International J.Math. Combin. Vol.3 (2008), 111-121

Actions of Multi-groups on Finite Sets

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Abstract: Classifying objects needs permutation groups in mathematics. Similarly, consideration should be also done for actions of multi-groups, i.e., permutation multi-groups. In this paper, we consider the action of multi-groups on a finite multi-set, its orbits, multi-transitive, primitive, etc. By choosing an element p in or not in a permutation group Γ, define a new operation $\circ_p$ enables us to finding permutation multi-groups. Considering such permutation multi-groups, some interesting results in finite permutation groups are generalized to permutation multi-groups.

Key Words: Action of multi-group, permutation multi-group, representation, transitive.

AMS(2000): 05E25, 20B05, 20B20, 51M15.

§1. Introduction

A bijection $f : X \to X$ is called a *permutation* of X. In the case of finite, there is a useful way for representing a permutation τ on X, $|X| = n$ by a $2 \times n$ table following,

$$\tau = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n, \end{pmatrix},$$

where, $x_i, y_i \in X$ and $x_i \neq x_j$, $y_i \neq y_j$ if $i \neq j$ for $1 \leq i, j \leq n$. For three sets X, Y and Z, let $f : X \to Y$ and $h : Y \to Z$ be mapping. Define a mapping $h \circ f : X \to Z$, called the *composition of f and h* by

$$h \circ f(x) = h(f(x))$$

for $\forall x \in X$. For example, let

$$\tau = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n, \end{pmatrix},$$

and

$$\varsigma = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n, \end{pmatrix}.$$

[1]Received July 8, 2008. Accepted September 28, 2008.

Then

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n, \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n, \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ z_1 & z_2 & \cdots & z_n, \end{pmatrix}.$$

It is well-known that all permutations form a group $\Pi(X)$ under the composition operation. For $\forall p \in \Pi(X)$, define an operation $\circ_p : \Pi(X) \times \Pi(X) \to \Pi(X)$ by

$$\sigma \circ_p \varsigma = \sigma p \varsigma, \quad \text{for } \forall \sigma, \varsigma \in \Pi(X).$$

Then we have

Theorem 1.1 *$(\Pi(X); \circ_p)$ is a group.*

Proof We check these conditions for a group hold in $(\Pi(X); \circ_p)$. In fact, for $\forall \tau, \sigma, \varsigma \in \Pi(X)$,

$$\begin{aligned}(\tau \circ_p \sigma) \circ_p \varsigma &= (\tau p \sigma) \circ_p \varsigma = \tau p \sigma p \varsigma \\ &= \tau p(\sigma \circ_p \varsigma) = \tau \circ_p (\sigma \circ_p \varsigma).\end{aligned}$$

The unit in $(\Pi(X); \circ_p)$ is $1_{\circ_p} = p^{-1}$. In fact, for $\forall \theta \in \Pi(X)$, we have that $p^{-1} \circ_p \theta = \theta \circ_p p^{-1} = \theta$.

For an element $\sigma \in \Pi(X)$, $\sigma_{\circ_p}^{-1} = p^{-1}\sigma^{-1}p^{-1} = (p\sigma p)^{-1}$. In fact,

$$\sigma \circ_p (p\sigma p)^{-1} = \sigma p p^{-1} \sigma^{-1} p^{-1} = p^{-1} = 1_{\circ_p},$$

$$(p\sigma p)^{-1} \circ_p \sigma = p^{-1}\sigma^{-1}p^{-1}p\sigma = p^{-1} = 1_{\circ_p}.$$

By definition, we know that $(\Pi(X); \circ_p)$ is a group. □

Notice that if $p = \mathbf{1}_X$, the operation $\circ_p$ is just the composition operation and $(\Pi(X); \circ_p)$ is the symmetric group $Sym(X)$ on X. Furthermore, Theorem 1.1 opens a general way for constructing multi-groups on permutations, which enables us to find the next result.

Theorem 1.2 *Let Γ be a permutation group on X, i.e., $\Gamma \prec Sim(X)$. For given m permutations $p_1, p_2, \cdots, p_m \in \Gamma$, $(\Gamma; \mathscr{O}_P)$ with $\mathscr{O}_P = \{\circ_p, p \in P\}, P = \{p_i, 1 \leq i \leq m\}$ is a permutation multi-group, denoted by $\mathscr{G}_X^P$.*

Proof First, we check that $(\Gamma; \{\circ_{p_i}, 1 \leq i \leq m\})$ is an associative system. Actually, for $\forall \sigma, \varsigma, \tau \in \mathscr{G}$ and $p, q \in \{p_1, p_2, \cdots, p_m\}$, we know that

$$\begin{aligned}(\tau \circ_p \sigma) \circ_q \varsigma &= (\tau p \sigma) \circ_q \varsigma = \tau p \sigma q \varsigma \\ &= \tau p(\sigma \circ_q \varsigma) = \tau \circ_p (\sigma \circ_q \varsigma).\end{aligned}$$

Similar to the proof of Theorem 1.1, we know that $(\Gamma;\circ_{p_i})$ is a group for any integer $i, 1 \leq i \leq m$. In fact, $1_{\circ_{p_i}} = p_i^{-1}$ and $\sigma_{\circ_{p_i}}^{-1} = (p_i\sigma p_i)^{-1}$ in $(\mathscr{G};\circ_{p_i})$. □

§2. Multi-permutation Representations

The construction for permutation multi-groups shown in Theorems $1.1-1.2$ can be also transferred to permutations on vector as follows, which is useful in some circumstances.

Choose m permutations $p_1, p_2, \cdots, p_m$ on X. An m-permutation on $x \in X$ is defined by

$$p^{(m)} : x \to (p_1(x), p_2(x), \cdots, p_m(x)),$$

i.e., an m-vector on x.

Denoted by $\Pi^{(s)}(X)$ all such s-vectors $p^{(m)}$. Let $\circ$ be an operation on X. Define a *bullet operation of two m-permutations*

$$P^{(m)} = (p_1, p_2, \cdots, p_m),$$

$$Q^{(sm)} = (q_1, q_2, \cdots, q_m)$$

on $\circ$ by

$$P^{(s)} \bullet Q^{(s)} = (p_1 \circ q_1, p_2 \circ q_2, \cdots, p_m \circ q_m).$$

Whence, if there are l-operations $\circ_i, 1 \leq i \leq l$ on X, we obtain an s-permutation system $\Pi^{(s)}(X)$ under these l bullet operations $\bullet_i, 1 \leq i \leq l$, denoted by $(\Pi^{(s)}(X);\odot_1^l)$, where $\odot_1^l = \{\bullet_i | 1 \leq i \leq l\}$.

Theorem 2.1 *Any s-operation system $(\mathscr{H}, \widetilde{O})$ on $\mathscr{H}$ with units $1_{\circ_i}$ for each operation $\circ_i, 1 \leq i \leq s$ in $\widetilde{O}$ is isomorphic to an s-permutation system $(\Pi^{(s)}(\mathscr{H});\odot_1^s)$.*

Proof For $a \in \mathscr{H}$, define an s-permutation $\sigma_a \in \Pi^{(s)}(\mathscr{H})$ by

$$\sigma_a(x) = (a \circ_1 x, a \circ_2 x, \cdots, a \circ_s x)$$

for $\forall x \in \mathscr{H}$.

Now let $\pi : \mathscr{H} \to \Pi^{(s)}(\mathscr{H})$ be determined by $\pi(a) = \sigma_a^{(s)}$ for $\forall a \in \mathscr{H}$. Since

$$\sigma_a(1_{\circ_i}) = (a \circ_1 1_{\circ_i}, \cdots, a \circ_{i-1} 1_{\circ_i}, a, a \circ_{i+1} 1_{\circ_i}, \cdots, a \circ_s 1_{\circ_i}),$$

we know that for $a, b \in \mathscr{H}$, $\sigma_a \neq \sigma_b$ if $a \neq b$. Hence, π is a $1-1$ and onto mapping. For $\forall i, 1 \leq i \leq s$ and $\forall x \in \mathscr{H}$, we find that

$$\begin{aligned}
\pi(a \circ_i b)(x) &= \sigma_{a \circ_i b}(x) \\
&= (a \circ_i b \circ_1 x, a \circ_i b \circ_2 x, \cdots, a \circ_i b \circ_s x) \\
&= (a \circ_1 x, a \circ_2 x, \cdots, a \circ_s x) \bullet_i (b \circ_1 x, b \circ_2 x, \cdots, b \circ_s x) \\
&= \sigma_a(x) \bullet_i \sigma_b(x) = \pi(a) \bullet_i \pi(b)(x).
\end{aligned}$$

Therefore, $\pi(a \circ_i b) = \pi(a) \bullet_i \pi(b)$, which implies that π is an isomorphism from $(\mathscr{H}, \widetilde{O})$ to $(\Pi^{(s)}(\mathscr{H}); \odot_1^s)$. □

According to Theorem 2.1, these algebraic multi-systems are the same as permutation multi-systems, particularly for multi-groups.

Corollary 2.1 *Any s-group $(\mathscr{H}, \widetilde{O})$ with $\widetilde{O} = \{\circ_i | 1 \leq i \leq s\}$ is isomorphic to an s-permutation multi-group $(\Pi^{(s)}(\mathscr{H}); \odot_1^s)$.*

Proof It can be shown easily that $(\Pi^{(s)}(\mathscr{H}); \odot_1^s)$ is a multi-group if $(\mathscr{H}, \widetilde{O})$ is a multi-group. □

For the special case of $s = 1$ in Corollary 2.1, we get the well-known Cayley result on groups.

Corollary 2.2(Cayley) *A group G is isomorphic to a permutation group.*

As shown in Theorem 1.2, many operations can be defined on a permutation group G, which enables it to be a permutation multi-group, and generally, these operations $\circ_i, 1 \leq i \leq s$ on permutations in Theorem 2.1 need not to be the composition of permutations. If we choose all $\circ_i, 1 \leq i \leq s$ to be just the composition of permutation, then all bullet operations in $\odot_1^s$ is the same, denoted by $\odot$. We find an interesting result following which also implies the Cayley's result on groups, i.e., Corollary 2.2.

Theorem 2.2 *$(\Pi^{(s)}(\mathscr{H}); \odot)$ is a group of order $\frac{(n!)!}{(n!-s)!}$.*

Proof By definition, we know that

$$P^{(s)}(x) \odot Q^{(s)}(x) = (P_1Q_1(x), P_2Q_2(x), \cdots, P_sQ_s(x))$$

for $\forall P^{(s)}, Q^{(s)} \in \Pi^{(s)}(\mathscr{H})$ and $\forall x \in \mathscr{H}$. Whence, $(1, 1, \cdots, 1)$ (l entries 1) is the unit and $(P^{-(s)}) = (P_1^{-1}, P_2^{-1}, \cdots, P_s^{-1})$ the inverse of $P^{(s)} = (P_1, P_2, \cdots, P_s)$ in $(\Pi^{(s)}(\mathscr{H}); \odot)$. Therefore, $(\Pi^{(s)}(\mathscr{H}); \odot)$ is a group.

Calculation shows that the order of $\Pi^{(s)}(\mathscr{H})$ is

$$\begin{pmatrix} n! \\ s \end{pmatrix} s! = \frac{(n!)!}{(n!-s)!},$$

which completes the proof. □

§3. Action of Multi-group

Let $(\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ be a multi-group, where $\widetilde{\mathscr{A}} = \bigcup_{i=1}^{m} \mathscr{H}_i$, $\widetilde{\mathscr{O}} = \bigcup_{i=1}^{m} \mathcal{O}_i$, and $\widetilde{X} = \bigcup_{i=1}^{m} X_i$ a multi-set. An *action* φ of $(\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ on $\widetilde{X}$ is defined to be a homomorphism

$$\varphi : (\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}}) \to \bigcup_{i=1}^{m} \mathscr{G}_{X_i}^{P_i}$$

for sets $P_1, P_2, \cdots, P_m \geq 1$ of permutations, i.e., for $\forall h \in \mathscr{H}_i$, $1 \leq i \leq m$, there is a permutation $\varphi(h) : x \to x^h$ with conditions following hold,

$$\varphi(h \circ g) = \varphi(h)\varphi(\circ)\varphi(g), \quad \text{for } h, g \in \mathscr{H}_i \text{ and } \circ \in \mathcal{O}_i.$$

Whence, we only need to consider the action of permutation multi-groups on multi-sets. Let $= (\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ be a multi-group action on a multi-set $\widetilde{X}$, denoted by $\mathscr{G}$. For a subset $\Delta \subset \widetilde{X}$, $x \in \Delta$, we define

$$x^{\mathscr{G}} = \{\ x^g \mid \forall g \in \mathscr{G}\} \ \text{ and } \ \mathscr{G}_x = \{\ g \mid x^g = x, g \in \mathscr{G}\},$$

called the *orbit* and *stabilizer* of x under the action of $\mathscr{G}$ and sets

$$\mathscr{G}_\Delta = \{\ g \mid x^g = x, g \in \mathscr{G} \ \text{ for } \ \forall x \in \Delta\},$$

$$\mathscr{G}_{(\Delta)} = \{\ g \mid \Delta^g = \Delta, g \in \mathscr{G} \ \text{ for } \ \forall x \in \Delta\},$$

respectively. Then we know the result following.

Theorem 3.1 *Let Γ be a permutation group action on X and $\mathscr{G}_X^P$ a permutation multi-group $(\Gamma; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}$ and $p_i \in \Gamma$ for integers $1 \leq i \leq m$. Then*

(i) $|\mathscr{G}_X^P| = |(\mathscr{G}_X^P)_x||x^{\mathscr{G}_X^P}|$, $\forall x \in X$;

(ii) for $\forall \Delta \subset X$, $((\mathscr{G}_X^P)_\Delta, \mathcal{O}_P)$ is a permutation multi-group if and only if $p_i \in P$ for $1 \leqslant i \leq m$.

Proof By definition, we know that

$$(\mathscr{G}_X^P)_x = \Gamma_x, \ \text{and } x^{\mathscr{G}_X^P} = x^\Gamma$$

for $x \in X$ and $\Delta \subset X$. Assume that $x^\Gamma = \{x_1, x_2, \cdots, x_l\}$ with $x^{g_i} = x_i$. Then we must have

$$\Gamma = \bigcup_{i=1}^{l} g_i \Gamma_x.$$

In fact, for $\forall h \in \Gamma$, let $x^h = x_k, 1 \leq k \leq m$. Then $x^h = x^{g_k}$, i.e., $x^{hg_k^{-1}} = x$. Whence, we get that $hg_k^{-1} \in \Gamma_x$, namely, $h \in g_k\Gamma_x$.

For integers $i, j, i \neq j$, there are must be $g_i\Gamma_x \cap g_j\Gamma_x = \emptyset$. Otherwise, there exist $h_1, h_2 \in \Gamma_x$ such that $g_ih_1 = g_jh_2$. We get that $x_i = x^{g_i} = x^{g_jh_2h_1^{-1}} = x^{g_j} = x_j$, a contradiction.

Therefore, we find that

$$|\mathscr{G}_X^P| = |\Gamma| = |\Gamma_x||x^\Gamma| = |(\mathscr{G}_X^P)_x||x^{\mathscr{G}_X^P}|.$$

This is the assertion (i). For (ii), notice that $(\mathscr{G}_X^P)_\Delta = \Gamma_\Delta$ and Γ_Δ is itself a permutation group. Applying Theorem 1.2, we find it. □

Particularly, for a permutation group Γ action on Ω, i.e., all $p_i = \mathbf{1}_X$ for $1 \leq i \leq m$, we get a consequence of Theorem 3.1.

Corollary 3.1 *Let Γ be a permutation group action on Ω. Then*

(*i*) $|\Gamma| = |\Gamma_x||x^\Gamma|$, $\forall x \in \Omega$;

(*ii*) *for* $\forall \Delta \subset \Omega$, Γ_Δ *is a permutation group.*

Theorem 3.2 *Let Γ be a permutation group action on X and $\mathscr{G}_X^P$ a permutation multi-group $(\Gamma; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}$, $p_i \in \Gamma$ for integers $1 \leq i \leq m$ and $Orb(X)$ the orbital sets of $\mathscr{G}_X^P$ action on X. Then*

$$|Orb(X)| = \frac{1}{|\mathscr{G}_X^P|} \sum_{p \in \mathscr{G}_X^P} |\Phi(p)|,$$

where $\Phi(p)$ is the fixed set of p, i.e., $\Phi(p) = \{x \in X | x^p = x\}$.

Proof Consider a set $E = \{(p, x) \in \mathscr{G}_X^P \times X | x^p = x\}$. Then we know that $E(p, *) = \Phi(p)$ and $E(*, x) = (\mathscr{G}_X^P)_x$. Counting these elements in E, we find that

$$\sum_{p \in \mathscr{G}_X^P} |\Phi(p)| = \sum_{x \in X} (\mathscr{G}_X^P)_x.$$

Now let $x_i, 1 \leq i \leq |Orb(X)|$ be representations of different orbits in $Orb(X)$. For an element y in $x_i^{\mathscr{G}_X^P}$, let $y = x_i^g$ for an element $g \in \mathscr{G}_X^P$. Now if $h \in (\mathscr{G}_X^P)_y$, i.e., $y^h = y$, then we find that $(x_i^g)^h = x_i^g$. Whence, $x_i^{ghg^{-1}} = x_i$. We obtain that $ghg^{-1} \in (\mathscr{G}_X^P)_{x_i}$, namely, $h \in g^{-1}(\mathscr{G}_X^P)_{x_i} g$. Therefore, $(\mathscr{G}_X^P)_y \subset g^{-1}(\mathscr{G}_X^P)_{x_i} g$. Similarly, we get that $(\mathscr{G}_X^P)_{x_i} \subset g(\mathscr{G}_X^P)_y g^{-1}$, i.e., $(\mathscr{G}_X^P)_y = g^{-1}(\mathscr{G}_X^P)_{x_i} g$. We know that $|(\mathscr{G}_X^P)_y| = |(\mathscr{G}_X^P)_{x_i}|$ for any element in $x_i^{\mathscr{G}_X^P}, 1 \leq i \leq |Orb(X)|$. This enables us to obtain that

$$\begin{aligned} \sum_{p \in \mathscr{G}_X^P} |\Phi(p)| &= \sum_{x \in X} (\mathscr{G}_X^P)_x \\ &= \sum_{i=1}^{|Orb(X)|} \sum_{y \in x_i^{\mathscr{G}_X^P}} |(\mathscr{G}_X^P)_{x_i}| \\ &= \sum_{i=1}^{|Orb(X)|} |x_i^{\mathscr{G}_X^P}||(\mathscr{G}_X^P)_{x_i}| \\ &= \sum_{i=1}^{|Orb(X)|} |\mathscr{G}_X^P| = |Orb(X)||\mathscr{G}_X^P| \end{aligned}$$

by applying Theorem 3.1. This completes the proof. □

For a permutation group Γ action on Ω, i.e., all $p_i = \mathbf{1}_X$ for $1 \leq i \leq m$, we get the famous *Burnside's Lemma* by Theorem 3.2.

Corollary 3.2(Burnside's Lemma) *Let Γ be a permutation group action on Ω. Then*

$$|Orb(\Omega)| = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} |\Phi(g)|.$$

§4. Transitive Multi-groups

A permutation multi-group $\mathscr{G}_X^P$ is *transitive* on X if $|Orb(X)| = 1$, i.e., for any elements $x, y \in X$, there is an element $g \in \mathscr{G}_X^P$ such that $x^g = y$. In this case, we know formulae following by Theorems 3.1 and 3.2.

$$|\mathscr{G}_X^P| = |X||(\mathscr{G}_X^P)_x| \quad \text{and} \quad |\mathscr{G}_X^P| = \sum_{p \in \mathscr{G}_X^P} |\Phi(p)|$$

Similarly, a permutation multi-group $\mathscr{G}_X^P$ is *k-transitive* on X if for any two k-tuples $(x_1, x_2, \cdots, x_k)$ and $(y_1, y_2, \cdots, y_k)$, there is an element $g \in \mathscr{G}_X^P$ such that $x_i^g = y_i$ for any integer $i, 1 \leq i \leq k$. It is well-known that $Sym(X)$ is $|X|$-transitive on a finite set X.

Theorem 4.1 *Let Γ be a transitive group action on X and $\mathscr{G}_X^P$ a permutation multi-group $(\Gamma; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}$ and $p_i \in \Gamma$ for integers $1 \leq i \leq m$. Then for an integer k,*

(i) $(\Gamma; X)$ is k-transitive if and only if $(\Gamma_x; X \setminus \{x\})$ is $(k-1)$-transitive;

(ii) $\mathscr{G}_X^P$ is k-transitive on X if and only if $(\mathscr{G}_X^P)_x$ is $(k-1)$-transitive on $X \setminus \{x\}$.

Proof If Γ is k-transitive on X, it is obvious that Γ is $(k-1)$-transitive on X itself. Conversely, if Γ_x is $(k-1)$-transitive on $X \setminus \{x\}$, then for two k-tuples $(x_1, x_2, \cdots, x_k)$ and $(y_1, y_2, \cdots, y_k)$, there are elements $g_1, g_2 \in \Gamma$ and $h \in \Gamma_x$ such that

$$x_1^{g_1} = x, \quad y_1^{g_2} = x \quad \text{and} \quad (x_i^{g_1})^h = y_i^{g_2}$$

for any integer $i, 2 \leq i \leq k$. Therefore,

$$x_i^{g_1 h g_2^{-1}} = y_i, \quad 1 \leq i \leq k.$$

We know that Γ is 'k-transitive on X. This is the assertion of (i).

By definition, $\mathscr{G}_X^P$ is k-transitive on X if and only if Γ is k-transitive, i.e., $(\mathscr{G}_X^P)_x$ is $(k-1)$-transitive on $X \setminus \{x\}$ by (i), which is the assertion of (ii). □

Applying Theorems 3.1 and 4.1 repeatedly, we get an interesting consequence for k-transitive multi-groups.

Theorem 4.2 *Let $\mathscr{G}_X^P$ be a k-transitive multi-group and $\Delta \subset X$ with $|\Delta| = k$. Then*

$$|\mathscr{G}_X^P| = |X|(|X| - 1) \cdots (|X| - k + 1|(\mathscr{G}_X^P)_\Delta|.$$

Particularly, a k-transitive multi-group $\mathscr{G}_X^P$ with $|\mathscr{G}_X^P| = |X|(|X| - 1) \cdots (|X| - k + 1|$ is called a *sharply k-transitive multi-group*. For example, choose $\Gamma = Sym(X)$ with $|X| = n$, i.e., the symmetric group S_n and permutations $p_i \in S_n$, $1 \leq i \leq m$, we get an n-transitive multi-group $(S_n; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}$.

Let Γ be a transitive group action on X and $\mathscr{G}_X^P$ a permutation multi-group $(\Gamma; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}, p_i \in \Gamma$ for integers $1 \leq i \leq m$. An equivalent relation R on X is *$\mathscr{G}_X^P$-admissible* if for $\forall (x, y) \in R$, there is $(x^g, y^g) \in R$ for $\forall g \in \mathscr{G}_X^P$. For a given set X and permutation multi-group $\mathscr{G}_X^P$, it can be shown easily by definition that

$$R = X \times X \quad \text{or} \quad R = \{(x,x)|x \in X\}$$

are $\mathscr{G}_X^P$-admissible, called *trivially $\mathscr{G}_X^P$-admissible relations*. A transitive multi-group $\mathscr{G}_X^P$ is *primitive* if there are no $\mathscr{G}_X^P$-admissible relations R on X unless $R = X \times X$ or $R = \{(x,x)|x \in X\}$, i.e., the trivially relations. The next result shows the existence of primitive multi-groups.

Theorem 4.3 *Every k-transitive multi-group $\mathscr{G}_X^P$ is primitive if $k \geq 2$.*

Proof Otherwise, there is a $\mathscr{G}_X^P$-admissible relations R on X such that $R \neq X \times X$ and $R \neq \{(x,x)|x \in X\}$. Whence, there must exists $(x,y) \in R$, $x, y \in X$ and $x \neq y$. By assumption, $\mathscr{G}_X^P$ is k-transitive on X, $k \geq 2$. For $\forall z \in X$, there is an element $g \in \mathscr{G}_X^P$ such that $x^g = x$ and $y^g = z$. This fact implies that $(x,z) \in R$ for $\forall z \in X$ by definition. Notice that R is an equivalence relation on X. For $\forall z_1, z_2 \in X$, we get $(z_1, x), (x, z_2) \in R$. Thereafter, $(z_1, z_2) \in R$, namely, $R = X \times X$, a contradiction. □

There is a simple criterion for determining which permutation multi-group is primitive by maximal stabilizers following.

Theorem 4.4 *A transitive multi-group $\mathscr{G}_X^P$ is primitive if and only if there is an element $x \in X$ such that $p \in (\mathscr{G}_X^P)_x$ for $\forall p \in P$ and if there is a permutation multi-group $(\mathcal{H}; \mathscr{O}_P)$ enabling $((\mathscr{G}_X^P)_x; \mathscr{O}_P) \prec (\mathcal{H}; \mathscr{O}_P) \prec \mathscr{G}_X^P$, then $(\mathcal{H}; \mathscr{O}_P) = ((\mathscr{G}_X^P)_x; \mathscr{O}_P)$ or $\mathscr{G}_X^P$.*

Proof If $(\mathcal{H}; \mathscr{O}_P)$ be a multi-group with $((\mathscr{G}_X^P)_x; \mathscr{O}_P) \prec (\mathcal{H}; \mathscr{O}_P) \prec \mathscr{G}_X^P$ for an element $x \in X$, define a relation

$$R = \{\ (x^g, x^{g \circ h})\ |\ g \in \mathscr{G}_X^P,\ h \in \mathcal{H}\ \}.$$

for a chosen operation $\circ \in \mathscr{O}_P$. Then R is a $\mathscr{G}_X^P$-admissible equivalent relation. In fact, it is $\mathscr{G}_X^P$-admissible, reflexive and symmetric by definition. For its transitiveness, let $(s,t) \in R$, $(t,u) \in R$. Then there are elements $g_1, g_2 \in \mathscr{G}_X^P$ and $h_1, h_2 \in \mathcal{H}$ such that

$$s = x^{g_1},\ t = x^{g_1 \circ h_1},\ t = x^{g_2},\ u = x^{g_2 \circ h_2}.$$

Hence, $x^{g_2^{-1} \circ g_1 \circ h_1} = x$, i.e., $g_2^{-1} \circ g_1 \circ h_1 \in \mathcal{H}$. Whence, $g_2^{-1} \circ g_1,\ g_1^{-1} \circ g_2 \in \mathcal{H}$. Let $g^* = g_1$, $h^* = g_1^{-1} \circ g_2 \circ h_2$. We find that $s = x^{g^*}$, $u = x^{g^* \circ h^*}$. Therefore, $(s,u) \in R$. This concludes that R is an equivalent relation.

Now if $\mathscr{G}_X^P$ is primitive, then $R = \{(x,x)|x \in X\}$ or $R = X \times X$ by definition. Assume $R = \{(x,x)|x \in X\}$. Then $s = x^g$ and $t = x^{g \circ h}$ implies that $s = t$ for $\forall g \in \mathscr{G}_X^P$ and $h \in \mathcal{H}$. Particularly, for $g = 1_\circ$, we find that $x^h = x$ for $\forall h \in \mathcal{H}$, i.e., $(\mathcal{H}; \mathscr{O}_P) = ((\mathscr{G}_X^P)_x; \mathscr{O}_P)$.

If $R = X \times X$, then $(x, x^f) \in R$ for $\forall f \in \mathscr{G}_X^P$. In this case, there must exist $g \in \mathscr{G}_X^P$ and $h \in \mathcal{H}$ such that $x = x^g, x^f = x^{g \circ h}$. Whence, $g \in ((\mathscr{G}_X^P)_x; \mathscr{O}_P) \prec (\mathcal{H}; \mathscr{O}_P)$ and $g^{-1} \circ h^{-1} \circ f \in ((\mathscr{G}_X^P)_x; \mathscr{O}_P) \prec (\mathcal{H}; \mathscr{O}_P)$. Therefore, $f \in \mathcal{H}$ and $(\mathcal{H}; \mathscr{O}_P) = ((\mathscr{G}_X^P); \mathscr{O}_P)$.

Conversely, assume R to be a $\mathscr{G}_X^P$-admissible equivalent relation and there is an element $x \in X$ such that $p \in (\mathscr{G}_X^P)_x$ for $\forall p \in P$, $((\mathscr{G}_X^P)_x; \mathscr{O}_P) \prec (\mathcal{H}; \mathscr{O}_P) \prec \mathscr{G}_X^P$ implies that $(\mathcal{H}; \mathscr{O}_P) = ((\mathscr{G}_X^P)_x; \mathscr{O}_P)$ or $((\mathscr{G}_X^P); \mathscr{O}_P)$. Define

$$\mathcal{H} = \{\ h \in \mathscr{G}_X^P\ |\ (x, x^h) \in R\ \}.$$

Then $(\mathcal{H};\mathscr{O}_P)$ is multi-subgroup of $\mathscr{G}_X^P$ which contains a multi-subgroup $((\mathscr{G}_X^P)_x;\mathscr{O}_P)$ by definition. Whence, $(\mathcal{H};\mathscr{O}_P) = ((\mathscr{G}_X^P)_x;\mathscr{O}_P)$ or $\mathscr{G}_X^P$.

If $(\mathcal{H};\mathscr{O}_P) = ((\mathscr{G}_X^P)_x;\mathscr{O}_P)$, then x is only equivalent to itself. Since $\mathscr{G}_X^P$ is transitive on X, we know that $R = \{(x,x)|x\in X\}$.

If $(\mathcal{H};\mathscr{O}_P) = \mathscr{G}_X^P$, by the transitiveness of $\mathscr{G}_X^P$ on X again, we find that $R = X\times X$. Combining these discussions, we conclude that $\mathscr{G}_X^P$ is primitive. □

Choose $p = 1_X$ for each $p\in P$ in Theorem 4.4, we get a well-known result in classical permutation groups following.

Corollary 4.1 *A transitive group Γ is primitive if and only if there is an element $x\in X$ such that a subgroup H with $\Gamma_x \prec H \prec \Gamma$ hold implies that $H = \Gamma_x$ or Γ.*

§5. Extended Permutation Multi-groups

Let Γ be a permutation group action on a set X and $P\subset \Pi(X)$. We have shown in Theorem 1.2 that $(\Gamma;\mathscr{O}_P)$ is a multi-group if $P\subset\Gamma$. Then *what can we say if not all $p\in P$ are in Γ*? For this case, we introduce a new multi-group $(\widetilde{\Gamma};\mathscr{O}_P)$ on X, the *permutation multi-group generated by P in Γ* by

$$\widetilde{\Gamma} = \{\ g_1\circ_{p_1} g_2\circ_{p_2}\cdots\circ_{p_l} g_{l+1}\ |\ g_i\in\Gamma, p_j\in P,\ 1\le i\le l+1, 1\le j\le l\ \},$$

denoted by $\left\langle\Gamma_X^P\right\rangle$. This multi-group has good behavior like $\mathscr{G}_X^P$, also a kind way of extending a group to a multi-group. For convenience, a group generated by a set S with the operation in Γ is denoted by $\langle S\rangle_\Gamma$.

Theorem 5.1 *Let Γ be a permutation group action on a set X and $P\subset\Pi(X)$. Then*

(*i*) $\left\langle\Gamma_X^P\right\rangle = \langle\Gamma\cup P\rangle_\Gamma$*, particularly,* $\left\langle\Gamma_X^P\right\rangle = \mathscr{G}_X^P$ *if and only if* $P\subset\Gamma$*;*

(*ii*) *for any subgroup Λ of Γ, there exists a subset $P\subset\Gamma$ such that*

$$\left\langle\Lambda_X^P;\mathscr{O}_P\right\rangle = \left\langle\Gamma_X^P\right\rangle.$$

Proof By definition, for $\forall a,b\in\Gamma$ and $p\in P$, we know that

$$a\circ_p b\ =\ apb.$$

Choosing $a = b = 1_\Gamma$, we find that

$$a\circ_p b\ =\ p,$$

i.e., $\Gamma\subset\widetilde{\Gamma}$. Whence,

$$\langle\Gamma\cup P\rangle_\Gamma\subset\left\langle\Gamma_X^P\right\rangle.$$

Now for $\forall g_i\in\Gamma$ and $p_j\in P$, $1\le i\le l+1$, $1\le j\le l$, we know that

$$g_1 \circ_{p_1} g_2 \circ_{p_2} \cdots \circ_{p_l} g_{l+1} = g_1 p_1 g_2 p_2 \cdots p_l g_{l+1},$$

which means that

$$\left\langle \Gamma_X^P \right\rangle \subset \langle \Gamma \cup P \rangle_\Gamma .$$

Therefore,

$$\left\langle \Gamma_X^P \right\rangle = \langle \Gamma \cup P \rangle_\Gamma .$$

Now if $\left\langle \Gamma_X^P \right\rangle = \mathscr{G}_X^P$, i.e., $\langle \Gamma \cup P \rangle_\Gamma = \Gamma$, there must be $P \subset \Gamma$. According to Theorem 1.2, this concludes the assertion (i).

For the assertion (ii), notice that if $P = \Gamma \setminus \Lambda$, we get that

$$\left\langle \Lambda_X^P \right\rangle = \left\langle \Lambda \bigcup P \right\rangle_\Gamma = \Gamma$$

by (i). Whence, there always exists a subset $P \subset \Gamma$ such that

$$\left\langle \Lambda_X^P ; \mathscr{O}_P \right\rangle = \left\langle \Gamma_X^P \right\rangle .$$

□

Theorem 5.2 *Let Γ be a permutation group action on a set X. For an integer $k \geq 1$, there is a set $P \in \Pi(X)$ with $|P| \leq k$ such that $\left\langle \Gamma_X^P \right\rangle$ is k-transitive.*

Proof Notice that the symmetric group $Sym(X)$ is $|X|$-transitive for any finite set X. If Γ is k-transitive on X, choose $P = \emptyset$ enabling the conclusion true. Otherwise, assume these orbits of Γ action on X to be $O_1, O_2, \cdots, O_s$, where $s = |Orb(X)|$. Construct a permutation $p \in \Pi(X)$ by

$$p = (x_1, x_2, \cdots, x_s),$$

where $x_i \in O_i$, $1 \leq i \leq s$ and let $P = \{p\} \subset Sym(X)$. Applying Theorem 5.1, we know that $\left\langle \Gamma_X^P \right\rangle = \langle \Gamma \cup P \rangle_\Gamma$ is transitive on X with $|P| = 1$.

Now for an integer k, if $\left\langle \Gamma_X^{P_1} \right\rangle$ is k-transitive with $|P_1| \leq k$, let $O_1', O_2', \cdots, O_l'$ be these orbits of the stabilizer $\left\langle \Gamma_X^{P_1} \right\rangle_{y_1 y_2 \cdots y_k}$ action on $X \setminus \{y_1, y_2, \cdots, y_k\}$. Construct a permutation

$$q = (z_1, z_2, \cdots, z_l),$$

where $z_i \in O_i'$, $1 \leq i \leq l$ and let $P_2 = P_1 \cup \{q\}$. Applying Theorem 5.1 again, we find that $\left\langle \Gamma_X^{P_2} \right\rangle_{y_1 y_2 \cdots y_k}$ is transitive on $X \setminus \{y_1, y_2, \cdots, y_k\}$, where $|P_2| \leq |P_1| + 1$. Therefore, $\left\langle \Gamma_X^{P_2} \right\rangle$ is $(k+1)$-transitive on X with $|P_2| \leq k+1$ by Theorem 2.5.7.

Applying the induction principle, we get the conclusion. □

Notice that any k-transitive multi-group is primitive if $k \geq 2$ by Theorem 4.3. We have a corollary following by Theorem 5.2.

Corollary 5.1 *Let Γ be a permutation group action on a set X. There is a set $P \in \Pi(X)$ such that $\left\langle \Gamma_X^P \right\rangle$ is primitive.*

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