, +^{(3)} \in \mathscr{O}$ and particularly, if $+_1 = +_2 = \cdots = +_m$, it is denoted by $\sum\limits_{i=1}^{m} x_i$ as usual. It can be checked easily that $\left\langle \widehat{S}|\mathscr{R}\right\rangle$ is a multi-submodule of $\mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$, call it *generated by* $\widehat{S}$ *in* $\mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$. If $\widehat{S}$ is finite, we also say that $\left\langle \widehat{S}|\mathscr{R}\right\rangle$ is *finitely generated*. Particularly, if $\widehat{S} = \{x\}$, then $\left\langle \widehat{S}|\mathscr{R}\right\rangle$ is called a *cyclic multi-submodule of* $\mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$, denoted by $\mathscr{R}x$. Notice that

$$\mathscr{R}x = \{ \bigoplus_{i=1}^{m} a_i \times_i x \mid a_i \in \mathscr{R} \}$$

by definition. For any finite set $\widehat{S}$, if for any integer $s, 1 \le s \le m$,

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{s_i} \alpha_{ij}\times_i x_{ij} = 0_{+_s}$$

implies that $\alpha_{ij} = 0_{\dot{+}_s}$ for $1 \le i \le m, 1 \le j \le n$, then we say that $\{x_{ij} | 1 \le i \le m, 1 \le j \le n\}$ is independent and $\widehat{S}$ a *basis of the multi-module* $\mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$, denoted by $\left\langle \widehat{S}|\mathscr{R}\right\rangle = \mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$.

For a multi-ring $(\mathscr{R}; \mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ with a unit 1. for $\forall \cdot \in \mathscr{O}_1$, where $\mathscr{O}_1 = \{\cdot_i | 1 \le i \le m\}$ and $\mathscr{O}_2 = \{\dot{+}_i | 1 \le i \le m\}$, let

$$\mathscr{R}^{(n)} = \{(x_1, x_2, \cdots, x_n) |\ x_i \in \mathscr{R}, 1 \le i \le n\}.$$

Define operations

$$(x_1, x_2, \cdots, x_n) +_i (y_1, y_2, \cdots, y_n) = (x_1 \dot{+}_i y_1, x_2 \dot{+}_i y_2, \cdots, x_n \dot{+}_i y_n)$$

and

$$a \times_i (x_1, x_2, \cdots, x_n) = (a \cdot_i x_1, a \cdot_i x_2, \cdots, a \cdot_i x_n)$$

for $\forall a \in \mathscr{R}$ and integers $1 \le i \le m$. Then it can be immediately known that $\mathscr{R}^{(n)}$ is a multi-module $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$. We construct a basis of this special multi-module in the following.

For any integer $k, 1 \le k \le n$, let

$$\mathbf{e}_1 = (1_{\cdot_k}, 0_{\dot{+}_k}, \cdots, 0_{\dot{+}_k});$$

$$\mathbf{e}_2 = (0_{\dot{+}_k}, 1_{\cdot_k}, \cdots, 0_{\dot{+}_k});$$

$$\cdots\cdots\cdots\cdots\cdots\cdots ;$$

$$\mathbf{e}_n = (0_{\dot{+}_k}, \cdots, 0_{\dot{+}_k}, 1_{\cdot_k}).$$

Notice that

$$(x_1, x_2, \cdots, x_n) = x_1 \times_k \mathbf{e}_1 +_k x_2 \times_k \mathbf{e}_2 +_k \cdots +_k x_n \times_k \mathbf{e}_n.$$

We find that each element in $\mathscr{R}^{(n)}$ is generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. Now since

$$(x_1, x_2, \cdots, x_n) = (0_{\dot{+}_k}, 0_{\dot{+}_k}, \cdots, 0_{\dot{+}_k})$$

implies that $x_i = 0_{\dot{+}_k}$ for any integer $i, 1 \leq i \leq n$. Whence, $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is a basis of $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$.

Theorem 6.2 *Let* $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)) = \left\langle \widehat{S} | \mathscr{R} \right\rangle$ *be a finitely generated multi-module with* $\widehat{S} = \{u_1, u_2, \cdots, u_n\}$. *Then*

$$\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)) \;\cong\; \mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)).$$

Proof Define a mapping $\vartheta : \mathscr{M}(\mathcal{O}) \to \mathscr{R}^{(n)}$ by $\vartheta(u_i) = \mathbf{e}_i$, $\vartheta(a \times_j u_i) = a \times_j \mathbf{e}_j$ and $\vartheta(u_i +_k u_j) = \mathbf{e}_i +_k \mathbf{e}_j$ for any integers i, j, k, where $1 \leq i, j, k \leq n$. Then we know that

$$\vartheta(\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} a_{ij} \times_i u_i) = \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} a_{ij} \times_i \mathbf{e}_i.$$

Whence, ϑ is a homomorphism. Notice that it is also $1-1$ and onto. We know that ϑ is an isomorphism between $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ and $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$. □

§7. Combinatorially Algebraic Systems

An *algebraic multi-system* is a pair $(\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ with

$$\widetilde{\mathscr{A}} = \bigcup_{i=1}^{m} \mathscr{H}_i \;\text{ and }\; \widetilde{\mathscr{O}} = \bigcup_{i=1}^{m} \mathcal{O}_i$$

such that for any integer $i, 1 \leq i \leq m$, $(\mathscr{H}_i; \mathcal{O}_i)$ is a multi-operation system. For an algebraic multi-operation system $(\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ and an integer $i, 1 \leq i \leq m$, a homomorphism $p_i : (\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}}) \to (\mathscr{H}_i; \mathcal{O}_i)$ is called a *sectional projection*, which is useful in multi-systems.

Two multi-systems $(\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}_1})$, $(\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}_2})$, where $\widetilde{\mathscr{A}_1} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i^k$ and $\widetilde{\mathscr{O}_1} = \bigcup\limits_{i=1}^{m} \mathcal{O}_i^k$ for $k = 1,\ 2$ are *homomorphic* if there is a mapping $o : \widetilde{\mathscr{A}_1} \to \widetilde{\mathscr{A}_2}$ such that op_i is a homomorphism for any integer $i, 1 \leq i \leq m$. By this definition, we know the existent conditions for homomorphisms on algebraic multi-systems following.

Theorem 7.1 *There exists a homomorphism from an algebraic multi-system* $(\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}_1})$ *to* $(\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}_2})$, *where* $\widetilde{\mathscr{A}_k} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i^k$ *and* $\widetilde{\mathscr{O}_k} = \bigcup\limits_{i=1}^{m} \mathcal{O}_i^k$ *for* $k = 1,\ 2$ *if and only if there are homomorphisms* $\eta_1, \eta_2, \cdots, \eta_m$ *on* $(\mathscr{H}_1^1; \mathcal{O}_1^1)$, $(\mathscr{H}_2^1; \mathcal{O}_2^1), \cdots$, $(\mathscr{H}_m^1; \mathcal{O}_m^1)$ *such that*

$$\eta_i|_{\mathscr{H}_i^1\cap\mathscr{H}_j^1} = \eta_j|_{\mathscr{H}_i^1\cap\mathscr{H}_j^1}$$

for any integer $1 \leq i, j \leq m$.

Proof By definition, if there is a homomorphism $o : (\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$, then op_i is a homomorphism on $(\mathscr{H}_i^1; \mathcal{O}_i^1)$ for any integer $i, 1 \leq i \leq m$.

On the other hand, if there are homomorphisms $\eta_1, \eta_2, \cdots, \eta_m$ on $(\mathscr{H}_1^1; \mathcal{O}_1^1)$, $(\mathscr{H}_2^1; \mathcal{O}_2^1), \cdots,$ $(\mathscr{H}_m^1; \mathcal{O}_m^1)$, define a mapping $o : (\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$ by $o(a) = \eta_i(a)$ if $a \in \mathscr{H}_i^1$. Then it can be checked immediately that o is a homomorphism. □

Let $o : (\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$ be a homomorphism with a unit $1_\circ$ for each operation $\circ \in \widetilde{\mathscr{O}}_2$. Similar to the case of multi-operation systems, we define the *multi-kernel* $\widetilde{\mathrm{Ker}}(o)$ by

$$\widetilde{\mathrm{Ker}}(o) = \{\ a \in \widetilde{\mathscr{A}_1} \mid o(a) = 1_\circ \ \text{ for } \ \forall \circ \in \widetilde{\mathscr{O}}_2\ \}.$$

Then we have the homomorphism theorem on algebraic multi-systems following.

Theorem 7.2 *Let* $(\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$ *be algebraic multi-systems, where* $\widetilde{\mathscr{A}_k} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i^k$, $\widetilde{\mathscr{O}_k} = \bigcup\limits_{i=1}^{m} \mathcal{O}_i^k$ *for* $k = 1, 2$ *and* $o : (\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$ *a onto homomorphism with a multi-group* $(\mathcal{I}_i^2; \mathcal{O}_i^2)$ *for any integer* $i, 1 \leq i \leq m$. *Then there are representation pairs* $(\widetilde{R}_1, \widetilde{P}_1)$ *and* $(\widetilde{R}_2, \widetilde{P}_2)$ *such that*

$$\frac{(\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1)}{(\widetilde{\mathrm{Ker}(o)}; \mathcal{O}_1)}\Big|_{(\widetilde{R}_1, \widetilde{P}_1)} \cong \frac{(\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)}{(\widetilde{\mathcal{I}}(\mathcal{O}_2); \mathcal{O}_2)}\Big|_{(\widetilde{R}_2, \widetilde{P}_2)}$$

where $(\widetilde{\mathcal{I}}(\mathcal{O}_2); \mathcal{O}_2) = \bigcup\limits_{i=1}^{m} (\mathcal{I}_i^2; \mathcal{O}_i^2)$.

Proof By definition, we know that $o|_{\mathscr{H}_i^1} : (\mathscr{H}_i^1; \mathcal{O}_i^1) \to (\mathscr{H}_{o(i)}^2; \mathcal{O}_{o(i)}^2)$ is also an onto homomorphism for any integer $i, 1 \leq i \leq m$. Applying Theorem 4.2 and Corollary 4.1, we can find representation pairs $(R_i^1, \widetilde{P}_i^1)$ and $(R_i^2, \widetilde{P}_i^2)$ such that

$$\frac{(\mathscr{H}_i^1; \mathcal{O}_i^1)}{(\mathrm{Ker}(o|_{\mathscr{H}_i^1}); \mathcal{O}_i^1)}_{(R_i^1, \widetilde{P}_i^1)} \cong \frac{(\mathscr{H}_{o(i)}^2; \mathcal{O}_{o(i)}^2)}{(\mathcal{I}_{o(i)}^2; \mathcal{O}_{o(i)}^2)}_{(R_{o(i)}^1, \widetilde{P}_{o(i)}^1)}.$$

Notice that

$$\widetilde{\mathscr{A}_k} = \bigcup_{i=1}^{m} \mathscr{H}_i^k, \quad \widetilde{\mathscr{O}_k} = \bigcup_{i=1}^{m} \mathcal{O}_i^k$$

for $k = 1, 2$ and

$$\widetilde{\mathrm{Ker}}(o) = \bigcup_{i=1}^{m} \mathrm{Ker}(o|_{\mathscr{H}_i^1}).$$

We finally get that

$$\frac{(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)}{(\widetilde{\mathrm{Ker}(o)};\mathcal{O}_1)}|_{(\widetilde{R}_1,\widetilde{P}_1)} \cong \frac{(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)}{(\widetilde{\mathcal{I}}(\mathcal{O}_2);\mathcal{O}_2)}|_{(\widetilde{R}_2,\widetilde{P}_2)}$$

with

$$\widetilde{R}_k = \bigcup_{i=1}^{m} R_i^k \quad \text{and} \quad \widetilde{P}_k = \bigcup_{i=1}^{m} \widetilde{P}_i^k$$

for $k = 1$ or 2. □

Let $(A;\circ)$ be an algebraic system with operation $\circ$. We associate a *labeled graph* $G^L[A]$ with $(A;\circ)$ by

$$V(G^L[A]) = A,$$

$$E(G^L[A]) = \{(a,c) \text{ with label } \circ b \mid \text{if } a \circ b = c \text{ for } \forall a,b,c \in A\},$$

as shown in Fig.7.1.

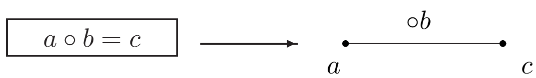

Fig.7.1

The advantage of this diagram on systems is that we can find a$\circ b = c$ for any edge in $G^L[A]$, if its vertices are a,c with a label $\circ b$ and vice versa immediately. For example, the labeled graph $G^L[Z_4]$ of an *Abelian* group Z_4 is shown in Fig.7.2.

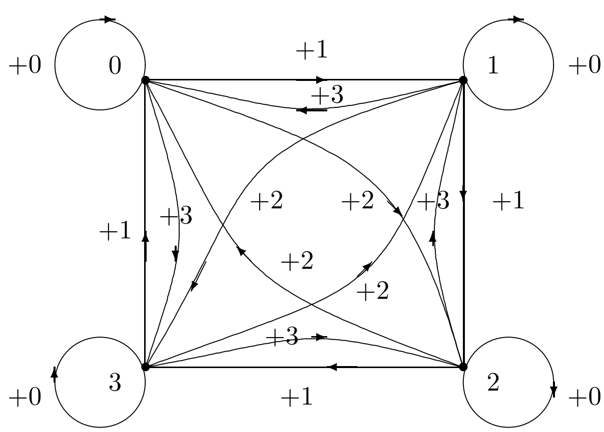

Fig.7.2

Some structure properties on these diagrams $G^L[A]$ of systems are shown in the following.

Property 1. *The labeled graph* $G^L[A]$ *is connected if and only if there are no partition* $A = A_1 \bigcup A_2$ *such that for* $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, *there are no definition for* $a_1 \circ a_2$ *in* $(A;\circ)$.

If $G^L[A]$ is disconnected, we choose one component C and let $A_1 = V(C)$. Define $A_2 = V(G^L[A]) \setminus V(C)$. Then we get a partition $A = A_1 \bigcup A_2$ and for $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, there are no definition for $a_1 \circ a_2$ in $(A;\circ)$, a contradiction.

Property 2. *If there is a unit 1_A in $(A;\circ)$, then there exists a vertex 1_A in $G^L[A]$ such that the label on the edge $(1_A, x)$ is $\circ x$.*

For a multiple 2-edge (a, b) in a directed graph, if two orientations on edges are both to a or both to b, then we say it a *parallel multiple* 2*-edge*. If one orientation is to a and another is to b, then we say it an *opposite multiple* 2*-edge*.

Property 3. *For $\forall a \in A$, if $a_\circ^{-1}$ exists, then there is an opposite multiple 2-edge $(\mathbf{1}_A, a)$ in $G^L[A]$ with labels $\circ a$ and $\circ a_\circ^{-1}$, respectively.*

Property 4. *For $\forall a, b \in A$ if $a \circ b = b \circ a$, then there are edges (a, x) and (b, x), $x \in A$ in $(A;\circ)$ with labels $w(a, x) = \circ b$ and $w(b, x) = \circ a$ in $G^L[A]$, respectively.*

Property 5. *If the cancellation law holds in $(A;\circ)$, i.e., for $\forall a, b, c \in A$, if $a \circ b = a \circ c$ then $b = c$, then there are no parallel multiple 2-edges in $G^L[A]$.*

These properties $2 - 5$ are gotten by definition. Each of these cases is shown in (1), (2), (3) and (4) in Fig.7.3.

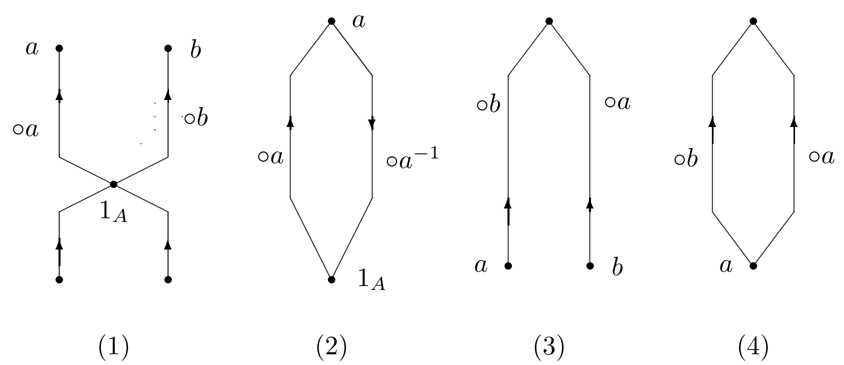

Fig.7.3

Now we consider the diagrams of algebraic multi-systems. Let $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ be an algebraic multi-system with

$$\widetilde{\mathscr{A}} = \bigcup_{i=1}^{m} \mathscr{H}_i \quad \text{and} \quad \widetilde{\mathscr{O}} = \bigcup_{i=1}^{m} \mathcal{O}_i$$

such that $(\mathscr{H}_i;\mathcal{O}_i)$ is a multi-operation system for any integer $i, 1 \le i \le m$, where the operation set $\mathcal{O}_i = \{\circ_{ij} | 1 \le j \le n_i\}$. Define a labeled graph $G^L[\widetilde{\mathscr{A}}]$ associated with $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ by

$$G^L[\widetilde{\mathscr{A}}] = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} G^L[(\mathscr{H}_i;\circ_{ij})],$$

where $G^L[(\mathscr{H}_i;\circ_{ij})]$ is the associated labeled graph of $(\mathscr{H}_i;\circ_{ij})$ for $1 \le i \le m$, $1 \le j \le n_{ij}$. The importance of $G^L[\widetilde{\mathscr{A}}]$ is displayed in the next result.

Theorem 7.3 *Let* $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ *be two algebraic multi-systems. Then*

$$(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1) \ \cong\ (\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$$

if and only if

$$G^L[\widetilde{\mathscr{A}_1}] \ \cong\ G^L[\widetilde{\mathscr{A}_2}].$$

Proof If $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1) \ \cong\ (\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$, by definition, there is a $1-1$ mapping $\omega:\widetilde{\mathscr{A}_1}\to\widetilde{\mathscr{A}_2}$ with $\omega:\widetilde{\mathscr{O}}_1\to\widetilde{\mathscr{O}}_2$ such that for $\forall a,b\in\widetilde{\mathscr{A}_1}$ and $\circ_1\in\widetilde{\mathscr{O}}_1$, there exists an operation $\circ_2\in\widetilde{\mathscr{O}}_2$ with the equality following hold,

$$\omega(a\circ_1 b)=\omega(a)\circ_2\omega(b).$$

Not loss of generality, assume $a\circ_1 b=c$ in $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$. Then for an edge (a,c) with a label $\circ_1 b$ in $G^L[\widetilde{\mathscr{A}_1}]$, there is an edge $(\omega(a),\omega(c))$ with a label $\circ_2\omega(b)$ in $G^L[\widetilde{\mathscr{A}_2}]$, i.e., ω is an equivalence from $G^L[\widetilde{\mathscr{A}_1}]$ to $G^L[\widetilde{\mathscr{A}_2}]$. Therefore, $G^L[\widetilde{\mathscr{A}_1}] \ \cong\ G^L[\widetilde{\mathscr{A}_2}]$.

Conversely, if $G^L[\widetilde{\mathscr{A}_1}] \ \cong\ G^L[\widetilde{\mathscr{A}_2}]$, let ϖ be a such equivalence from $G^L[\widetilde{\mathscr{A}_1}]$ to $G^L[\widetilde{\mathscr{A}_2}]$, then for an edge (a,c) with a label $\circ_1 b$ in $G^L[\widetilde{\mathscr{A}_1}]$, by definition we know that $(\omega(a),\omega(c))$ with a label $\omega(\circ_1)\omega(b)$ is an edge in $G^L[\widetilde{\mathscr{A}_2}]$. Whence,

$$\omega(a\circ_1 b)=\omega(a)\omega(\circ_1)\omega(b),$$

i.e., $\omega:\widetilde{\mathscr{A}_1}\to\widetilde{\mathscr{A}_2}$ is an isomorphism from $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$ to $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$. $\square$

Generally, let $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ be two algebraic multi-systems associated with labeled graphs $G^L[\widetilde{\mathscr{A}_1}],G^L[\widetilde{\mathscr{A}_2}]$. A *homomorphism* $\iota:G^L[\widetilde{\mathscr{A}_1}]\to G^L[\widetilde{\mathscr{A}_2}]$ is a mapping $\iota:V(G^L[\widetilde{\mathscr{A}_1}])\to V(G^L[\widetilde{\mathscr{A}_2}])$ and $\iota:\widetilde{\mathscr{O}}_1\to\widetilde{\mathscr{O}}_2$ such that $\iota(a,c)=(\iota(a),\iota(c))$ with a label $\iota(\circ)\iota(b)$ for $\forall(a,c)\in E(G^L[\widetilde{\mathscr{A}_1}])$ with a label $\circ b$. We get a result on homomorphisms of labeled graphs following.

Theorem 7.4 *Let* $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ *be algebraic multi-systems, where* $\widetilde{\mathscr{A}_k}=\bigcup\limits_{i=1}^{m}\mathscr{H}_i^k$, $\widetilde{\mathscr{O}}_k=\bigcup\limits_{i=1}^{m}\mathcal{O}_i^k$ *for* $k=1,2$ *and* $\iota:(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)\to(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ *a homomorphism. Then there is a homomorphism* $\iota:G^L[\mathscr{A}_1]\to G^L[\mathscr{A}_2]$ *from* $G^L[\mathscr{A}_1]$ *to* $G^L[\mathscr{A}_2]$ *induced by* ι.

Proof By definition, we know that $o:V(G^L[\mathscr{A}_1])\to V(G^L[\mathscr{A}_2])$. Now if $(a,c)\in E(G^L[\mathscr{A}_1])$ with a label $\circ b$, then there must be $a\circ b=c$ in $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$. Hence, $\iota(a)\iota(\circ)\iota(b)=\iota(c)$ in $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$, where $\iota(\circ)\in\widetilde{\mathscr{O}}_2$ by definition. Whence, $(\iota(a),\iota(c))\in E(G^L[\mathscr{A}_1])$ with a label $\iota(\circ)\iota(b)$ in $G^L[\mathscr{A}_2]$, i.e., ι is a homomorphism between $G^L[\mathscr{A}_1]$ and $G^L[\mathscr{A})_2]$. Therefore, ι induced a homomorphism from $G^L[\mathscr{A}_1]$ to $G^L[\mathscr{A}_2]$. $\square$

Notice that an algebraic multi-system $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ is a combinatorial system $\mathscr{C}_\Gamma$ with an underlying graph Γ, called a Γ-*multi-system*, where

$$V(\Gamma)=\{\mathscr{H}_i|1\leq i\leq m\},$$

$$E(\Gamma)=\{(\mathscr{H}_i,\mathscr{H}_j)|\exists a\in\mathscr{H}_i,b\in\mathscr{H}_j \text{ with } (a,b)\in E(G^L[\widetilde{\mathscr{A}}]) \text{ for } 1\leq i,j\leq m\}.$$

We obtain conditions for an algebraic multi-system with a graphical structure in the following.

Theorem 7.5 *Let $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ be an algebraic multi-system. Then it is*

(i) a circuit multi-system if and only if there is arrangement $\mathscr{H}_{l_i}, 1 \le i \le m$ for $\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_m$ such that

$$\mathscr{H}_{l_{i-1}} \bigcap \mathscr{H}_{l_i} \neq \emptyset,\ \mathscr{H}_{l_i} \bigcap \mathscr{H}_{l_{i+1}} \neq \emptyset$$

for any integer $i(mod\ m),\ 1 \le i \le m$ but

$$\mathscr{H}_{l_i} \bigcap \mathscr{H}_{l_j} = \emptyset$$

for integers $j \neq i-1, i, i+1(mod\ m)$;

(ii) a star multi-system if and only if there is arrangement $\mathscr{H}_{l_i}, 1 \le i \le m$ for $\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_m$ such that

$$\mathscr{H}_{l_1} \bigcap \mathscr{H}_{l_i} \neq \emptyset \ \ but \ \ \mathscr{H}_{l_i} \bigcap \mathscr{H}_{l_j} = \emptyset$$

for integers $1 < i, j \le m, i \neq j$.

(iii) a tree multi-system if and only if any subset of $\widetilde{\mathscr{A}}$ is not a circuit multi-system under operations in $\widetilde{\mathscr{O}}$.

Proof By definition, these conditions really ensure a circuit, star, or a tree multi-system. Conversely, a circuit, star, or a tree multi-system constrains these conditions, respectively. □

Now if an associative system $(\mathscr{A};\circ)$ has a unit and inverse element $a_\circ^{-1}$ for any element $a \in \mathscr{A}$, i.e., a group, then for any elements $x, y \in \mathscr{A}$, there is an edge $(x,y) \in E(G^L[\mathscr{A}])$. In fact, by definition, there is an element $z \in \mathscr{A}$ such that $x_\circ^{-1} \circ y = z$. Whence, $x \circ z = y$. By definition, there is an edge (x,y) with a label $\circ z$ in $G^L[\mathscr{A}]$, and an edge (y,x) with label $z_\circ^{-1}$. Thereafter, the diagram of a group is a complete graph attached with a loop at each vertex, denoted by $K[\mathscr{A};\circ]$. As a by-product, the diagram $G^L[\widetilde{G}]$ of a m-group $\widetilde{G}$ is a union of m complete graphs with the same vertices, each attached with m loops.

Summarizing previous discussion, we can sketch the diagram of a multi-group as follows.

Theorem 7.6 *Let $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ be a multi-group with $\widetilde{\mathscr{A}} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$, $\widetilde{\mathscr{O}} = \bigcup\limits_{i=1}^{m} \mathcal{O}_i$, $\mathcal{O}_i = \{\circ_{ij}, 1 \le j \le n_i\}$ and $(\mathscr{H}_i;\circ_{ij})$ a group for integers $i, j,\ 1 \le i \le m, 1 \le i \le n_i$. Then its diagram $G^L[\mathscr{A}]$ is*

$$G^L[\mathscr{A}] = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} K[\mathscr{H}_i;\circ_{ij}].$$

Corollary 7.1 *The diagram of a field $(\mathscr{H};+,\circ)$ is a union of two complete graphs attached with 2 loops at each vertex.*

Corollary 7.2 *Let $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ be a multi-group. Then $G^L[\mathscr{A}]$ is hamiltonian if and only if $\mathscr{C}_\Gamma$ is hamiltonian.*

Proof Notice that $\mathscr{C}_\Gamma$ is an resultant graph in $G^L[\mathscr{A}]$ shrinking each $\bigcup_{j=1}^{n_i} K[\mathscr{H}_i;\circ_{ij}]$ to a vertex $\mathscr{H}_i$ for $1 \leq i \leq m$ by definition. Whence, $\mathscr{C}_\Gamma$ is hamiltonian if $G^L[\mathscr{A}]$ is hamiltonian.

Conversely, if $\mathscr{C}_\Gamma$ is hamiltonian, we can easily find a hamiltonian circuit in $G^L[\mathscr{A}]$ by applying Theorem 7.6. □

§8. Remarks

8.1 These conceptions of multi-group, multi-ring, multi-field and multi-vector space are first presented in [11]-[14] introduced by Smarandache multi-spaces. In Sections $4-5$, we consider their general case, i.e., *multi-operation systems* and extend the homomorphism theorem to this multi-system. Section 6 is a generalization of works in [13] to multi-modules. There are many trends or topics in multi-systems should be researched, such as extending those of results in groups, rings or linear spaces to multi-systems.

8.2 The topic discussed in Section 7 can be seen as an application of combinatorial speculation([16]) to classical algebra. In fact, there are many research trends in *combinatorially algebraic systems*, in algebra or combinatorics. For example, *given an underlying combinatorial structure G, what can we say about its algebraic behavior*? Similarly, *what can we know on its graphical structure, such as in what condition it has a hamiltonian circuit, or a* 1*-factor*? *When it is regular*? $\cdots$, etc..

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