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Subclasses of Analytic Functions Associated with q-Derivative

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Abstract: In this paper, we define the classes $\mathcal{T}_q(A, B, \lambda)$ and $\mathcal{C}_q(A, B, \lambda)$ using Janowski class and q-derivative also we study coefficient estimates, extreme points and many more properties.

Key Words: Janowski class, extreme points, convex linear combination, q-derivative.

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§1. Introduction

Let $\mathcal{A}$ denote the family of analytic functions defined in the open unit disc

$$\mathcal{U} = \{z : |z| < 1\},$$

which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathcal{U}$, consisting of analytic functions whose non-zero coefficients from the second term onwards are negative. That is, an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \tag{1.2}$$

which are univalent in the open unit disc $\mathcal{U}$.

The q-shifted factorial is defined for α, $q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & \text{n=0;} \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & \text{n} \in \mathbb{N}, \end{cases} \tag{1.3}$$

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and in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \quad (n>0), \tag{1.4}$$

where the q-gamma functions [2], [3] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty}, \quad (0<q<1). \tag{1.5}$$

Note that, if $|q|<1$, the q-shifted factorial (1.3), remains meaningful for $n=\infty$ as a convergent infinite product

$$(\alpha;q)_\infty = \prod_{m=0}^{\infty}(1-\alpha q^m).$$

Now recall the following q-analogue definitions given by Gasper and Rahman [2]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \text{ where } [x]_q = \frac{(1-q^x)}{(1-q)} \tag{1.6}$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [2]

$$D_q f(z) = \frac{f(z)-f(zq)}{z(1-q)}, \quad (z\neq 0,\, q\neq 0). \tag{1.7}$$

$$\int_0^z f(t)d_q(t) = z(1-q)\sum_{m=0}^{\infty} q^m f(zq^m). \tag{1.8}$$

In view of the relation

$$\lim_{q\to 1^-}\frac{(q^{\alpha};q)_n}{(1-q)^n} = (\alpha)_n, \tag{1.9}$$

we observe that the q-shifted fractional (1.2) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

For $-1\leq A<B\leq 1, \mathcal{P}_1(A,B)$ [4] denotes the class of analytic functions in $\mathcal{U}$ which are of the form $\dfrac{1+A\omega(z)}{1+B\omega(z)}$, where ω is a bounded analytic function satisfying the conditions $\omega(0)=0$ and $|\omega(z)|<1$.

Now we define the subclass $\mathcal{T}_q(A,B,\lambda)$ consisting of functions $f\in\mathcal{T}$ such that

$$\frac{zD_q(f(z))}{\lambda zD_q(f(z))+(1-\lambda)f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \tag{1.10}$$

where, $-1\leq A<B\leq 1,\ 0<q<1,\ \lambda>0,\ z\in\mathcal{U}$.

Let $\mathcal{C}_q(A,B,\lambda)$ denote the class of functions $f\in\mathcal{T}$ such that $zf'\in\mathcal{T}_q(A,B,\lambda)$. For $\lambda=0$ and $q\to 1^-$ we get the well-known classes $\mathcal{T}^*(A,B)$ and $\mathcal{C}(A,B)$ studied by Ganesan in [1].

For parametric values $A = 2\alpha - 1$ and $B = 1$ and as $q \to 1^-$ we get the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ studied by Mostafa [5]. In particular, if $q \to 1^-$ we get the classes studided by Ravikumar et al. [6].

In the next section we obtain the characterization properties for the classes $\mathcal{T}_q(A, B, \lambda)$ and $\mathcal{C}_q(A, B, \lambda)$.

§2. Main Results

Theorem 2.1 *A function* $f \in \mathcal{T}_q(A, B, \lambda)$ *if and only if*

$$\sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} a_n \leq B - A \tag{2.1}$$

for $-1 \leq A < B \leq 1$, $0 < q < 1$, $\lambda > 0$ *and* $z \in \mathcal{U}$.

Proof Suppose $f \in \mathcal{T}_q(A, B, \lambda)$. Then

$$\begin{aligned} \Re\left\{\frac{zD_q(f(z))}{\lambda z D_q(f(z)) + (1-\lambda)f(z)}\right\} &> \frac{1+A}{1+B}, \\ \Re\left\{\frac{z - \sum_{n=2}^{\infty}[n]_q a_n z^n}{z - \sum_{n=2}^{\infty}[\lambda([n]_q - 1) + 1]a_n z^n}\right\} &> \frac{1+A}{1+B}. \end{aligned}$$

Letting $z \to 1$, then we get,

$$\left[1 - \sum_{n=2}^{\infty}[n]_q a_n z^n\right](1+B) > (1+A)\left[1 - \sum_{n=2}^{\infty}[\lambda([n]_q - 1) + 1]a_n\right].$$

Hence

$$\sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} a_n \leq B - A.$$

Conversely, if (2.1) holds, it suffices to show that $|\omega(z)| < 1$. From (1.10), we have

$$\begin{aligned} |\omega(z)| &= \left|\frac{\sum_{n=2}^{\infty}[(\lambda - 1)([n]_q - 1)]a_n z^n}{(B-A)z - \sum_{n=2}^{\infty}[[n]_q B - A(\lambda - 1 - [n]_q\lambda)]a_n z^n}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}[(\lambda - 1)([n]_q - 1)]a_n}{(B-A) - \sum_{n=2}^{\infty}[[n]_q B - A(\lambda - 1 - [n]_q\lambda)]a_n}. \end{aligned}$$

The last expression is bounded by 1 if

$$\sum_{n=2}^{\infty}[(\lambda-1)([n]_q-1)]a_n \leq (B-A)-\sum_{n=2}^{\infty}[[n]_qB-A(\lambda-1-[n]_q\lambda)]a_n$$

which is equivalent to (2.1). Hence the proof. □

Analogous to Theorem 2.1 we get the following result.

Theorem 2.2 *A function* $f \in \mathcal{C}_q(A,B,\lambda)$ *if and only if*

$$\sum_{n=2}^{\infty}[n]_q\left\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\right\}a_n \leq B-A. \tag{2.2}$$

Corollary 2.3 *If function* $f(z) \in \mathcal{T}_j$ *is in the class* $\mathcal{T}_q(A,B,\lambda)$ *then*

$$|a_n| \leq \frac{(B-A)}{\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}}$$

for some $-1 \leq A < B \leq 1$, $\lambda > 0$, $0 < q < 1$, *and* $z \in \mathcal{U}$.

Now we determine extreme points for the class $\mathcal{T}_q(A,B,\lambda)$.

Theorem 2.4 *Let* $f(z) \in \mathcal{T}_q(A,B,\lambda)$. *Define* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{B-A}{\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}}z^n, \quad n \geq 2$$

for some $-1 \leq A < B \leq 1$, $\lambda > 0$, $0 < q < 1$, *and* $z \in \mathcal{U}$. *Then* $f \in \mathcal{T}_q(A,B,\lambda)$ *if and only if* f *can be expressed as*

$$f(z) = \sum_{n=2}^{\infty}\mu_n f_n(z),$$

where $\mu_n \geq 0$ *and* $\displaystyle\sum_{n=1}^{\infty}\mu_n = 1$.

Proof If

$$f(z) = \sum_{n=2}^{\infty}\mu_n f_n(z) \text{ with } \sum_{n=1}^{\infty}\mu_n = 1, \quad \mu_n \geq 0,$$

then

$$\sum_{n=2}^{\infty}\frac{\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}}{\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}}\mu_n(B-A)$$
$$=\sum_{n=2}^{\infty}\mu_n(B-A) = (1-\mu_j)(B-A) \leq (B-A).$$

Hence, $f(z) \in \mathcal{T}_q(A,B,\lambda)$.

Conversely, let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}_q(A, B, \lambda),$$

define

$$\mu_n = \frac{\{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\}|a_k|}{(B-A)}, \quad n \geq 2$$

and

$$\mu_n = 1 - \sum_{n=2}^{\infty} \mu_n.$$

From Theorem 2.1, $\sum_{n=2}^{\infty} \mu_n \leq 1$ and hence $\mu_1 \geq 0$.

Since $\mu_n f_n(z) = \mu_n f(z) + a_n z^n$, we get that

$$\sum_{n=1}^{\infty} \mu_n f_n(z) = z - \sum_{n=2}^{\infty} a_n z^n = f(z). \qquad \square$$

Theorem 2.5 *The class $\mathcal{T}_q(A, B, \lambda)$ is closed under convex linear combination.*

Proof Let $f(z),\ g(z) \in \mathcal{T}_q(A, B, \lambda)$ and let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z - \sum_{n=2}^{\infty} b_n z^n.$$

For a number η such that $0 \leq \eta \leq 1$, it suffices to show that the function defined by $h(z) = (1-\eta)f(z) + \eta g(z),\ z \in \mathcal{U}$ belongs to $\mathcal{T}_q(A, B, \lambda)$. Now

$$h(z) = z - \sum_{n=2}^{\infty} [(1-\eta)a_n + \eta b_n] z^n.$$

Applying Theorem 2.1 to $f(z),\ g(z) \in \mathcal{T}_q(A, B, \lambda)$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} [(1-\eta)a_n + \eta b_n] \\
&= (1-\eta) \sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} a_n \\
&\quad + \eta \sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} b_n \\
&\leq (1-\eta)(B-A) + \eta(B-A) = (B-A).
\end{aligned}$$

This implies that $h(z) \in \mathcal{T}_q(A, B, \lambda)$. $\square$

Theorem 2.6 *For integers $i = 1, 2, \cdots, n$, let $f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \in \mathcal{T}_q(A, B, \lambda)$ and*

$0<\beta_i<1$ such that $\sum_{i=1}^{n}\beta_i=1$, then the function $F(z)$ defined by

$$F(z)=\sum_{i=1}^{n}\beta_i f_i(z)$$

is also in $\mathcal{T}_q(A,B,\lambda)$.

Proof For each integer $i\in\{1,2,3,\cdots,n\}$, we obtain

$$\sum_{n=2}^{\infty}\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}\,|a_{n,i}|<(B-A).$$

Since

$$F(z)=\sum_{i=1}^{n}\beta_i(z-\sum_{n=2}^{\infty}a_{n,i}z^n)=z-\sum_{n=2}^{\infty}(\sum_{i=1}^{n}\beta_i a_{n,i})z^n$$

and

$$\begin{aligned}&\sum_{n=2}^{\infty}\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}\left[\sum_{i=1}^{n}\beta_i a_{n,i}\right]\\ =&\sum_{i=1}^{n}\beta_i\left[\sum_{n=2}^{\infty}\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}\right]\\ <&\sum_{i=1}^{n}\beta_i(B-A)<(B-A),\end{aligned}$$

we therefore know that $F(z)\in\mathcal{T}_q(A,B,\lambda)$. □

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